Lecture 26

Most of this lecture was a condensed exposition of the basic theory of right derived functors, following Section 7.2 of my "Derived Functors" notes. I say condensed, because the theory is parallel to what we did for left derived functors. These notes contain the material on Ext which I explained, but is not contained in the "Derived Functors" notes.

Let R be a ring and M an R-module, and consider the functor

$$T = \mathrm{Hom}_R(M,-) : R\underline{\mathrm{Mod}} \longrightarrow \underline{\mathrm{Ab}}.$$

This is a left exact functor, so we have $R^0T \cong T$. For $n>0$ we denote the right derived functors of T by

$$\mathrm{Ext}^n_R(M,-) := R^nT : R\underline{\mathrm{Mod}} \longrightarrow \underline{\mathrm{Ab}}.$$

$$\text{i.e.} \quad \mathrm{Ext}^n_R(M,N) = H^n(\mathrm{Hom}_R(M, I_N))$$

where I_N is an injective resolution of N. If $\phi : M \longrightarrow M'$ is an R-linear map it gives a natural transformation

$$\phi^* : \underset{\substack{\| \\ T'}}{\mathrm{Hom}_R(M',-)} \longrightarrow \underset{\substack{\| \\ T}}{\mathrm{Hom}_R(M,-)} \qquad (\phi^*)_N(\alpha) = \alpha \circ \phi.$$

and hence a natural transformation

$$\mathrm{Ext}^n_R(M',-) = R^nT' \xrightarrow{R^n\phi^*} R^nT = \mathrm{Ext}^n_R(M,-)$$

and for an R-module N we set $\mathrm{Ext}^n_R(\phi, N) := (R^n\phi^*)_N$.

Ex 1 Check that this defines a bifunctor

$$\mathrm{Ext}^n_R(-,-) : (R\underline{\mathrm{Mod}})^{op} \times R\underline{\mathrm{Mod}} \longrightarrow \underline{Ab}$$

$$(M,N) \longmapsto \mathrm{Ext}^n_R(M,N)$$

$$\begin{aligned}(\phi : M \to M', \alpha : N \to N') \longmapsto\ & \mathrm{Ext}^n_R(\phi, N') \circ \mathrm{Ext}^n_R(M', \alpha) \\ =\ & \mathrm{Ext}^n_R(M, \alpha) \circ \mathrm{Ext}^n_R(\phi, N)\end{aligned}$$

Remark If $0 \to M' \xrightarrow{\phi} M \xrightarrow{\varphi} M'' \to 0$ is exact then the sequence of natural transformations

$$\mathrm{Hom}_R(M'',-) \xrightarrow{\varphi^*} \mathrm{Hom}_R(M,-) \xrightarrow{\phi^*} \mathrm{Hom}_R(M',-)$$

is exact on injectives (see the Derived Functors note, Definition 12) and thus by Proposition 45 there we have a long exact sequence for any R-module N

$$\begin{aligned}0 \longrightarrow \mathrm{Hom}_R(M'',N) \longrightarrow \mathrm{Hom}_R(M,N) \longrightarrow \mathrm{Hom}_R(M',N) \\ \longrightarrow \mathrm{Ext}^1_R(M'',N) \longrightarrow \mathrm{Ext}^1_R(M,N) \longrightarrow \mathrm{Ext}^1_R(M',N) \longrightarrow \cdots\end{aligned}$$

In particular if a module M has a projective presentation $0 \to R \xrightarrow{\mu} P \xrightarrow{\varepsilon} M \to 0$ we have an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(M,N) \longrightarrow \mathrm{Hom}_R(P,N) \xrightarrow{\mu^*} \mathrm{Hom}_R(R,N) \longrightarrow \mathrm{Ext}^1_R(M,N) \longrightarrow 0$$

and thus an isomorphism $\mathrm{Ext}^1_R(M,N) \cong \mathrm{Ext}^1_R(M,N)^{\text{old}}$, where the RHS is $\mathrm{Coker}\,\mu^*$, our original definition of Ext.

Ex 2 We made $\mathrm{Ext}^1_R(-,-)^{old}$ into a bifunctor. Prove that the isomorphism of the previous remark is natural in both variables, i.e. a natural iso. of bifunctors

$$\mathrm{Ext}^1_R(-,-) \cong \mathrm{Ext}^1_R(-,-)^{old}.$$