MAST90068 – Lecture 5

1/8/16

In the previous lecture we introduced the simplex category Δ, and in today's lecture we prove the fundamental theorem about it.

<u>Theorem</u> Any morphism $\mu : [q] \to [p]$ in Δ can be written uniquely as

$$\mu = \underbrace{\varepsilon^{i_1} \cdots \varepsilon^{i_s}}_{\text{split mono}} \underbrace{\eta^{j_1} \cdots \eta^{j_t}}_{\text{split epi}}$$

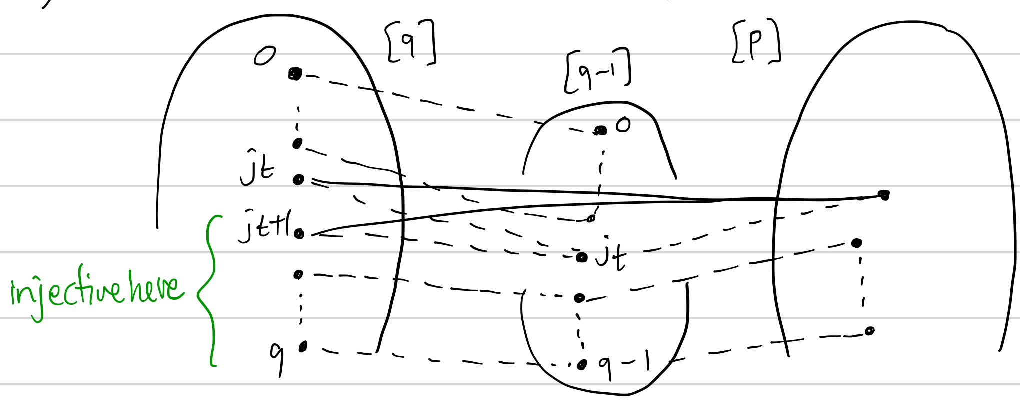
with $p \geq i_1 > i_2 > \cdots > i_s \geq 0$ and $0 \leq j_1 < \cdots < j_t < q$ and $q - t + s = p$. The identity $\mu = 1$ corresponds to $s = t = 0$.

<u>Note</u> $\underbrace{[q] \xrightarrow{\eta} [q-1] \xrightarrow{\eta} \cdots \xrightarrow{\eta} [q-t]}_{\text{responsible for "kernel" of } \mu} \xrightarrow{\varepsilon} [q-t+1] \longrightarrow \cdots \xrightarrow{\varepsilon} [q-t+s=p]$

<u>Proof</u> Let the elements j of $[q]$ with $\mu(j) = \mu(j+1)$ be

$$0 \leq j_1 < \cdots < j_t < q$$

Observe that μ may be written as $\mu = \mu' \eta^{j_t}$ for $\mu' : [q-1] \to [p]$ where $\mu'(a) = \mu(a)$ for all $0 \leq a \leq j_t$ and $\mu'(a) = \mu(a+1)$ for $a > j_t$



Note For $a = j_t + 1$ the second def^{n} $\mu'(a-1) = \mu(a)$

Check For $0 \le a \le j_t$ we have

$$\mu' \eta^{j_t}(a) = \mu'(a) = \mu(a)$$

while for $a = j_t + 1$

by hypothesis!

$$\mu' \eta^{j_t}(j_t + 1) = \mu'(j_t) = \mu(j_t) = \mu(j_t + 1)$$

and for $a > j_t + 1$

$$\mu' \eta^{j_t}(a) = \mu'(a-1) = \mu(a)$$

Claim The list of j in $[q-1]$ with $\mu'(j) = \mu'(j+1)$ is $j_1 < \cdots < j_{t-1}$.

