MAST90068 – Lecture 3

28/7/16

In this lecture we continue with examples of functors, and introduce the notion of a <u>representing object</u> for certain functors.

<u>Example</u> Let $\mathcal{C}$ be a category and A an object of $\mathcal{C}$. There is a functor

$$h^A : \mathcal{C} \longrightarrow \underline{\mathrm{Set}}$$

$$\begin{tikzcd} & A \arrow[dl, "x"'] \\ B \arrow[r, "f"'] & B' \end{tikzcd}$$

- $h^A(B) = \mathcal{C}(A, B)$
- for $f : B \to B'$, $h^A(f) : \mathcal{C}(A,B) \longrightarrow \mathcal{C}(A,B')$ is defined by $h^A(f)(x) = f \circ x$

<u>Ex 1</u> Check h^A is a functor.

in the sense of p. 7 of Lecture 2 i.e. $\mathbb{Z}(M\times N) := V(M\times N)$ there

<u>Example</u> Let M, N be abelian groups, and recall that their <u>tensor product</u> $M \otimes N$ is defined to be the quotient of the free abelian group $\mathbb{Z}(M\times N)$ generated by the set

$$R = \{(m_1+m_2, n) - (m_1, n) - (m_2, n)\}_{m_1, m_2 \in M, n\in N} \cup \{(m, n_1+n_2) - (m, n_1) - (m, n_2)\}_{m\in M, n_1, n_2 \in N}$$

Let μ denote the composite (ι as in Lecture 2)

$$M\times N \overset{\iota}{\hookrightarrow} \mathbb{Z}(M\times N) \twoheadrightarrow \mathbb{Z}(M\times N)/\langle R\rangle =: M\otimes N$$

$$\text{i.e. } m\otimes n := \mu(m,n).$$

This is a definition by "construction". Arguably more natural is the following style of definition "universal property + existence proof". The point is that μ is the <u>universal</u> bilinear form on $M\times N$.

Example Consider the functor

$$F : \underline{Ab} \longrightarrow \underline{Set}$$

- $F(T) = \{ \alpha : M \times N \to T \mid \alpha$ is bilinear, i.e. $\alpha(m_1 + m_2, n) = \alpha(m_1, n) + \alpha(m_2, n)$ and $\alpha(m, n_1 + n_2) = \alpha(m, n_1) + \alpha(m, n_2)$ for all $m_1, m_2, m \in M$ and $n_1, n_2, n \in N \}$

- $F(\beta : T \to T') : F(T) \longrightarrow F(T')$ is defined by sending $\alpha : M \times N \to T$ to $\beta \circ \alpha : M \times N \to T'$.

Lemma There is for every abelian group T a bijection

$$\Phi_T : \mathrm{Hom}_{\underline{Ab}}(M \otimes N, T) \longrightarrow F(T) \qquad (2.1)$$

defined by $\Phi_T(\gamma) = \gamma \circ \mu$. Moreover this bijection is natural, i.e. for any morphism $\beta : T \to T'$ the following diagram commutes

$$\begin{tikzcd}
\mathrm{Hom}_{\underline{Ab}}(M \otimes N, T) \arrow[r, "\Phi_T"] \arrow[d, "\beta \circ -"'] & F(T) \arrow[d, "F(\beta)"] \\
\mathrm{Hom}_{\underline{Ab}}(M \otimes N, T') \arrow[r, "\Phi_{T'}"'] & F(T')
\end{tikzcd} \qquad (2.2)$$

Proof By construction $\mu : M \times N \to M \otimes N$ is bilinear. We claim it is universal, i.e. that if $\alpha : M \times N \to T$ is some other bilinear form then there is a unique $\tilde{\alpha} : M \otimes N \to T$ such that

morphism in $\underline{Ab}$

$$
(3.1) \qquad \begin{tikzcd} M \times N \arrow[r, "\mu"] \arrow[dr, bend right, "\alpha"'] & M \otimes N \arrow[d, "\tilde{\alpha}"] \\ & T \end{tikzcd}
$$

[where $\hat{\alpha} : \mathbb{Z}(M \times N) \longrightarrow T$ is the morphism determined by α according to Lemma on p.8 of Lecture 2, so $\hat{\alpha} \circ \iota = \alpha$ and $\hat{\alpha}$ is unique with this property]

commutes. The existence of $\tilde{\alpha}$ is clear: bilinearity of α means $\hat{\alpha}(R) = 0$, so $\hat{\alpha}$ factors via $\mathbb{Z}(M \times N)/\langle R \rangle = M \otimes N$, and we call this factorisation $\tilde{\alpha}$. By construction $\tilde{\alpha} \circ \mu = \alpha$, and $\tilde{\alpha}$ is unique making (3.1) commute (why?). We conclude that

$$\Phi_T^{-1}(\alpha) = \tilde{\alpha}$$

defines a two-sided inverse to Φ_T, completing the proof (naturality is an easy exercise). $\square$

