Lecture 28

Our aim in today's lecture is to prove one half of the theorem of Serre quoted at the end of last lecture. Specifically, we aim to prove

<u>Theorem</u> Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring. Then $\text{gl.dim}(A) = \text{proj.dim}_A(k)$.

Some reminders: in this lecture all rings are commutative, a <u>local</u> ring is a ring with a unique maximal ideal, a ring R is <u>Noetherian</u> if every chain of ideals $\cdots \subseteq \mathfrak{a}_n \subseteq \mathfrak{a}_{n+1} \subseteq \cdots$ satisfies $\mathfrak{a}_n = \mathfrak{a}_{n+1}$ for $n \gg 0$. When we say "let $(A, \mathfrak{m}, k)$ be a Noetherian local ring" we mean $\mathfrak{m} \subseteq A$ is the unique maximal ideal and $k = A/\mathfrak{m}$. The <u>dimension</u> $\dim(A)$ is the length d of the longest chain $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_d$ of prime ideals in A. If $A = \mathcal{O}_{X,x}$ is the ring of germs of regular functions at x on an algebraic variety X then $\dim(A)$ matches the geometric notion of dimension of X at x.

<u>Remark</u> Most of what follows is from Matsumura's book on Commutative Algebra, and my notes on his book from <u>therisingsea.org</u>.

From now on let $(A, \mathfrak{m}, k)$ be a Noetherian local ring. The strategy to prove Theorem I is express gl.dim$A$ in terms of Tor <u>against k</u>, then compute invariants of k.

<u>Theorem</u> Let M be a finitely generated A-module. Then M is flat iff. it is projective iff. it is free.

<u>Proof</u> Let $m = \dim_k(M/\mathfrak{m}M)$. By Nakayama's lemma we can find generators $u_1, \ldots, u_m$ for M as an A-module ($\because$ map to a k-basis of $M/\mathfrak{m}M$). Let $\varepsilon : A^{\oplus m} \longrightarrow M$ have ith component $A \longrightarrow M$, $1 \longmapsto u_i$ so that we have an exact sequence

$$0 \longrightarrow K \longrightarrow A^{\oplus m} \xrightarrow{\varepsilon} M \longrightarrow 0 \qquad (1.1)$$

Now suppose M is flat (in fact $\mathrm{Tor}_1(M,k)=0$ is enough). Then we have exactness of the first row of

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K\otimes_A k & \longrightarrow & A^{\oplus m}\otimes_A k & \longrightarrow & M\otimes_A k & \longrightarrow & 0 \\
 & & \wr\| & & \wr\| & & \wr\| & & \\
0 & \longrightarrow & K/\mathfrak{m}K & \longrightarrow & k^{\oplus m} & \xrightarrow[\mu]{} & M/\mathfrak{m}M & \longrightarrow & 0
\end{array}$$

By construction μ in this diagram is an isomorphism, so $K/\mathfrak{m}K=0$ and hence by Nakayama $K=0$, so $M\cong A^{\oplus m}$ is free. $\square$

<u>Proposition</u> If M is a f.g. A-module then for $n\geq 0$

$$\mathrm{proj.dim}_A M\leq n \iff \mathrm{Tor}^A_{n+1}(M,-)=0.$$

<u>Proof</u> By dimension shifting if $0\to K\to F\to M\to 0$ is exact with F flat then $\mathrm{Tor}_{i+1}(K,N)\cong\mathrm{Tor}_i(M,N)$ for $i\geq 1$, and since projectives are flat we can conclude ($\Rightarrow$) holds by arguments similar to those in Lecture 27.

For ($\Leftarrow$) suppose $\mathrm{Tor}^A_{n+1}(M,-)=0$ and that $0\to K\to P_{n-1}\to\cdots\to P_0\to M\to 0$ is exact with P_i projective and f.g. By the familiar arguments K is flat, hence projective, so $\mathrm{proj.dim}_A M\leq n$. $\square$

<u>Ex 1</u> $\mathrm{gl.dim}(A)=\inf\{n\geq -1 \mid \mathrm{Tor}^A_{n+1}(-,-)=0\}$.