Proof of claim If $0 \le j < j_t$ with $\mu'(j) = \mu'(j+1)$ then $\mu(j) = \mu(j+1)$ so $j \in \{j_1, \ldots, j_{t-1}\}$.
If $j = j_t$ and $\mu'(j) = \mu'(j+1)$ then $\mu(j_t) = \mu(j_t + 2)$ implies $\mu(j_t + 1) = \mu(j_t + 2)$ contradicting that the original j_t was maximal.
If $j > j_t$ then $\mu'(j) = \mu'(j+1)$ again contradicts maximality of j_t.

Applying the same argument as above to $\mu' : [q-1] \longrightarrow [p]$ we may continue writing $\mu' = \mu'' \eta^{j_{t-1}}$ and so on, until we have eventually

$$\mu = \tilde{\mu}\, \eta^{j_1} \cdots \eta^{j_t} \qquad \tilde{\mu} \text{ injective}$$

Notice that two injective morphisms in Δ with the same image are equal, and for any sequence $i_1 > \cdots > i_s$ in $\{0, .., p\}$ the map

$$\varepsilon^{i_1} \cdots \varepsilon^{i_s} : [p-s] \longrightarrow [p]$$

has image $\{0,..,p\} \setminus \{i_1,..,i_s\}$ by constuction (the latter ε's don't mess with earlier ones because $i_1 > \cdots > i_s$). Hence, if we take $i_1 > \cdots > i_s$ in $\{0,\dots,p\}$ such that $\{0,..,p\} \setminus \{i_1,..,i_s\} = \mathrm{Im}(\mu) = \mathrm{Im}(\tilde{\mu})$ then $\tilde{\mu} = \varepsilon^{i_1} \cdots \varepsilon^{i_s}$. Hence $\mu = \varepsilon^{i_1} \cdots \varepsilon^{i_s} \eta^{j_1} \cdots \eta^{j_t}$ as claimed.

If we have (for $\underline{i}', \underline{j}'$ ordered as above)

$$\varepsilon^{i_1} \cdots \varepsilon^{i_s} \eta^{j_1} \cdots \eta^{j_t} = \varepsilon^{i_1'} \cdots \varepsilon^{i_{s'}'} \eta^{j_1'} \cdots \eta^{j_{t'}'}$$

Then since both sides have the same image, $s = s'$ and $i_b' = i_b$ for $1 \le b \le s$. Then $\varepsilon := \varepsilon^{i_1} \cdots \varepsilon^{i_s}$ is mono and $\varepsilon \eta^{j_1} \cdots \eta^{j_t} = \varepsilon \eta^{j_1'} \cdots \eta^{j_{t'}'}$ implies $\eta^{j_1} \cdots \eta^{j_t} = \eta^{j_1'} \cdots \eta^{j_{t'}'}$. But the ordered set $\{j_1, \dots, j_t\}$ is recovered from $\eta = \eta^{j_1} \cdots \eta^{j_t}$ as the indices j s.t. $\eta(j) = \eta(j+1)$, so $t = t'$ and $j_b = j_b'$ all b.

$\square$

<u>Ex 1</u> Prove that every epi in Δ is split, and every mono is split

<u>Ex 2</u> Is $\mathrm{Hom}_\Delta([q],[p])$ finite? If so, how many elements does it contain? If not, why not?

Def^N Let $\mathcal{C}$ be a category. The <u>opposite category</u> $\mathcal{C}^{op}$ has

- $ob(\mathcal{C}^{op}) = ob(\mathcal{C})$
- $\mathcal{C}^{op}(A,B) := \mathcal{C}(B,A)$
- $\mathcal{C}^{op}$ has the same identities as $\mathcal{C}$
- $c_{\mathcal{C}^{op}} : \mathcal{C}^{op}(B,C) \times \mathcal{C}^{op}(A,B) \longrightarrow \mathcal{C}^{op}(A,C)$ is defined by
  $c_{\mathcal{C}^{op}}(g,f) = c_{\mathcal{C}}(f,g)$, where $c_{\mathcal{C}} : \mathcal{C}(B,A) \times \mathcal{C}(C,B) \longrightarrow \mathcal{C}(C,A)$
  is the composition in $\mathcal{C}$.

