Lecture 24

We have defined the left derived functors $L_nT : R\underline{\text{Mod}} \longrightarrow S\underline{\text{Mod}}$ for $n \geq 0$ and any additive functor $T: R\underline{\text{Mod}} \longrightarrow S\underline{\text{Mod}}$. In the next two lectures we examine the example of Tor, which is the left derived functor of the <u>tensor product</u>. Throughout we fix a ring R (not necessarily commutative).

<u>Def^n</u> Given a right R-module M, set $T = M \otimes_R - : R\underline{\text{Mod}} \longrightarrow \underline{Ab}$ and denote the left derived functors by

$$\mathrm{Tor}_n^R(M, -) := L_n T(-) : R\underline{\text{Mod}} \longrightarrow \underline{Ab}$$

so that $\mathrm{Tor}_n^R(M,N) = H_n(\cdots \longrightarrow M \otimes_R P_1 \longrightarrow M \otimes_R P_0)$
where $P \longrightarrow N$ is a projective resolution of N.

<u>Example</u> If P is projective then $\mathrm{Tor}_n^R(M,P) = 0$ for $n > 0$ as we may take $P_0 = P$, $P_n = 0$ for $n > 0$.

<u>Example</u> The tensor product is right exact so for any N

$$M \otimes_R P_1 \longrightarrow M \otimes_R P_0 \longrightarrow M \otimes_R N \longrightarrow 0$$

is exact, hence there is a natural iso $\mathrm{Tor}_0^R(M,N) \cong M \otimes_R N$ of abelian groups (at this point, we claim naturality in N only).

<u>Example</u> If $R = \mathbb{Z}$ then $\mathrm{Tor}_n^R(M,N) = 0$ for $n \geq 2$, since any N has a projective resolution $0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow N \longrightarrow 0$. This is a consequence of submodules of projective modules being projective (in fact this vanishing for $n \geq 2$ is true over any PID for the same reason).

This means for a PID R we need only worry about $\mathrm{Tor}_1^R(M,N)$, and naturally we start with N f.g. (in fact since any N is a direct limit of its f.g. submodules and Tor commutes with direct limits, this is essentially enough to compute any Tor over a PID, but we will not use this). By the classification of f.g. modules over R and the fact that

<u>Ex 1</u> Prove that if $T: \underline{R\mathrm{Mod}} \longrightarrow \underline{S\mathrm{Mod}}$ is additive then $T(\bigoplus_{i=1}^n M_i) \cong \bigoplus_{i=1}^n T(M_i)$ for any collection $\{M_i\}_{i\in I}$ of R-modules.

<u>Lemma</u> For any ring R and non-zerodivisor $r \in R$ there is a natural isomorphism

$$\mathrm{Tor}_1^R(M, R/(r)) \cong \{x \in M \mid rx = 0\}. \qquad (2.1)$$

<u>Proof</u> The sequence

$$0 \longrightarrow R \xrightarrow{r} R \longrightarrow R/(r) \longrightarrow 0$$

is exact since r is a non-zerodivisor, and gives a projective resolution of $R/(r)$. Hence we compute Tor_1 from

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & M\otimes_R R \arrow[r, "1\otimes r"] \arrow[d, phantom, "\cong" rotate=90] & M\otimes_R R \arrow[d, phantom, "\cong" rotate=90] & & R \arrow[r] & R \\
\cdots \arrow[r] & 0 \arrow[r] & M \arrow[r, "r"] & M \\
 & & 1 &
\end{tikzcd}$$

i.e. $\mathrm{Tor}_1^R(M, R/(r)) = \mathrm{Ker}(M \xrightarrow{r} M)$ as claimed. $\square$

<u>Note</u> This explains the notation: $\mathrm{Tor}_1^{\mathbb{Z}}(M, \mathbb{Z}a)$ computes the <u>a-torsion</u> of an abelian group.

