MAST90068 – Lecture 4

1/8/16

We have seen the definition of categories and functors, and some simple examples. This lecture enumerates some more examples, leading up to simplicial sets as a motivation for functor categories.

Def^N A category $\mathcal{C}$ is small if $ob(\mathcal{C})$ is a set (not just a class).

Def^N A small category $\mathcal{C}$ is ordered if for every pair of objects A, B the set $\mathcal{C}(A,B)$ contains at most one element.

Lemma If $\mathcal{C}$ is ordered then $(ob(\mathcal{C}), \leq)$ is a preorder, where we define $A \leq B$ if and only if $\mathcal{C}(A,B)$ is nonempty.

Proof A preorder is a reflexive, transitive relation, so this is clear from the axioms. $\square$

Lemma If $(X, \leq)$ is a preorder then there is a small category $\mathcal{C}_X$ with $ob(\mathcal{C}_X) = X$ and

$$\mathcal{C}_X(x_1, x_2) = \begin{cases} \{*\} & x_1 \leq x_2 \\ \phi & \text{else} \end{cases}.$$

Proof Also clear. $\square$

Def^N A morphism of preorders $f : (X_1, \leq) \longrightarrow (X_2, \leq)$ is a function $f : X_1 \to X_2$ such that whenever $x \leq x'$ in X_1, then $f(x) \leq f(x')$ in X_2. This defines the category of preorders $\underline{\text{Pre}}$.

Def^N Let <u>Cat</u> denote the category of small categories and functors.
(the existence of this follows from exercises in Lecture 2).

Def^N A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is

- <u>faithful</u> if for all pairs $A, B \in ob(\mathcal{C})$,

$$\mathcal{C}(A,B) \longrightarrow \mathcal{D}(FA, FB) \qquad (2.1)$$

  is <u>injective</u>, and

- <u>full</u> if for all pairs $A, B \in ob(\mathcal{C})$, (2.1) is <u>surjective</u>, and

- <u>essentially surjective</u> if for every $D \in ob(\mathcal{D})$ there exists $C \in ob(\mathcal{C})$ and an isomorphism $F(C) \cong D$.

F is <u>fully faithful</u> if it is both full and faithful, and an <u>equivalence</u> if it is fully faithful and essentially surjective.

Lemma There is a fully faithful functor $F : \underline{Pre} \longrightarrow \underline{Cat}$ defined by $F(X, \leq) = \mathcal{C}_X$ on objects and for $f : (X_1, \leq) \longrightarrow (X_2, \leq)$ in <u>Pre</u> by

$$F(f) : \mathcal{C}_{X_1} \longrightarrow \mathcal{C}_{X_2}$$
$$F(f)(x) = f(x)$$
$$F(* : x_1 \to x_2) = * : f(x_1) \longrightarrow f(x_2).$$

Ex 1 Prove the lemma.

Note This defines an equivalence $\underline{Pre} \longrightarrow \underline{Ord}$ where <u>Ord</u> denotes ordered categories and functors.

Def^N Let $\mathscr{C}$ be a category. A <u>subcategory</u> $\mathscr{J}$ of $\mathscr{C}$ is a class $ob(\mathscr{J}) \subseteq ob(\mathscr{C})$ and for each pair $A, B \in ob(\mathscr{J})$ a subset $\mathscr{J}(A,B) \subseteq \mathscr{C}(A,B)$ s.t. for all $A \in ob(\mathscr{J})$ we have $1_A \in \mathscr{J}(A,A)$, and $\mathscr{J}$ is closed under composition in $\mathscr{C}$. Thus $\mathscr{J}$ is itself a category, and there is a faithful functor $\mathscr{J} \longrightarrow \mathscr{C}$.

Def^N For $p \in \mathbb{N}$ consider the preorder $[p] := \{0, 1, \dots, p\}$ with the usual relation $\leq$. We denote by Δ the full subcategory of <u>Pre</u> with objects $\{[p]\}_{p \geq 0}$. That is,

$$ob(\Delta) = \{[0], [1], \dots \}$$

$$\begin{aligned} \mathrm{Hom}_{\Delta}([p],[q]) &= \mathrm{Hom}_{\underline{\mathrm{Pre}}}([p],[q]) \\ &= \{ f : \{0,1,\dots,p\} \longrightarrow \{0,1,\dots,q\} \mid i \leq j \\ &\qquad \text{implies } f(i) \leq f(j) \}. \end{aligned}$$

This is quite an interesting and important category, for both topology and homological algebra. It is called (for reasons that will become clear) the <u>simplex category</u>. By the above, there is a fully faithful functor $\Delta \longrightarrow$ <u>Cat</u>, if we choose to think of $[p]$ as a category.

