Lecture 27

In the last week we develop one of the first and most impressive applications of homological algebra to algebraic geometry: dimension theory. Our aim is to prove a theorem of Serre which characterises non-singularity of a point x an algebraic variety X in terms of the bifunctors

$$\mathrm{Ext}^n_R(-,-) : (R\underline{\mathrm{Mod}})^{op} \times R\underline{\mathrm{Mod}} \longrightarrow \underline{\mathrm{Ab}}$$

where $R = \mathcal{O}_{X,x}$ is the ring of germs of regular functions on X at x. We begin with the basic theory of homological dimensions.

<u>Def^n</u> Let R be a ring and M a nonzero left R-module. Let D_M be the set of all integers $n \geq 0$ for which there exists an exact sequence

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0 \qquad (1.1)$$

with all P_i projective. We call this a projective resolution of <u>length n</u>. We define the <u>projective dimension</u> of M to be the length of the shortest such resolution, if one exists, and ∞ otherwise:

$$\mathrm{proj.dim}_R(M) := \begin{cases} \infty & D_M = \phi \\ \inf D_M & \text{otherwise.} \end{cases}$$

<u>Remarks</u> (1) We set $\mathrm{proj.dim}_R(0) = -1$

(2) If $M \neq 0$ then $\mathrm{proj.dim}_R(M) = 0 \iff M$ is projective

(3) If $n \in D_M$ then $n+1 \in D_M$ as we can make from (1.1) an exact sequence

$$0 \longrightarrow P \xrightarrow{\begin{pmatrix}1\\0\end{pmatrix}} P \oplus P_n \xrightarrow{(0\ \partial_n)} P_{n-1} \longrightarrow \cdots$$ for any projective P.

<u>Lemma</u> (Dimension shifting) Given R-modules M, N and an exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$$

with P projective, we have for all $n \geq 1$

$$\mathrm{Ext}^n_R(K,N) \cong \mathrm{Ext}^{n+1}_R(M,N) \qquad (2.1)$$

<u>Proof</u> From the long exact sequence for Ext we have for $n \geq 1$ a piece

$$\cdots \longrightarrow \mathrm{Ext}^n_R(P,N) \longrightarrow \mathrm{Ext}^n_R(K,N) \longrightarrow \mathrm{Ext}^{n+1}_R(M,N) \longrightarrow \mathrm{Ext}^{n+1}_R(P,N) \longrightarrow \cdots$$

Since $\mathrm{Ext}^n_R(P,N) = 0$ for $n > 0$ we deduce (2.1). $\square$

<u>Example</u> Let R be a PID. As submodules of projectives are projective for PIDs, every R-module M has a projective resolution of length 1. Hence $\mathrm{proj.dim}_R M \leq 1$ for all M, and

$$\mathrm{proj.dim}_R M = \begin{cases} -1 & \text{if } M = 0 \\ 0 & \text{if } M \text{ is projective} = \text{free} \\ 1 & \text{otherwise.} \end{cases}$$

<u>Example</u> Let $R = k[x]/x^2$, and consider the projective resolution of $k = R/xR$,

$$\cdots \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R \longrightarrow k \longrightarrow 0. \qquad (2.1)$$

We claim this module k has <u>no</u> finite projective resolution, so $\mathrm{proj.dim}_R(k) = \infty$. How to justify this? Suppose to the contrary that k <u>did</u> have a finite projective resolution

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow k \longrightarrow 0 \qquad (3.1)$$

Define $R_i = \mathrm{Ker}(\partial_i : P_i \longrightarrow P_{i-1})$ with $P_{-1} = k$, so that we have exact sequences

$$\begin{gathered}
0 \longrightarrow R_0 \longrightarrow P_0 \longrightarrow k \longrightarrow 0 \\
0 \longrightarrow R_1 \longrightarrow P_1 \longrightarrow R_0 \longrightarrow 0 \\
\vdots \\
0 \longrightarrow R_{n-1} \longrightarrow P_{n-1} \longrightarrow R_{n-2} \longrightarrow 0
\end{gathered}$$

By dimension shifting we have for any module N and $m \geq 1$

$$\begin{aligned}
0 &= \mathrm{Ext}^m(R_{n-1}, N) && (R_{n-1} = P_{n-1} \text{ is projective}) \\
&\cong \mathrm{Ext}^{m+1}(R_{n-2}, N) \\
&\cong \cdots \\
&\cong \mathrm{Ext}^{m+n-1}(R_0, N) && (*) \\
&\cong \mathrm{Ext}^{m+n}(k, N)
\end{aligned}$$

Thus $\mathrm{Ext}^i(k,N) = 0$ for all $i > n$ and R-modules N.

