Lecture 29

In this final lecture we sketch the proof of our main application of the homological algebra that has been developed over the course of the semester:

Theorem (Serre) Let (A, m, k) be a noetherian local ring. Then A is regular (i.e. non-singular) if and only if $gl.dim A < \infty$, and in this case

$$gl.dim A = \dim A$$

The part left to prove is that A is regular $\Leftrightarrow$ $gl.dim(A) < \infty$. We begin with a reminder on regularity:

Theorem / Definition (See Atiyah & MacDonald "Introduction to Commutative Algebra" 11.22)
Let (A, m, k) be a Noetherian local ring. Then A is regular if the following equivalent conditions are satisfied, with $d = \dim(A)$,

(i) $G_m(A) := A/m \oplus m/m^2 \oplus \cdots$ is isomorphic as a graded k-algebra to $k[x_1, \dots, x_d]$.

(ii) $\dim_k(m/m^2) = d$

(iii) m can be generated by d elements as an ideal.

Remark If $X = Z(f_1, \dots, f_p) \subseteq k^m$ where $f_i \in k[x_1, \dots, x_m]$ and $P \in X$, $A = \mathcal{O}_{X,P}$ then regularity of A is equivalent (for k alg. closed) to the Jacobian $\left(\partial f_i / \partial x_j\right)$ having rank $m - \dim X$ at P, i.e. the usual notion of nonsingularity in geometry.

<u>Example</u> $A = k[[x_1,\dots,x_n]]$ is local of dimension n, and

$$\mathfrak{m}/\mathfrak{m}^2 = (x_1,\dots,x_n)\big/(x_1,\dots,x_n)^2 \cong kx_1 \oplus \cdots \oplus kx_n$$

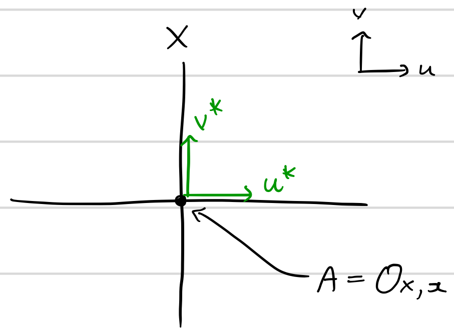
so $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = n$ and A is regular.

<u>Non-example</u> $A = k[[u,v]]/uv$

$$\mathfrak{m}/\mathfrak{m}^2 = (u,v)\big/(u,v)^2 = \begin{pmatrix} ku \oplus ku^2 \oplus \cdots \\ \oplus kv \oplus kv^2 \oplus \cdots \end{pmatrix} \Big/ \begin{matrix} ku^2 \oplus \cdots \\ \oplus kv^2 \oplus \cdots \end{matrix}$$

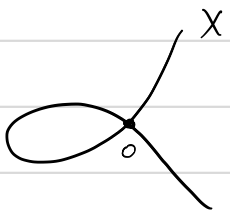
$$\cong ku \oplus kv$$

So $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = 2 \neq 1 = \dim A$, so A is <u>not</u> regular. Geometrically, $(\mathfrak{m}/\mathfrak{m}^2)^*$ is the tangent space to the origin, and A is the completion of the curve $uv = 0$ in $\mathbb{A}^2$ at the origin, i.e.



and the basis u^*, v^* of the tangent space corresponds to the two local tangents at O.

<u>Remark</u> Geometrically, A occurs as the completion of the local ring of a nodal curve



$$A \cong \hat{\mathcal{O}}_{X,0}$$

$y^2 - x^2(x+1) = 0$

We proved last lecture that if A is regular, $\text{gl.dim} A = \dim A$, so it only remains to show

$$\text{gl.dim}(A) < \infty \implies A \text{ regular.}$$

For any Noetherian local ring we have $\dim(A) \leq \dim_k(\mathfrak{m}/\mathfrak{m}^2)$ so the strategy is, assuming $\text{gl.dim}(A) < \infty$, to prove two inequalities

$$\dim_k(\mathfrak{m}/\mathfrak{m}^2) \underset{(1)}{\leq} \text{gl.dim}(A) \underset{(2)}{\leq} \dim(A)$$

The second is a consequence of the Auslander-Buchsbaum formula, which says for any f.g. A-module M with $\text{proj.dim}(M) < \infty$ we have

$$\text{proj.dim}(M) + \text{depth}(M) = \text{depth}(A) \leq \dim(A),$$

and $\text{gl.dim}(A) = \text{proj.dim}(k)$ from last lecture. We will focus our attention instead on the proof of (1). It will follow from

<u>Theorem A</u> Let $s = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$. Then

$$\dim_k \text{Tor}_i^A(k,k) \geqslant \binom{s}{i} \qquad 0 \leq i \leq s.$$

We will sketch the proof in a moment. But first let us see how it implies the claim