<u>Notes</u> We can rewrite (2.1) as a natural bijection

$$h^{M \otimes N}(T) \xrightarrow{\cong} F(T).$$

We say there is a <u>natural isomorphism of functors</u> $h^{M \otimes N} \cong F$, and say that $(M \otimes N, \mu)$ is the solution of a <u>universal problem</u>, namely: what is the universal bilinear form on $M \times N$? This universal problem is encoded by F.

<u>Note</u> We would like to make the connection between a family of bijections (2.1) which is natural (2.2) and the notion of a "universal bilinear form on $M \times N$". Suppose to this end that we had never heard of the tensor product, but suppose we were given an abelian group P and a family of bijections, one for each $T \in \underline{Ab}$

$$\Phi_T : \mathrm{Hom}_{\underline{Ab}}(P, T) \xrightarrow{\cong} F(T) \qquad (4.1)$$

which are natural in the sense that the diagrams (2.2) commute for all β with P in place of $M \otimes N$. Applying (4.1) to $T = P$ we have

$$\mathrm{Hom}_{\underline{Ab}}(P, P) \xrightarrow{\Phi_P} F(P)$$

$$\therefore \quad \mu := \Phi_P(1_P) \text{ is some bilinear form } \mu : M \times N \to P.$$

Now observe that if $\alpha : M \times N \to T$ is any bilinear form, $\alpha \in F(T)$, so $\Phi_T^{-1}(\alpha)$ is a morphism $P \to T$ to which naturality (2.2) may be applied. That is, with $\beta = \Phi_T^{-1}(\alpha)$,

$$\begin{tikzcd}
\mathrm{Hom}_{\underline{Ab}}(P,P) \arrow[r, "\Phi_P"] \arrow[d, "\beta \circ -"'] & F(P) \arrow[d, "F(\beta)"] \\
\mathrm{Hom}_{\underline{Ab}}(P,T) \arrow[r, "\Phi_T"'] & F(T)
\end{tikzcd} \qquad (4.2)$$

commutes. But this means

$$\begin{aligned}
\alpha &= \Phi_T(\Phi_T^{-1}(\alpha)) \\
&= \Phi_T(\Phi_T^{-1}(\alpha) \circ 1_P) \\
&= \Phi_T(\beta \circ -)(1_P) \\
&= F(\beta)\Phi_P(1_P) \quad \text{by commutativity of (4.2)} \\
&= \beta \circ \mu.
\end{aligned}$$

$$\begin{tikzcd}
M \times N \arrow[rr, "\alpha"] \arrow[dr, "\mu"'] & & T \\
& P \arrow[ur, "\beta = \Phi_T^{-1}(\alpha)"'] &
\end{tikzcd}$$

That is, α factors through μ. Moreover, this factorisation is <u>unique</u>:

If $\beta' : P \to T$ were another morphism with $\beta' \circ \mu = \alpha$ then by naturality for β'

$$
\begin{tikzcd}
\mathrm{Hom}_{\underline{Ab}}(P, P) \arrow[r, "\Phi_P"] \arrow[d, "\beta' \circ -"'] & F(P) \arrow[d, "F(\beta')"] \\
\mathrm{Hom}_{\underline{Ab}}(P, T) \arrow[r, "\Phi_T"'] & F(T)
\end{tikzcd}
$$

commutes, and

$$
\begin{array}{l}
F(\beta')\Phi_P(1_P) = \beta' \circ \mu = \alpha = \Phi_T(\beta) \\
\quad \| \\
\Phi_T(\beta' \circ -)(1_P) \\
\quad \| \\
\Phi_T(\beta')
\end{array}
$$

So we conclude $\beta = \beta'$.

<u>Upshot</u> The bilinear form $\mu : M \times N \to P$ is <u>universal</u> among bilinear forms on $M \times N$, in the precise sense that any other bilinear form $\alpha : M \times N \to T$ factors uniquely through μ.

<u>Ex 2</u> Deduce in the above situation a unique iso $M \otimes N \xrightarrow[\cong]{\theta} P$ such that the diagram

$$
\begin{tikzcd}
M \otimes N \arrow[rr] & & P \\
& M \times N \arrow[ul, "\mu"] \arrow[ur, "\mu"'] &
\end{tikzcd}
$$

commutes.