MAST90068 – Lecture 6

Recall that last lecture we defined a functor from topological spaces to simplicial sets

$$S : \underline{Top} \longrightarrow \underline{SSet} \qquad (1.1)$$

on objects by defining SX to be the functor $SX := h_X \circ (\Delta^\bullet)^{op}$, as in

$$\Delta^{op} \xrightarrow{(\Delta^\bullet)^{op}} \underline{Top}^{op} \xrightarrow{h_X} \underline{Set} \qquad (1.2)$$

where $\Delta^\bullet : \Delta \longrightarrow \underline{Top}$ was the standard simplices $[n] \mapsto \Delta^n \subseteq \mathbb{R}^{n+1}$. Given continuous $\varphi : X \longrightarrow Y$ we defined a natural transformation $S\varphi : SX \longrightarrow SY$ "by hand" and in today's lecture we re-examine this more abstractly via (1.2).

<u>Lemma</u> Let $\mathscr{C}, \mathscr{D}_1, \mathscr{D}_2$ be categories with $\mathscr{C}$ small and let $Q : \mathscr{D}_1 \longrightarrow \mathscr{D}_2$ be a functor. Then there is an induced functor

$$A_Q : [\mathscr{C}, \mathscr{D}_1] \longrightarrow [\mathscr{C}, \mathscr{D}_2]$$

$$A_Q(F) = Q \circ F$$

$$\begin{cases} A_Q(\alpha : F \longrightarrow F') : Q \circ F \longrightarrow Q \circ F' \\ A_Q(\alpha)_C = Q(\alpha_C) \end{cases}$$

<u>Proof</u> First we need to check $A_Q(\alpha)$, as defined, is a natural transformation $Q \circ F \longrightarrow Q \circ F'$. So let $f : C \longrightarrow C'$ be a morphism in $\mathscr{C}$. We have to check that

$$\begin{tikzcd}
(Q \circ F)(C) \arrow[r, "A_Q(\alpha)_C"] \arrow[d, "(Q \circ F)(f)"'] & (Q \circ F')(C) \arrow[d, "(Q \circ F)(f)"] \\
(Q \circ F)(C') \arrow[r, "A_Q(\alpha)_{C'}"'] & (Q \circ F')(C')
\end{tikzcd}$$

commutes. But this is the image under Q of the diagram

$$\begin{tikzcd} FC \arrow[r, "\alpha_C"] \arrow[d, "Ff"'] & F'C \arrow[d, "F'f"] \\ FC' \arrow[r, "\alpha_{C'}"'] & F'C' \end{tikzcd}$$

which commutes by naturality of α, so we've done: $A_Q(\alpha)$ is a natural transformation $Q \circ F \longrightarrow Q \circ F'$ and thus a morphism in $[\mathcal{C}, \mathcal{D}_2]$.

So we have defined a mapping A_Q on objects and morphisms, it remains to check that it is functorial: $A_Q(id : F \longrightarrow F)_C = Q(id_C) = id_{QC}$ so $A_Q(id) = id$, and

$$\begin{aligned} A_Q(g \circ f)_C &= Q((g \circ f)_C) && \text{(def}^N\ A_Q) \\ &= Q(g_C \circ f_C) && \text{(def}^N\text{ composition of nat. trans)} \\ &= Q(g_C) \circ Q(f_C) && (Q \text{ is a functor}) \\ &= A_Q(g)_C \circ A_Q(f)_C \\ &= [A_Q(g) \circ A_Q(f)]_C \end{aligned}$$

Hence $A_Q(g \circ f) = A_Q(g) \circ A_Q(f)$ and we're done. $\square$

<u>Lemma</u> Let $\mathcal{C}_1, \mathcal{C}_2, \mathcal{D}$ be categories with $\mathcal{C}_1, \mathcal{C}_2$ small and let $Q : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ be a functor. Then there is an induced functor $B_Q : [\mathcal{C}_2, \mathcal{D}] \longrightarrow [\mathcal{C}_1, \mathcal{D}]$ defined by $B_Q(F) = F \circ Q$, $B_Q(\alpha : F \longrightarrow F')_C = \alpha_{QC}$.