<u>Example</u> With the notation of the lemma, there are canonical R-linear maps

$$\varphi : R/(r^2) \longrightarrow R/(r) \qquad \psi : R/(r) \longrightarrow R/(r^2)$$
$$\varphi(\bar{s}) = \bar{s} \qquad \psi(\bar{s}) = \overline{rs}$$

which lift to maps on the resolutions

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r, "r^2"] \arrow[d, "r"'] & R \arrow[r] \arrow[d, "1"] & R/(r^2) \arrow[r] \arrow[d, "\varphi"] & 0 \\ 0 \arrow[r] & R \arrow[r, "r"] & R \arrow[r] & R/(r) \arrow[r] & 0 \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & R \arrow[r, "r"] \arrow[d, "1"'] & R \arrow[r] \arrow[d, "r"] & R/(r) \arrow[r] \arrow[d, "\psi"] & 0 \\ 0 \arrow[r] & R \arrow[r, "r^2"'] & R \arrow[r] & R/(r^2) \arrow[r] & 0 \end{tikzcd}$$

and hence induce maps on $M \otimes_R -$ of the resolutions

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Tor}_1^R(M, R/r^2) \arrow[r] \arrow[d, dashed, "\mathrm{Tor}(M,\varphi)"] & M \arrow[r, "r^2"] \arrow[d, "r"] & M \arrow[d, "1"] \\ 0 \arrow[r] & \mathrm{Tor}_1^R(M, R/r) \arrow[r] & M \arrow[r, "r"'] & M \end{tikzcd}$$

i.e. $\mathrm{Tor}(M,\varphi)$ corresponds under (2.1) to

$$\begin{tikzcd} \{x \in M \mid r^2 x = 0\} \arrow[d] & x \arrow[d, mapsto] \\ \{x \in M \mid rx = 0\} & rx \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Tor}_1^R(M, R/r) \arrow[r] \arrow[d, dashed, "\mathrm{Tor}(M,\psi)"] & M \arrow[r, "r"] \arrow[d] & M \arrow[d] \\ 0 \arrow[r] & \mathrm{Tor}_1^R(M, R/r^2) \arrow[r] & M \arrow[r, "r^2"'] & M \end{tikzcd}$$

i.e. $\mathrm{Tor}(M,\psi)$ corresponds to

$$\begin{tikzcd} \{x \in M \mid rx = 0\} \arrow[d] & x \arrow[d, mapsto] \\ \{x \in M \mid r^2 x = 0\} & x \end{tikzcd}$$

This raises a natural question: for what R-module (if any) Q_r do we have

$$\mathrm{Tor}_1^R(M, Q_r) = \{ x \in M \mid r^i x = 0 \text{ for some } i \geq 0 \} \,?$$

We will answer this for $R = \mathbb{Z}$ but the same thing works for any PID (and even more generally, with suitable tweaks). But first we need a general theorem, which we will prove later.

Theorem For any ring M, right R-module M and exact sequence of left R-modules

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$$

we have a long exact sequence of abelian groups

$$\cdots \longrightarrow \operatorname{Tor}_n^R(M,N') \longrightarrow \operatorname{Tor}_n^R(M,N) \longrightarrow \operatorname{Tor}_n^R(M,N'')$$
$$\longrightarrow \operatorname{Tor}_{n-1}^R(M,N') \longrightarrow \operatorname{Tor}_{n-1}^R(M,N) \longrightarrow \operatorname{Tor}_{n-1}^R(M,N'') \longrightarrow \cdots$$

$$\cdots \longrightarrow \operatorname{Tor}_1^R(M,N') \longrightarrow \operatorname{Tor}_1^R(M,N) \longrightarrow \operatorname{Tor}_1^R(M,N'')$$
$$\longrightarrow M \otimes_R N' \longrightarrow M \otimes_R N \longrightarrow M \otimes_R N'' \longrightarrow 0$$

Theorem If R is commutative then $\operatorname{Tor}_n^R(M,N) \cong \operatorname{Tor}_n^R(N,M)$, for all $n \geq 0$ and R-modules M, N.

Example If R is commutative and F is flat, then for $n > 0$

$$\operatorname{Tor}_n^R(M,F) \cong \operatorname{Tor}_n^R(F,M)$$

$$= H_n(\cdots \longrightarrow F \otimes_R P_1^M \longrightarrow F \otimes_R P_0^M)$$

$$= 0 \quad \text{since } F \otimes_R - \text{ is exact}$$

Ex 2 If R is commutative and $S \subseteq R$ is multiplicatively closed (i.e. $1 \in S$ and $s, t \in S \Rightarrow st \in S$) then $S^{-1}R$ is a flat R-module. In particular, $\mathbb{Q}$ is a flat $\mathbb{Z}$-module.