i.e.

$$\boxed{\mathcal{C}^{op} \quad \begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "c_{\mathcal{C}^{op}}(g,f) := f \circ g"'] & & C \end{tikzcd}} \longleftrightarrow \boxed{\mathcal{C} \quad \begin{tikzcd} & B \arrow[dl, "f"'] & \\ A & & C \arrow[ul, "g"'] \arrow[ll, "f \circ g"] \end{tikzcd}}$$

<u>Example</u> Let $\mathcal{C}$ be a category and $A \in ob(\mathcal{C})$. Then there is a functor

$$h_A : \mathcal{C}^{op} \longrightarrow \underline{Set}$$

- $h_A(B) = Hom_{\mathcal{C}}(B, A)$

- for $f : B \to B'$ in $\mathcal{C}$, so $f : B' \longrightarrow B$ in $\mathcal{C}^{op}$,
  $h_A(f) : h_A(B') \longrightarrow h_A(B)$ is $h_A(f)(x) = x \circ f$ (in $\mathcal{C}$).

<u>Lemma</u> If $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a functor, the same data defines a functor
$F^{op} : \mathcal{C}^{op} \longrightarrow \mathcal{D}^{op}$, with $(id_{\mathcal{C}})^{op} = id_{\mathcal{C}^{op}}$, $(G \circ F)^{op} = G^{op} \circ F^{op}$.

Def^N A *simplicial set* is a functor $\Delta^{op} \longrightarrow \underline{Set}$.

Example For $n \geq 0$ define the *standard n-simplex* to be the topological space

$$\Delta^n = \{ x = (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid \textstyle\sum_i x_i = 1, \ x_i \geq 0 \}$$

and for any morphism $f : [q] \longrightarrow [p]$ in Δ define the cts map

$$\Delta^f : \Delta^q \longrightarrow \Delta^p$$

$$\text{i.e. } \Delta^f = \left. \begin{array}{c|c} & 0 \; 1 \cdots\cdots q \\ \begin{array}{c} 0 \\ \vdots \\ p \end{array} & \begin{pmatrix} & & \\ & f & \\ & & \end{pmatrix} \end{array} : \mathbb{R}^{q+1} \longrightarrow \mathbb{R}^{p+1} \right|_{\Delta^q}$$

matrix has (i,j) entry equal to $\delta_{i, f(j)}$

restricted to Δ^q

Ex 3 Check that for $[q] \xrightarrow{f} [p] \xrightarrow{g} [t]$ in Δ we have $\Delta^g \circ \Delta^f = \Delta^{g \circ f}$, i.e. that the above defines a functor $\Delta \longrightarrow \underline{Top}$, by $[n] \longmapsto \Delta^n$ and $f \longmapsto \Delta^f$. Denote this functor by $\Delta^\bullet$.

Def^N The *simplicial singular complex* of a topological space X is the simplicial set $SX : \Delta^{op} \longrightarrow \underline{Set}$ defined by

$$(SX)([n]) = \mathrm{Hom}_{\underline{Top}}(\Delta^n, X)$$

maps $\Delta^n \longrightarrow X$ are called singular simplices

$$(SX)(f : [p] \longrightarrow [q]) : \mathrm{Hom}_{\underline{Top}}(\Delta^p, X) \longrightarrow \mathrm{Hom}_{\underline{Top}}(\Delta^q, X)$$
$$\gamma \longmapsto \gamma \circ \Delta^f$$

this is a morphism in Δ^{op}, so really f is $[q] \longrightarrow [p]$ in Δ

<u>Note</u> We can write SX as the composite functor

$$\Delta^{op} \xrightarrow{(\Delta^\bullet)^{op}} \underline{Top}^{op} \xrightarrow{h_X} \underline{Set}$$

<u>Next</u> we want to make a category $\underline{SSet}$ of simplicial sets and promote $X \mapsto SX$ to a functor $\underline{Top} \longrightarrow \underline{SSet}$. For this we need the notion of a natural transformation.