<u>Ex 1</u> Prove the Lemma.

<u>Example</u> From $\Delta^\bullet : \Delta \longrightarrow \underline{Top}$ we get a functor $(\Delta^\bullet)^{op} : \Delta^{op} \longrightarrow \underline{Top}^{op}$ and thus

$$\begin{aligned}[\underline{Top}^{op}, \underline{Set}] &\longrightarrow [\Delta^{op}, \underline{Set}] = \underline{SSet} \\ F &\longmapsto F \circ (\Delta^\bullet)^{op}\end{aligned}$$

(modulo $\underline{Top}$ not being small. Either ignore this or use a small, full subcategory $\mathcal{C} \subseteq \underline{Top}$ including all the Δ^n's). The next step is to construct a functor as marked by the dotted line in:

$$\begin{array}{ccccc} \underline{Top} & \dashrightarrow & [\underline{Top}^{op}, \underline{Set}] & \longrightarrow & [\Delta^{op}, \underline{Set}] = \underline{SSet} \\ X & \longmapsto & h_X & \longmapsto & h_X \circ (\Delta^\bullet)^{op} =: SX \end{array}$$

<u>Lemma</u> Let $\mathcal{C}$ be a small category. There is a functor $h_{(-)} : \mathcal{C} \longrightarrow [\mathcal{C}^{op}, \underline{Set}]$ defined on objects by $X \longmapsto h_X$ and on morphisms $f : X \longrightarrow X'$ by the natural transformation

$$h_f : h_X \longrightarrow h_{X'}$$

$$\begin{array}{cccc} (h_f)_C : & h_X(C) & \longrightarrow & h_{X'}(C) \\ & \| & & \| \\ & \mathrm{Hom}_{\mathcal{C}}(C, X) & & \mathrm{Hom}_{\mathcal{C}}(C, X') \end{array}$$

$$(h_f)_C(x) := f \circ x$$

$$\begin{tikzcd} & C \arrow[dl, dashed, "x"'] \arrow[dr, dashed, "f \circ x"] & \\ X \arrow[rr, "f"'] & & X' \end{tikzcd}$$

<u>Proof</u> ① <u>h_f is a natural transformation</u> of functors $h_X, h_{X'} : \mathcal{C}^{op} \to \underline{Set}$

Suppose $t : C' \to C$ in $\mathcal{C}$, thus $C \to C'$ in $\mathcal{C}^{op}$. We need to show commutativity of the outer square in

$$
\begin{tikzcd}
h_X(C) \arrow[rrr, "h_X(t)"] \arrow[ddd, "(h_f)_C"'] \arrow[dr, equal] & & & h_X(C') \arrow[ddd, "(h_f)_{C'}"] \arrow[dl, equal] \\
& \mathcal{C}(C,X) \arrow[r, "-\circ t"] \arrow[d, "f\circ -"'] & \mathcal{C}(C',X) \arrow[d, "f\circ -"] & \\
& \mathcal{C}(C,X') \arrow[r, "-\circ t"] & \mathcal{C}(C',X') & \\
h_{X'}(C) \arrow[rrr, "h_{X'}(t)"'] \arrow[ur, equal] & & & h_{X'}(C') \arrow[ul, equal]
\end{tikzcd}
$$

But all the marked squares commute by def^N, so it is enough to show the inner square commutes, which is true since $\urcorner\!\downarrow$ is $x \longmapsto f\circ(x\circ t)$ and $\llcorner\!\to$ is $x \longmapsto (f\circ x)\circ t$ which are equal by associativity.

