Lecture 21

We now know that Λ-Mod has enough injectives. Building on this one proves that any Grothendieck abelian category has enough injectives (e.g. quasi-coherent sheaves on a scheme). This is of fundamental importance because while projectives may seem more "natural" to us, in practice (at least in algebraic geometry) it is the <u>injectives</u> that one ends up using to define derived functors like Ext, for the very simple reason that while every Grothendieck abelian category $\mathcal{A}$ has enough injectives, $\mathcal{A}$ may <u>fail to have enough projectives</u>:

<u>Ex 1</u> (Hard) Prove that a quasi-coherent sheaf on $\mathbb{P}^1(\mathbb{C})$ is projective in $\mathrm{Qcoh}(\mathbb{P}^1)$ if and only if it is zero.

This is typical of non-affine schemes. With this in mind, the aim of today's lecture is to better understand injective modules over commutative Noetherian rings. We begin however with some general theory of <u>injective envelopes</u>, following Mitchell "Theory of categories" III.2.

<u>Ex 2</u> Prove that $\mathrm{Ext}(\mathbb{Q}, \mathbb{Z}) \cong \mathbb{R}$ as abelian groups.

Let Λ be a ring, $\mathcal{C} = \Lambda$-Mod.

<u>Def^n</u> An <u>essential extension</u> of a Λ-module A' is a monomorphism $u: A' \longrightarrow A$ such that for any nonzero submodule B of A we have $B \cap A' \neq 0$. Equivalently, for every $0 \neq a \in A$ there is $\lambda \in \Lambda$ with $\lambda a \in A'$, $\lambda a \neq 0$.

A <u>proper</u> extension is $u: A' \hookrightarrow A$ which is mono but not an isomorphism.

<u>Lemma 1</u> A mono $u : A' \longrightarrow A$ is an essential extension if and only if every morphism $f : A \longrightarrow B$ such that fu is mono, f itself is mono.

<u>Proof</u> f not mono $\Rightarrow Ker(f) \neq 0$, so if u is essential, $Ker(f) \cap A' \neq 0$, hence fu is not mono.

If u is not essential, let $C \subseteq A$ be a submodule with $C \cap A' = 0$, $C \neq 0$, and let f be $A \xrightarrow{f} A/C$. Then $A' \longrightarrow A \longrightarrow A/C$ is mono but f is not. $\square$

<u>Lemma 2</u> Q is injective if and only if it admits no proper essential extension.

<u>Proof</u> If Q is injective and $u : Q \longrightarrow A$ essential then u is split mono, say $A \cong Q \oplus Q'$. But then $Q' \cap Q = 0$ implies $Q' = 0$ so u is not proper.

Suppose every essential extension of Q is an iso. Since Λ-Mod has enough injectives, let $Q \xrightarrow{\iota} I$ be mono with I injective. We show ι is split, from which we get Q injective.

Let $\mathcal{C}$ be the set of submodules $M \subseteq I$ with $M \cap Q$ 0. Then is a poset, and if $\{M_i\}_{i \in I}$ is a chain then $\bigcup_i M_i = \sum_i M_i$ satisfies

$$\left(\bigcup_i M_i\right) \cap Q = \bigcup_i \left(M_i \cap Q \right) = 0$$

so every chain in $\mathcal{C}$ has an upper bound. By Zorn's Lemma there is a maximal element $\overline{M}$. Now, since $\overline{M} \cap Q = 0$ the map $Q \longrightarrow I \xrightarrow{\pi} I/\overline{M}$ is mono, but not epi (if it is, $Q \longrightarrow I$ is split and we are done already) hence by hypothesis <u>not</u> an essential extension. But if $0 \neq \overline{B} \subseteq I/\overline{M}$ satisfies $\overline{B} \cap Q = 0$ then $B := \pi^{-1}(B)$ a submodule of I with $B \supsetneq \overline{M}$ and $B \cap Q = 0$ in I, contradicting maximality of $\overline{M}$. $\square$

<u>Def^n</u> A <u>direct family of submodules</u> $\{M_i\}_{i\in I}$ of a Λ-module M is a family of submodules $M_i \subseteq M$ s.t. for every pair $i, j \in I$ there is $k \in I$ with both $M_i \subseteq M_k$ and $M_j \subseteq M_k$.

<u>Lemma 3</u> If every element of a direct family of submodules $\{M_i\}_{i\in I}$ of M is an essential extension of some fixed $A \subseteq M$ then $\bigcup_i M_i = \sum_i M_i$ is also an essential extension of A.

<u>Lemma 4</u> If $u: A \longrightarrow B$ and $v: B \longrightarrow C$ are mono, then vu is an essential extension iff. both u and v are essential extensions.

<u>Ex 3</u> Prove Lemma 3 and Lemma 4.

<u>Def^n</u> An <u>injective envelope</u> for a Λ-module M is an essential extension $M \longrightarrow I$ with I injective.

<u>Example</u> $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an injective envelope in <u>Ab</u>.