<u>Def^{n}</u> Let $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ be functors. A <u>natural transformation</u> $\alpha : F \longrightarrow G$ is the following data:

(1) for every $C \in ob(\mathcal{C})$ a morphism $\alpha_C : FC \longrightarrow GC$ in $\mathcal{D}$

subject to the axiom

(2) for every morphism $f : C \longrightarrow C'$ in $\mathcal{C}$ the diagram

$$\begin{tikzcd} FC \arrow[r, "\alpha_C"] \arrow[d, "F(f)"'] & GC \arrow[d, "G(f)"] \\ FC' \arrow[r, "\alpha_{C'}"'] & GC' \end{tikzcd}$$

commutes.

<u>Remark</u> $1_F : F \longrightarrow F$ defined by $(1_F)_C = 1_{F(C)}$ is always a natural transformation (the <u>identity natural transformation</u>)

<u>Ex 4</u> If $F_1, F_2, F_3 : \mathcal{C} \longrightarrow \mathcal{D}$ are functors and $\alpha : F_1 \longrightarrow F_2$ and $\beta : F_2 \longrightarrow F_3$ are natural transformations then so is $\beta \circ \alpha : F_1 \longrightarrow F_3$ defined by $(\beta \circ \alpha)_C = \beta_C \circ \alpha_C$.

<u>Ex 5</u> Given functors $F_i : \mathcal{C} \to \mathcal{D}$ for $i \in \{1,2,3,4\}$ and natural transformations

$$F_1 \xrightarrow{\alpha} F_2 \xrightarrow{\beta} F_3 \xrightarrow{\gamma} F_4$$

we have $\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha$. Further, if $\alpha : F_1 \to F_2$ is a natural transformation

$$1_{F_2} \circ \alpha = \alpha = \alpha \circ 1_{F_1}$$

as natural transformations.

These exercises prove that the next definition makes sense.

<u>Def^N</u> Let $\mathcal{C}, \mathcal{D}$ be categories with $\mathcal{C}$ small. There is a category $[\mathcal{C}, \mathcal{D}]$ whose objects are functors $F : \mathcal{C} \longrightarrow \mathcal{D}$ and where morphisms $F \to F'$ are natural transformations, with the composition defined above.

<u>Def^N</u> The <u>category of simplicial sets</u> is <u>sSet</u> $= [\Delta^{op}, \underline{Set}]$.

<u>Example</u> Given a topological space X let $L(X)$ denote the poset whose elements are open subsets of X with $U \leq V$ if U is a subset of V. Then $L(X)$ is a preorder, and we view it as a category. The category of <u>presheaves of abelian groups</u> on X is simply $[L(X)^{op}, \underline{Ab}]$.

Lemma The singular simplicial complex construction $X \mapsto SX$ is part of a functor $S : \underline{\mathrm{Top}} \longrightarrow \underline{\mathrm{SSet}}$.

Proof We have defined S on objects. Let $\varphi : X \longrightarrow X'$ be a cts map. Then we have a function, denoted $\varphi \circ (-)$, for any $n \geq 0$

$$\varphi \circ (-) : \mathrm{Hom}_{\underline{\mathrm{Top}}}(\Delta^n, X) \longrightarrow \mathrm{Hom}_{\underline{\mathrm{Top}}}(\Delta^n, X')$$
$$\gamma \longmapsto \varphi \circ \gamma$$

We claim these fit into a natural transformation

$$S(\varphi) : SX \longrightarrow SX'$$

both are functors $\underline{\Delta}^{op} \longrightarrow \underline{\mathrm{Set}}$

$$\begin{array}{ccc} S(\varphi)_{[n]} : (SX)([n]) & \longrightarrow & (SX')([n]) \\ \| & & \| \\ \mathrm{Hom}_{\underline{\mathrm{Top}}}(\Delta^n, X) & & \mathrm{Hom}_{\underline{\mathrm{Top}}}(\Delta^n, X') \end{array}$$