Example Take $R = \mathbb{Z}$ and consider the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

and corresponding long exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(M, \mathbb{Z}) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(M, \mathbb{Q}) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \longrightarrow M \otimes_{\mathbb{Z}} \mathbb{Z} \longrightarrow M \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow M \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

Since $\mathrm{Tor}_1^{\mathbb{Z}}(M, \mathbb{Q}) = 0$ as $\mathbb{Q}$ is flat, we obtain an exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \longrightarrow M \xrightarrow{\varphi} M \otimes_{\mathbb{Z}} \mathbb{Q}$$

where $\varphi(m) = m \otimes 1$ is the canonical map, hence $\mathrm{Tor}_1^{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is identified with $\mathrm{Tors}(M) = \{x \in M \mid nx = 0 \text{ for some } n \neq 0 \text{ in } \mathbb{Z}\}$ the <u>torsion submodule</u> of M.

Example Let $S = \{1, p, p^2, \dots\}$ for p a prime and write $\mathbb{Z}[p^{-1}]$ for $S^{-1}\mathbb{Z}$. There is an exact sequence $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}[p^{-1}] \longrightarrow \mathbb{Z}[p^{-1}]/\mathbb{Z} \longrightarrow 0$ from which we deduce an isomorphism

call this $\mathrm{Tors}_p(M)$

$$\mathrm{Tor}_1^{\mathbb{Z}}(M, \mathbb{Z}[p^{-1}]/\mathbb{Z}) \cong \{x \in M \mid p^k x = 0 \text{ some } k \geqslant 0\}$$

of the Tor groups against $\mathbb{Z}[p^{-1}]/\mathbb{Z}$ with the <u>p-power torsion of M</u>. If M is finite this is the unique Sylow p-subgroup of M.

<u>Ex 3</u> Prove that $\mathbb{Z}[p^{-1}]/\mathbb{Z} \cong \mathbb{Z}_{p^\infty}$, the injective envelope of $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$, so we may write $\mathrm{Tor}_1^{\mathbb{Z}}(M, E(\mathbb{Z}_p)) \cong \{ x \in M \mid p^k x = 0 \text{ for some } k \}$.

<u>Ex 4</u> Prove that for any set $\{N_i\}_{i \in I}$ of R-modules that $\mathrm{Tor}_n^{\mathbb{Z}}(M, \bigoplus_i N_i) \cong \bigoplus_i \mathrm{Tor}_n^{\mathbb{Z}}(M, N_i)$.

From the above we deduce that the decomposition

$$\mathbb{Q}/\mathbb{Z} \cong \bigoplus_{p \text{ prime}} E(\mathbb{Z}_p)$$

from earlier in lectures gives rise to a decomposition of $\mathrm{Tors}(M)$ for every abelian group M,

$$\begin{aligned} \mathrm{Tors}(M) &\cong \mathrm{Tor}_1^{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \\ &\cong \mathrm{Tor}_1^{\mathbb{Z}}(M, \textstyle\bigoplus_p E(\mathbb{Z}_p)) \\ &\cong \textstyle\bigoplus_p \mathrm{Tor}_1^{\mathbb{Z}}(M, E(\mathbb{Z}_p)) \\ &\cong \textstyle\bigoplus_p \mathrm{Tors}_p(M). \end{aligned}$$

Of course this can be seen much more easily by direct calculation. But the Tor language allows us to deduce, for example, that if $0 \to M' \to M \to M'' \to 0$ is exact then there is an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Tors}_p M' \arrow[r] & \mathrm{Tors}_p M \arrow[r] & \mathrm{Tors}_p M'' \arrow[dll, out=-10, in=170] & \\ & M'[p^{-1}]/M' \arrow[r] & M[p^{-1}]/M \arrow[r] & M''[p^{-1}]/M'' \arrow[r] & 0 \end{tikzcd}$$

which is less immediately obvious.