② <u>$h_{(-)}$ is functorial</u> i.e. $h_{id} = id$, $h_{g\circ f} = h_g \circ h_f$.

$$(h_{id_X})_C(x) = id_X \circ x = x \qquad \therefore\ (h_{id_X})_C = id_{h_X(C)}$$

$$\therefore\ h_{id_X} = id_{h_X}$$

$$
\begin{aligned}
(h_{g\circ f})_C(x) &= (g\circ f)\circ x \\
&= g\circ(f\circ x) \\
&= (h_g)_C\big((h_f)_C(x)\big) \\
&= \big[(h_g)_C \circ (h_f)_C\big](x) \\
&= \big[h_g \circ h_f\big]_C(x)
\end{aligned}
$$

$\therefore\ h_{g\circ f} = h_g \circ h_f$ as natural transformations.

$\square$

Ex 2 (Yoneda Lemma) Let $\mathcal{C}$ be a category, and $F : \mathcal{C}^{op} \to \underline{Set}$ a functor. Writing

$$\mathrm{Nat}(F, F') := \{\text{natural transformations } F \to F'\}$$

prove that the map

$$\Phi_{C,F} : \mathrm{Nat}(h_C, F) \longrightarrow F(C), \quad \Phi_{C,F}(\alpha) = \alpha_C(id_C)$$

is a bijection with the following properties

(a) $\Phi_{C,F}$ is natural in C: for any morphism $f : C \to C'$ in $\mathcal{C}$ there is a commutative diagram

$$\begin{tikzcd} \mathrm{Nat}(h_C, F) \arrow[r, "\Phi_{C,F}"] & F(C) \\ \mathrm{Nat}(h_{C'}, F) \arrow[u, "- \circ h_f"] \arrow[r, "\Phi_{C',F}"'] & F(C') \arrow[u, "F(f)"'] \end{tikzcd}$$

(b) $\Phi_{C,F}$ is natural in F: for any natural transformation $\alpha : F \to F'$ there is a commutative diagram

$$\begin{tikzcd} \mathrm{Nat}(h_C, F) \arrow[r, "\Phi_{C,F}"] \arrow[d, "\alpha \circ -"'] & F(C) \arrow[d, "\alpha_C"] \\ \mathrm{Nat}(h_C, F') \arrow[r, "\Phi_{C,F'}"'] & F'(C) \end{tikzcd}$$

(c) Is there a <u>unique</u> family $\{\Phi_{C,F}\}_{C,F}$ of bijections satisfying the naturality in (a), (b)? Discuss.

<u>Corollary</u> Let $\mathscr{C}$ be a small category. Then the functor

$$h_{(-)} : \mathscr{C} \longrightarrow [\mathscr{C}^{op}, \underline{Set}]$$

is fully faithful. It is called the <u>Yoneda embedding</u>.

<u>Proof</u> Given $A, B \in ob(\mathscr{C})$ consider $F = h_B$ in the above, and the maps

$$\begin{tikzcd} \mathscr{C}(A,B) \arrow[r, bend left, "f \mapsto h_f"] \arrow[r, phantom, "\cong"] & Nat(h_A, h_B) \arrow[l, bend left, "\Phi_{A,h_B} \text{ i.e. } \alpha \mapsto (\alpha_A)(id_A)"] \end{tikzcd}$$

These are mutually inverse since

$$(h_f)_A(id_A) = f \circ id_A = f$$

and we already know by Ex 2 that Φ_{A,h_B} is a bijection. $\square$

<u>Conclusion</u> Let $\underline{Top}'$ be a small subcategory of $\underline{Top}$ containing $\{\Delta^n\}_{n \geq 0}$ (e.g. finite CW-complexes). Then we have a composite functor

$$\underline{Top}' \hookrightarrow [\underline{Top}'^{op}, \underline{Set}] \longrightarrow [\Delta^{op}, \underline{Set}] = \underline{SSet} \qquad (6.1)$$

Yoneda   from $\Delta \longrightarrow \underline{Top}'$

which sends $X \in \underline{Top}'$ to the simplicial set SX defined earlier, i.e. (6.1) recovers (more abstractly) the functor on p.⑦ of Lecture 5.