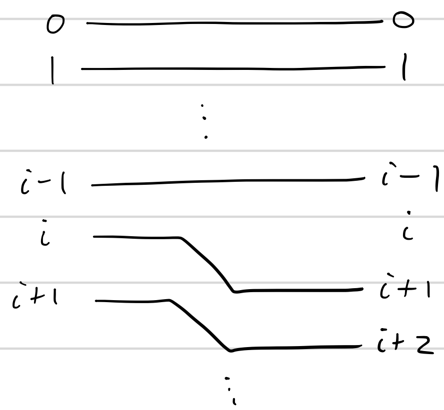
Def^N Given $0 \leq i \leq q$ integers, define morphisms in Δ

$$\varepsilon^i = \varepsilon^i_q : [q-1] \longrightarrow [q] \qquad \eta^i = \eta^i_q : [q+1] \longrightarrow [q]$$

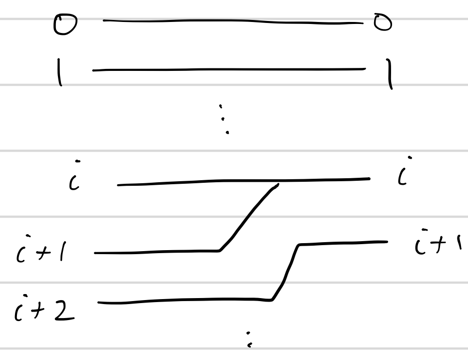
$$\varepsilon^i(a) = \begin{cases} a & a < i \\ a+1 & a \geq i \end{cases} \qquad \eta^i(a) = \begin{cases} a & a \leq i \\ a-1 & a > i \end{cases} \tag{3.1}$$

where ε^i_q is defined for $q > 0$. (3.2)

$\boxed{\varepsilon^i}$



$\boxed{\eta^i}$



Ex 2 Prove

$$\varepsilon^j_{q+1}\,\varepsilon^i_q = \varepsilon^i_{q+1}\,\varepsilon^{j-1}_q \qquad i<j \qquad (4.1)$$

$$\eta^j_q\,\eta^i_{q+1} = \eta^i_q\,\eta^{j+1}_{q+1} \qquad i\le j \qquad (4.2)$$

$$\eta^j_{q-1}\,\varepsilon^i_q = \begin{cases} \varepsilon^i_{q-1}\,\eta^{j-1}_{q-2} & i<j \\ 1 & i=j,\ i=j+1 \\ \varepsilon^{i-1}_{q-1}\,\eta^j_{q-2} & i>j+1 \end{cases} \qquad (4.3)$$

Note Observe that the ε^i are injective, and since composition in Δ is of functions between sets, this implies ε^i are monomorphisms. Similarly all the η^i are epimorphisms. In fact something stronger can be read from (4.3).

Def^N In a category $\mathcal{C}$ a morphism $f: A \to B$ is a retraction (or split epimorphism) if there exists $g: B \to A$ with $f \circ g = 1_B$. Dually, f is a coretraction (or split monomorphism) if there exists $g: B \to A$ with $g \circ f = 1_A$.

Lemma Split epi $\Rightarrow$ epi and split mono $\Rightarrow$ mono.

Proof Suppose $f \circ g = 1_B$ and $uf = vf$, then

$$u = u 1_B = ufg = vfg = v 1_B = v.$$

Similarly for split monos. $\square$

[for $0 \leq i \leq q$, and $\eta^q \varepsilon^{q+1} = 1$ as well. Thanks to an audience member for pointing out this oversight]

Lemma In Δ every ε^i is split mono and every η^i is split epi.

Proof By (4.3), $\eta^i \varepsilon^i = 1$, i.e. $[q] \xrightarrow{\varepsilon^i} [q+1] \xrightarrow{\eta^i} [q] = id_{[q]}$. $\square$

Ex 3 Any composite of epis (resp. split epis) in a category $\mathcal{C}$ is again an epi (resp. split epi) and the same for monos & split monos.

Theorem Any morphism $\mu : [q] \to [p]$ in Δ can be written uniquely as

$$\mu = \underbrace{\varepsilon^{i_1} \cdots \varepsilon^{i_s}}_{\text{split mono}} \underbrace{\eta^{j_1} \cdots \eta^{j_t}}_{\text{split epi}}$$

with $p \geq i_1 > i_2 > \cdots > i_s \geq 0$ and $0 \leq j_1 < \cdots < j_t < q$ and $q - t + s = p$. The identity $\mu = 1$ corresponds to $s = t = 0$.

Note $\underbrace{[q] \xrightarrow{\eta} [q-1] \xrightarrow{\eta} \cdots \xrightarrow{\eta} [q-t]}_{\text{responsible for "kernel" of } \mu} \xrightarrow{\varepsilon} [q-t+1] \longrightarrow \cdots \xrightarrow{\varepsilon} [q-t+s=p]$

Proof Next time!