That is, we set $S(\varphi)_{[n]} := \varphi \circ (-)$. We need to check naturality, i.e. that for any $f : [q] \longrightarrow [p]$ in $\underline{\Delta}$ (thus $[p] \longrightarrow [q]$ in $\underline{\Delta}^{op}$) the diagram

$$\begin{tikzcd} (SX)([p]) \arrow[r, "S(\varphi)_{[p]}"] \arrow[d, "(SX)(f)"'] & (SX')([p]) \arrow[d, "(SX')(f)"] \\ (SX)([q]) \arrow[r, "S(\varphi)_{[q]}"'] & (SX)([q]) \end{tikzcd} \qquad (7.2)$$

commutes. But (7.2) is just

$$\begin{tikzcd} \mathrm{Hom}_{\underline{Top}}(\Delta^p, X) \arrow[r, "\varphi \circ -"] \arrow[d, "- \circ \Delta^f"'] & \mathrm{Hom}_{\underline{Top}}(\Delta^p, X') \arrow[d, "- \circ \Delta^f"] \\ \mathrm{Hom}_{\underline{Top}}(\Delta^q, X) \arrow[r, "\varphi \circ -"'] & \mathrm{Hom}_{\underline{Top}}(\Delta^q, X') \end{tikzcd} \tag{8.1}$$

where recall $\Delta^f : \Delta^q \longrightarrow \Delta^p$.

But it is clear that (8.1) commutes, e.g. for $\gamma : \Delta^p \longrightarrow X$ we have

$$\begin{tikzcd} \gamma \arrow[r, mapsto] \arrow[d, mapsto] & \varphi \circ \gamma \arrow[d, mapsto] \\ \gamma \circ \Delta^f \arrow[r, mapsto] & (\varphi \circ \gamma) \circ \Delta^f = \varphi \circ (\gamma \circ \Delta^f) \end{tikzcd}$$

so we conclude $S(f)$ is natural. Clearly $S(1_X) = 1_{SX}$ and $S(g \circ f) = S(g) \circ S(f)$, so S is a functor. $\square$

<u>Note</u> This model of spaces in simplicial sets is the starting point of <u>homological algebra</u>. If $V : \underline{Set} \longrightarrow \underline{Ab}$ denotes the free group functor from Lecture 2, then the functor

$$\Delta^{op} \xrightarrow{SX} \underline{Set} \xrightarrow{V} \underline{Ab}$$

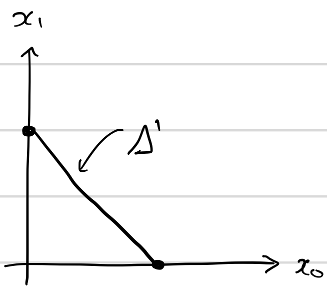
is the data from which ones defines the (integral) <u>singular homology</u> $H_*(X)$, the principal algebraic invariant of a topological space X. (we will see this construction in detail later). The functoriality of S we have just shown leads to linear maps $H_*X \longrightarrow H_*Y$ for any $\varphi : X \longrightarrow Y$, which are fundamental.

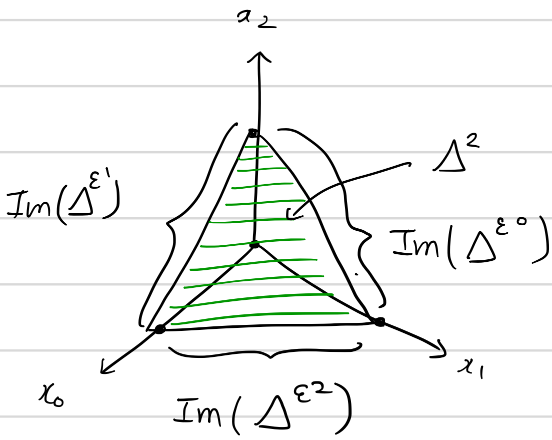
<u>Example</u> Consider $\varepsilon^0, \varepsilon^1, \varepsilon^2 : [1] \longrightarrow [2]$ and the induced