<u>Proof of Serre's Theorem</u> As we have discussed, it is enough to prove $s \leq \text{gl.dim}(A)$. But recall $\text{gl.dim}(A) \leq n \iff \text{Tor}_{i+1}(k,k) = 0$, hence $\text{Tor}_i(k,k) \neq 0 \Rightarrow \text{gl.dim}(A) \geqslant i$. But

$$\dim_k \text{Tor}_s(k,k) \geqslant \binom{s}{s} = 1$$

so $\text{gl.dim}(A) \geqslant s$. $\square$

<u>Remark</u> One of the first applications of Serre's homological characterisation of regularity was to prove that if A is regular then so is $A_{\mathfrak{p}}$ for any prime $\mathfrak{p}$. This is not obvious from the definition, but is easy once you think in terms of global dimensions (as $A_{\mathfrak{p}}$ is flat over A).

<u>Proof of Theorem A</u> (sketch) The proof plays off two complexes. The first is a <u>minimal free resolution</u>, which is a projective resolution

$$\cdots \xrightarrow{\partial_2} L_1 \xrightarrow{\partial_1} L_0 \xrightarrow{\varepsilon} k \longrightarrow 0$$

in which all L_i are f.g. free and the map $\varepsilon : L_0 \to k$ and $L_i \longrightarrow \operatorname{Im}\partial_i$ are <u>minimal</u> for $i \geq 1$. A morphism $\alpha : M \longrightarrow N$ of f.g. A-modules is <u>minimal</u> if it induces an isomorphism $M \otimes_A k \xrightarrow{\cong} N \otimes_A k$. Since

$$L_{i+1} \xrightarrow{\partial_{i+1}} L_i \longrightarrow \operatorname{Im}\partial_i$$

is zero, we deduce that the matrices ∂_i for $i \geq 1$ all have entries in $\mathfrak{m}$, i.e. $\partial_i \otimes_A k = 0$, and consequently

$$\begin{aligned} \operatorname{Tor}_i^A(k,k) &= H_i\left(\cdots \xrightarrow{0} L_1 \otimes_A k \xrightarrow{0} L_0 \otimes_A k \right) \\ &\cong L_i \otimes_A k \end{aligned}$$

which implies $\dim_k \operatorname{Tor}_i^A(k,k) = \text{free-rank}_A(L_i)$. Minimal free resolutions exist: if $M/\mathfrak{m}M$ has basis $\overline{u}_1, \ldots, \overline{u}_a$ for $u_i \in M$ then $A^{\oplus a} \longrightarrow M$ with components u_i is minimal, so we can build up minimal resolutions in the usual way.

The second complex is the <u>Koszul complex</u> of a set of generators of $\mathfrak{m}$. Since $\dim_k \mathfrak{m}/\mathfrak{m}^2 = s$ we may generate $\mathfrak{m}$ by elements $a_1, \dots, a_s$ (<u>not</u> necessarily a regular sequence!) and we define a complex K by

$$K_i = \bigwedge^i (A\theta_1 \oplus \cdots \oplus A\theta_s)$$

$$\partial_i = \left(\textstyle\sum_{j=1}^s a_i \theta_i^*\right) \lrcorner (-)$$

contraction

i.e.

$$\partial_i(\theta_{j_1} \wedge \cdots \wedge \theta_{j_i}) = \sum_{\ell=1}^{i} (-1)^{\ell-1} a_{j_\ell}\, \theta_{j_1} \wedge \cdots \wedge \widehat{\theta_{j_\ell}} \wedge \cdots \wedge \theta_{j_i}$$

This complex ends in

$$\begin{tikzcd}
\cdots\cdots \arrow[r] & K_1 \arrow[r, "\partial_1"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & K_0 \arrow[d, equal] \\
& A\theta_1 \oplus \cdots \oplus A\theta_s \arrow[r, "(a_1 \, \cdots \, a_s)"'] & A
\end{tikzcd}$$

so $H_0 K \cong A/(a_1, \dots, a_s) = A/\mathfrak{m} = k$. By one of our early results, there is a chain map $f : K \longrightarrow L$ lifting the identity on $H_0 K \cong k \xrightarrow{1} k \cong H_0 L$. Some technical work (using the fact that we understand the Koszul complex very well) leads us to conclude each $f_i : K_i \longrightarrow L_i$ is a <u>split</u> mono. Hence

$$\begin{aligned}
\binom{s}{i} &= \text{free-rank}\left(\textstyle\bigwedge^i (A\theta_1 \oplus \cdots \oplus A\theta_s)\right) \\
&= \text{free-rank}(K_i) \\
&\leq \text{free-rank}(L_i) = \dim_k \operatorname{Tor}_i(k, k). \quad \square
\end{aligned}$$