<u>Ex 4</u> For p a prime let $\mathbb{Z}_{p^\infty} \subseteq \mathbb{Q}/\mathbb{Z}$ denote the subgroup of cosets $\left[\frac{a}{p^s}\right]$ with $a \in \mathbb{Z}$ and $s \geqslant 1$. Prove that $\mathbb{Z}_{p^\infty}$ is an injective envelope of $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ in <u>Ab</u>.

<u>Lemma 5</u> If $u_1: M \longrightarrow I_1$ and $u_2: M \longrightarrow I_2$ are injective envelopes then there is an isomorphism $\theta: I_1 \longrightarrow I_2$ with $\theta \circ u_1 = u_2$ (θ is <u>not</u> unique!).

<u>Proof</u> Since I_2 is injective, there is θ with $\theta \circ u_1 = u_2$. Now, θ is mono by Lemma 1, and an essential extension by Lemma 4, hence an iso by Lemma 2. $\square$

Even though injective envelopes are only unique up to <u>non-unique</u> iso, we sometimes abuse notation and say "the" injective envelope.

Theorem Every Λ-module M has an injective envelope, denoted $E(M)$.

Proof Let $M \longrightarrow I$ be mono with I injective, and $\mathcal{C}$ the poset of all submodules $A \subseteq I$ which are essential extensions of M. Any chain in $\mathcal{C}$ is a direct family, so the union is by Lemma 3 again an essential extension, so by Zorn $\mathcal{C}$ has a maximal element $\bar{A}$. We claim $\bar{A}$ is injective. It suffices by Lemma 2 to show $\bar{A}$ admits no proper essential extension. Suppose for a contradiction $u : \bar{A} \longrightarrow T$ were such an extension. By injectivity there is f making

$$\begin{tikzcd} \bar{A} \arrow[r, hook] \arrow[dr, "u"'] & I \\ & T \arrow[u, dashed, "f"'] \end{tikzcd}$$

⌜Note this also shows that if $M \subseteq I$ with I injective, then
$$M \subseteq E(M) \subseteq I,$$
properly understood⌟

commute. By Lemma 1, f is mono. But then by Lemma 4, $M \longrightarrow \bar{A} \longrightarrow T$ is an essential extension bigger than $\bar{A}$, a contradiction. $\square$

Theorem (Matlis) Suppose R is a commutative Noetherian ring. Then every injective R-module is a coproduct of injective R-modules of the form $E(R/\mathfrak{p})$, as $\mathfrak{p}$ varies over all prime ideals $\mathfrak{p} \subseteq R$.

Ex 5 Prove directly (i.e. without the Theorem) that $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_{p \text{ prime}} \mathbb{Z}_{p^\infty} = \bigoplus_p E(\mathbb{Z}_p)$. As we already know $\mathbb{Q} = E(\mathbb{Z})$, we have essentially accounted for all the injective (= divisible) abelian groups. Observe that $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ can be written now as

$$0 \longrightarrow \mathbb{Z} \longrightarrow E(\mathbb{Z}) \longrightarrow \bigoplus_p E(\mathbb{Z}_p) \longrightarrow 0.$$

This is actually typical: the injective resolution of a (nice) ring R puts the $E(R/\mathfrak{p})$'s in order according to the dimension of $R/\mathfrak{p}$.

<u>Example</u> Since $R = \mathbb{C}[x]$ is a PID, injective = divisible. Clearly the field of fractions $\mathbb{C}(x)$ is divisible, as is $\mathbb{C}(x)/\mathbb{C}[x]$, whence both are injective. The prime ideals are (0) and $(x-\lambda)$ for $\lambda \in \mathbb{C}$ and with $\mathbb{C}_\lambda := \mathbb{C}[x]/(x-\lambda)$

$$E(\mathbb{C}[x]) = \mathbb{C}(x)$$

$$E(\mathbb{C}_\lambda) = \left\{ \frac{a(x)}{(x-\lambda)^s} \;\middle|\; a(x) \in \mathbb{C}[x],\ s \geqslant 1 \right\} \subseteq \mathbb{C}(x)/\mathbb{C}[x].$$

By Matlis, any injective is a coproduct of copies of these. In particular,

$$\mathbb{C}(x)/\mathbb{C}[x] \cong \bigoplus_{\lambda \in \mathbb{C}} E(\mathbb{C}_\lambda).$$

Observe that in the exact sequence

$$0 \longrightarrow \mathbb{C}[x] \longrightarrow \mathbb{C}(x) \xrightarrow{\pi} \mathbb{C}(x)/\mathbb{C}[x] \longrightarrow 0,$$

a rational function $F = p(x)/q(x)$ is sent to components $\pi(F)_\lambda \in E(\mathbb{C}_\lambda)$ with $\pi(F)_\lambda = 0 \iff F$ is defined at λ, and otherwise if $\pi(F)_\lambda = a(x)/(x-\lambda)^s$ in lowest common denominator, s is the order of the pole of F at λ.