$$\begin{array}{ccc} \Delta^{\varepsilon^0} : \Delta^1 & \longrightarrow & \Delta^2 \\ \cap\!| & & \cap\!| \\ \mathbb{R}^2 & & \mathbb{R}^3 \end{array}
\qquad
\Delta^{\varepsilon^0}(x_0, x_1) = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = (0, x_0, x_1)$$

$$\Delta^{\varepsilon^1}(x_0, x_1) = (x_0, 0, x_1)$$

$$\Delta^{\varepsilon^2}(x_0, x_1) = (x_0, x_1, 0)$$



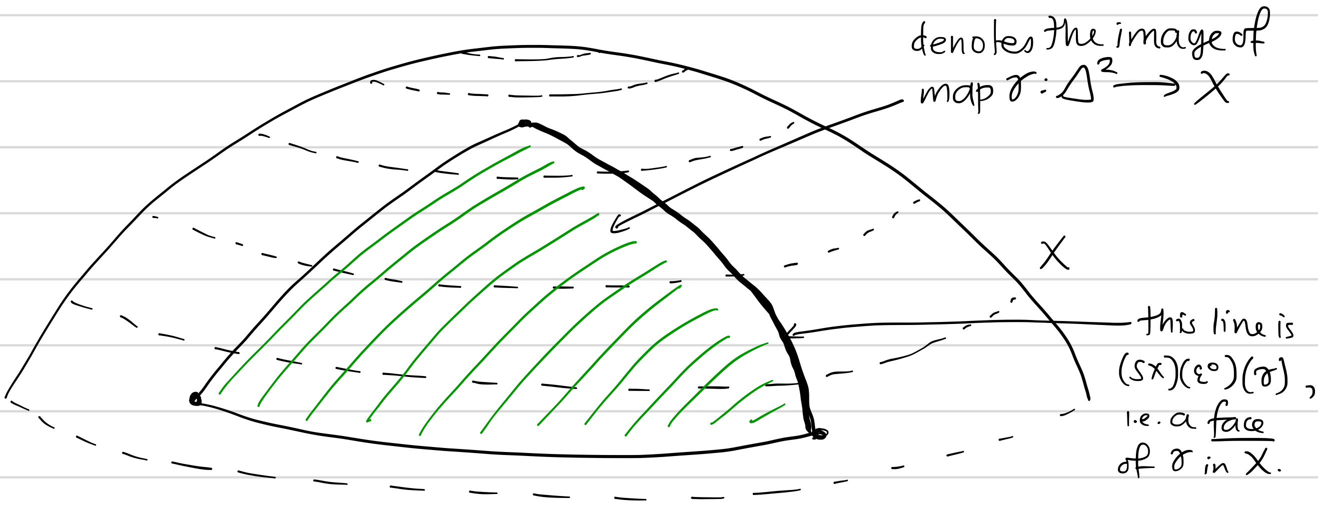


Let X be a topological space. Then

$$(SX)(\varepsilon^i)$$

$$(SX)([1]) = \mathrm{Hom}_{\underline{Top}}(\Delta^1, X) = \text{line segments in } X$$

$$(SX)([2]) = \mathrm{Hom}_{\underline{Top}}(\Delta^2, X) = \text{triangles in } X$$



$$(SX)(\varepsilon^0)(\gamma) : \Delta^1 \longrightarrow X \qquad (SX)(\varepsilon^0)(\gamma)(x_0, x_1) = (\gamma \circ \Delta^{\varepsilon^0})(x_0, x_1)$$

$$= \gamma(0, x_0, x_1)$$