But now consider (2.1), which is made up of short exact sequences of the form $0 \longrightarrow k \xrightarrow{\varphi} R \longrightarrow k \longrightarrow 0$ where $\varphi(1) = x \in R$.

From which we deduce, again by dimension shifting, that $\mathrm{Ext}^m(k,N) \cong \mathrm{Ext}^{m+1}(k,N)$ for any $m \geq 1$. But then we are led to a contradiction, since this would imply in particular,

$$\mathrm{Ext}^1(k,N) \cong \mathrm{Ext}^2(k,N) \cong \cdots \cong \mathrm{Ext}^{n+1}(k,N) = 0$$

dimension shift

from $(*)$

which implies k is projective (which is false, for instance $0 \to k \to R \to k \to 0$ above is not split!). This contradiction proves $\mathrm{proj.dim}_R\, k = \infty$.

What we have done above in ⊛ didn't depend on k, so actually we have proven one direction in

<u>Lemma</u> The following are equivalent for an R-module M and $n \geq 0$

(i) $\text{proj.dim}_R(M) \leq n$
(ii) $\text{Ext}^i_R(M,-) = 0$ for $i > n$
(iii) $\text{Ext}^{n+1}_R(M,-) = 0$

<u>Proof</u> We have done (i)⇒(ii). Suppose $\text{Ext}^{n+1}_R(M,N) = 0$ for every R-module N, and let $0 \longrightarrow K \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ be an exact sequence with P_i projective and R arbitrary. Then by dimension shifting

$$\text{Ext}^1_R(K,-) \cong \text{Ext}^{n+1}_R(M,-) = 0$$

as functors, so K is projective and $\text{proj.dim}_R(M) \leq n$. $\square$

<u>Upshot</u> We have for $M \neq 0$,

$$\begin{aligned} \text{proj.dim}_R(M) &= \inf\{ n \mid \text{exists a projective resolution of } M \text{ of length } n \} \\ &= \inf\{ n \mid \text{Ext}^{n+1}_R(M,-) = 0 \} \end{aligned}$$

The Ext-characterisation of dimension is <u>much</u> easier to use:

<u>Ex 1</u> Use $\text{Ext}^i_R(M \oplus M', N) \cong \text{Ext}^i_R(M,N) \oplus \text{Ext}^i_R(M',N)$ to prove that $\text{proj.dim}_R(M \oplus M') = \sup\{ \text{proj.dim}\, M, \text{proj.dim}\, M' \}$.

<u>Def^n</u> The <u>global dimension</u> of a ring R, denoted $gl.\dim(R)$, is

$$gl.\dim(R) = \sup\{\text{proj.dim}_R M \mid M \text{ is an } R\text{-module}\} \in \mathbb{N} \cup \{\infty\} \qquad (5.1)$$

<u>Ex 2</u> Prove that $gl.\dim(R) = \inf\{n \geq -1 \mid \text{Ext}_R^{n+1}(-,-) = 0\}$ where the infimum of the empty set is ∞.

<u>Ex 3</u> Prove that $gl.\dim(R) = \sup\{\text{proj.dim}_R M \mid M \text{ is a } \underline{\text{finitely generated}}\ R\text{-module}\}$.

<u>Examples</u> (1) If R is a PID then $gl.\dim(R) = 1$ (e.g. $k[x]$, k a field)
(2) If k is a field then $gl.\dim(R) = 0$.
(3) $gl.\dim(k[x]/x^2) = \infty$.

An important theorem that we will not have time to prove is

<u>Theorem</u> (Hilbert's Syzygy Theorem) If k is a field, $gl.\dim k[x_1, \dots, x_n] = n$.

This says that if you start constructing a projective resolution of any $k[x_1, \dots, x_n]$-module $\cdots \to P_1 \to P_0 \to M \to 0$ then the kernel of $P_{n-1} \to P_{n-2}$ is <u>always</u> projective! The above examples suggest that at least for smooth things (identify $k[x_1, \dots, x_n]$ with $\mathbb{A}_k^n$) the global dimension <u>recovers the geometric dimension</u> from just the category $R\underline{\text{Mod}}$. This doesn't work for $k[x]/x^2$, which is singular. Indeed, a beautiful theorem of Serre proves that this is the case

<u>Theorem</u> (Serre) Let (A, m, k) be a noetherian local ring. Then A is regular (i.e. non-singular) if and only if $gl.\dim A < \infty$, and in this case

$$gl.\dim A = \dim A$$