Corollary If M is a f.g. A-module then for $n \geq 0$ (see Lemma 123 of my Matsumura notes)

$$\text{proj.dim}_A M \leq n \iff \operatorname{Tor}^A_{n+1}(M,k) = 0. \quad (*)$$

Proof We prove by induction on n, the statement P_n: $\forall$ f.g. M $(*)$ holds. The base case P_0 was our Theorem earlier, so it remains to prove the inductive step. Let M be a f.g. module with $\operatorname{Tor}_{n+2}(M,k) = 0$ and suppose P_n holds. From the long exact Ext sequence of (1.1) we have $\operatorname{Ext}^{i+1}(K,-) \cong \operatorname{Ext}^i(M,-)$ for $i \geq 1$ and

$$\text{proj.dim}\, M \leq \text{proj.dim}\, K + 1. \quad (**)$$

For the same dimension shift reasons,

$$\operatorname{Tor}_{n+1}(M,k) \cong \operatorname{Tor}_n(K,k).$$

Hence if P_n holds, since $(\Rightarrow)$ in $(*)$ is already proven, to prove P_{n+1} we need only show $\operatorname{Tor}_{n+2}(M,k) = 0 \Rightarrow \text{proj.dim}\, M \leq n+2$. But if $\operatorname{Tor}_{n+2}(M,k) = 0$ then $\operatorname{Tor}_{n+1}(K,k) = 0$ and so by P_n for K, $\text{proj.dim}\, K \leq n+1$, whence by $(**)$ $\text{proj.dim}\, M \leq n+2$ and we are done. $\square$

<u>Theorem</u> $gl.\dim A = proj.\dim_A(k)$.

<u>Proof</u> We prove first that for $n \geq 0$

$$gl.\dim A \leq n \iff Tor_{n+1}(k,k) = 0.$$

The implication $\Rightarrow$ follows from our earlier observations. Suppose $Tor_{n+1}(k,k)=0$. Then $proj.\dim(k) \leq n$ by $\circledast$. Computing $Tor_{n+1}(M,k)$ using a projective resolution of k of length $\leq n$ we see that it vanishes for any module M, and thus $proj.\dim(M) \leq n$ as well. Hence $gl.\dim A \leq n$ as claimed.

Since $proj.\dim(k) \leq n \iff Tor_{n+1}(k,k)=0$ it follows that $gl.\dim A = proj.\dim(k)$. $\square$

So at least we have reduced the calculation of $gl.\dim A$ to that of $proj.\dim(k)$, for Noetherian local rings, and it remains to show that this number $proj.\dim(k)$ has some geometric content. Recall that $x \in A$ is <u>regular</u> or a non-zerodivisor if the multiplication map $x : A \longrightarrow A$ is injective. There is a generalisation to sequences which we need.

<u>Def^n</u> Let M be an A-module and $a_1, \dots, a_n \in A$. We say this is an <u>M-regular sequence</u> if $(a_1, \dots, a_n)M \neq M$ and

- $a_1 : M \longrightarrow M$ is injective
- for all $1 \leq i < n$, the map

$$M/(a_1,\dots,a_i)M \xrightarrow{a_{i+1}} M/(a_1,\dots,a_i)M$$

is injective.

Remark (1) If a sequence $a_1,\dots,a_n$ is A-regular (i.e. $M=A$) we simply say it is regular.

(2) If $\{a_1,\dots,a_n\}\subseteq \mathfrak{m}$ and M is f.g. then $a_1,\dots,a_n$ is M-regular iff. $a_{\sigma(1)},\dots,a_{\sigma(n)}$ is M-regular for any permutation σ (this not obvious, see Matsumura for the proof).

(3) If $a_1,\dots,a_n$ is M-regular so is $a_1,\dots,a_m$ for $m\leq n$.

Example The variables $x_1,\dots,x_n\in k[[x_1,\dots,x_n]]=A$ form a regular sequence, as do any powers $x_1^{a_1},\dots,x_n^{a_n}$ for $a_i\geq 1$.

Example Suppose A is regular, so that $\dim_k(\mathfrak{m}/\mathfrak{m}^2)=\dim A=d$. Then any set of generators $x_1,\dots,x_d$ for $\mathfrak{m}$ as an ideal gives a regular sequence (for context see the statement about $G_\mathfrak{m}(A)$ on p. 1 of Lecture 28 and the material on quasi-regularity in Matsumura).

Understanding the relationship between regular sequences and projective dimension is the key to proving Serre's theorem. Note that projective dimension measures how far a module is from being projective. While $k[x_1,\dots,x_n]$ is obviously a projective $k[x_1,\dots,x_n]$-module, $k[x_1,\dots,x_{n-1}]\cong k[x_1,\dots,x_n]/(x_n)$ is not, as it is torsion.

Proposition I Let M be a f.g. A-module. If $\text{proj.dim}\, M=r<\infty$ and $x\in\mathfrak{m}$ is M-regular, $\text{proj.dim}(M/xM)=r+1$.

Proof By p. 4 of Lecture 28 it suffices to prove

$$\text{Tor}_{r+2}(M/xM,k)=0,\quad\text{and}\quad \text{Tor}_{r+1}(M/xM,k)\neq 0.$$

The long exact Tor sequence associated to $0\to M\xrightarrow{x}M\to M/xM\to 0$ past Tor_r looks as follows, using that $\text{Tor}_i(M,k)=0$ for $i>r$

$$\cdots \to \overset{0}{\overset{\shortparallel}{\mathrm{Tor}_{r+1}(M,k)}} \xrightarrow{x} \overset{0}{\overset{\shortparallel}{\mathrm{Tor}_{r+1}(M,k)}} \to \mathrm{Tor}_{r+1}(M/xM,k) \to \mathrm{Tor}_r(M,k) \xrightarrow{x=0} \mathrm{Tor}_r(M,k) \to \cdots$$

from which we deduce $\mathrm{Tor}_i(M/xM, k) = 0$ for $i > r+1$, and $\mathrm{Tor}_{r+1}(M/xM, k)$ is isomorphic to $\mathrm{Tor}_r(M,k) \neq 0$. $\square$

<u>Proposition II</u> Let M be a f.g. A-module and $a_1, \dots, a_s$ an M-regular sequence. If $\mathrm{proj.dim}\, M = r < \infty$ then $\mathrm{proj.dim}\left(M/(a_1,\dots,a_s)M\right) = r+s$.

<u>Proof</u> By induction on s. $\square$

<u>Proposition III</u> Suppose $\mathfrak{m}$ can be generated by a regular sequence $a_1, \dots, a_d$. Then $\mathrm{gl.dim}(A) = d$.

<u>Proof</u> We have $k = A/\mathfrak{m} = A/(a_1, \dots, a_d)$ hence by Prop II and the Theorem

$$\begin{aligned} \mathrm{gl.dim}(A) &= \mathrm{proj.dim}_A(k) \\ &= \mathrm{proj.dim}_A\left(A/(a_1,\dots,a_d)\right) \\ &= \mathrm{proj.dim}_A(A) + d \\ &= d. \quad \square \end{aligned}$$

In this case $d = \dim(A)$ so we have moreover shown that $\mathrm{gl.dim}(A)$ computes the <u>geometric dimension</u>, which is therefore an intrinsic homological invariant of the abelian category $A\underline{\mathrm{Mod}}$. This is a motivating example of the paradigm

$$\begin{array}{ccccc} \boxed{\text{Space}} & \xrightarrow{\text{modules}} & \boxed{\text{Category}} & \xrightarrow{\text{hom.alg.}} & \boxed{\text{Invariants}} \\ X & & \mathcal{O}_{X,x}\text{-}\underline{\mathrm{Mod}} & & \mathrm{Ext}^*(-,-),\ \mathrm{Tor}_*(-,-) \end{array}$$