# MAST90068 – Lecture 7

In this lecture we start to develop the ideas of limits and colimits using the functor categories of the previous lecture.

**Def$^N$** An <u>oriented graph</u> is a tuple $G = (V, E, s, t)$ consisting of a set $V$ of <u>vertices</u> (possibly empty) a set $E$ of <u>edges</u> (possibly empty) and functions $s, t : E \longrightarrow V$ (<u>source</u> and <u>target</u>). Given $e \in E$ with $s(e) = v_1$, $t(e) = v_2$ we write $e : v_1 \longrightarrow v_2$.

**Example** (a) $(\{\bullet\}, \phi, \text{unique}, \text{unique})$ $\qquad \bullet$

(b) $G(\ulcorner)$ defined by the diagram

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd}$$

i.e. $\{v_1, v_2, v_3\} =: V$

$\{e, f\} =: E$

$e : v_1 \longrightarrow v_2$, $f : v_1 \longrightarrow v_3$.

(c) $G(\lrcorner)$ defined by

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

**Def$^N$** A <u>morphism</u> of graphs $G_1 \xrightarrow{f} G_2$ where $G_i = (V_i, E_i, s_i, t_i)$ is a pair of functions $f_V : V_1 \longrightarrow V_2$ and $f_E : E_1 \longrightarrow E_2$ with the property that if $e : v_1 \longrightarrow v_2$ is an edge in $G_1$ then $f_E(e) : f_V(v_1) \longrightarrow f_V(v_2)$. We denote the category of oriented graphs by <u>Graph</u>.

**Lemma** There is a functor $U : \underline{Cat} \longrightarrow \underline{Graph}$ defined by

$$U(\mathscr{C}) = \Big(ob(\mathscr{C}), \coprod_{A,B \in ob(\mathscr{C})} \mathscr{C}(A,B), \text{source}, \text{target}\Big)$$

and in the obvious way on functors. call this the <u>underlying graph</u> of $\mathscr{C}$.

**Proof** Clear. $\square$

<u>Proposition</u> There is a functor $P : \underline{\text{Graph}} \longrightarrow \underline{\text{Cat}}$ together with a bijection for every oriented graph $G$ and small category $\mathcal{C}$

$$\Phi_{G,\mathcal{C}} : \mathrm{Hom}_{\underline{\text{Cat}}}(PG, \mathcal{C}) \longrightarrow \mathrm{Hom}_{\underline{\text{Graph}}}(G, U\mathcal{C})$$

which is natural in $G, \mathcal{C}$. That is, for every morphism of graphs $f : G \longrightarrow G'$ and functor $F : \mathcal{C} \to \mathcal{C}'$ the following diagrams commute

$$\begin{tikzcd}
\mathrm{Hom}_{\underline{\text{Cat}}}(PG, \mathcal{C}) \arrow[r, "\Phi_{G,\mathcal{C}}"] \arrow[d, "F\circ -"'] & \mathrm{Hom}_{\underline{\text{Graph}}}(G, U\mathcal{C}) \arrow[d, "UF\circ -"] \\
\mathrm{Hom}_{\underline{\text{Cat}}}(PG, \mathcal{C}') \arrow[r, "\Phi_{G,\mathcal{C}'}"'] & \mathrm{Hom}_{\underline{\text{Graph}}}(G, U\mathcal{C}')
\end{tikzcd} \qquad (2.1)$$

$$\begin{tikzcd}
\mathrm{Hom}_{\underline{\text{Cat}}}(PG, \mathcal{C}) \arrow[r, "\Phi_{G,\mathcal{C}}"] & \mathrm{Hom}_{\underline{\text{Graph}}}(G, U\mathcal{C}) \\
\mathrm{Hom}_{\underline{\text{Cat}}}(PG', \mathcal{C}) \arrow[r, "\Phi_{G',\mathcal{C}}"'] \arrow[u, "-\circ Pf"] & \mathrm{Hom}_{\underline{\text{Graph}}}(G', U\mathcal{C}) \arrow[u, "-\circ f"']
\end{tikzcd} \qquad (2.2)$$

<u>Proof</u> Let $G = (V, E, s, t)$ be a graph. The <u>path category</u> $PG$ has objects $\mathrm{ob}(PG) = V$. Given $v, v' \in V$ and $n > 0$ we define <u>paths of length $n$</u> from $v$ to $v'$ in $G$ to be sequences

$$\mathrm{Path}_n(v, v') := \{ (e_1, \dots, e_n) \in E^n \mid s(e_1) = v,\ t(e_n) = v' \text{ and } t(e_i) = s(e_{i+1}) \text{ for } 1 \le i \le n-1 \}$$

We define separately paths of length zero:

$$\mathrm{Path}_0(v,v') = \begin{cases} \phi & v \neq v' \quad \text{(no paths)} \\ \{\phi\} & v = v' \quad \text{(\underline{one} path, the empty path)} \end{cases}$$

Given $m,n \geq 0$ and $v,v',v'' \in V$ we have a concatenation map

$$\begin{aligned} c: \mathrm{Path}_m(v',v'') \times \mathrm{Path}_n(v,v') &\longrightarrow \mathrm{Path}_{m+n}(v,v'') \\ ((f_1,\ldots,f_m),(e_1,\ldots,e_n)) &\longmapsto (e_1,\ldots,e_n,f_1,\ldots,f_m) \end{aligned} \tag{3.1}$$

The <u>morphisms</u> in $PG$ are all paths:

$$\mathrm{Hom}_{PG}(v,v') = \coprod_{n \geq 0} \mathrm{Path}_n(v,v') \tag{3.2}$$

with composition defined by the concatenation function $c$ of (3.1). Composition is clearly associative, with identity at $v \in V$ being the empty path $\mathrm{id}_v = \{\phi\} \in \mathrm{Path}_0(v,v)$. So $PG$ is a small category.

<u>Functoriality</u> let $h: G \longrightarrow G'$ be a morphism of graphs, and write $h$ for both $h_V, h_E$. We define a functor $Ph: PG \longrightarrow PG'$ on objects by $(Ph)(v) = h(v)$ and on morphisms by $(Ph)(\{\phi\}) = \{\phi\}$ for each $\{\phi\} \in \mathrm{Hom}_{PG}(v,v)$ and

$$(Ph)(e_1,\ldots,e_n) = (h(e_1),\ldots,h(e_n))$$

$$v = v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} \cdots \longrightarrow v_n \xrightarrow{e_n} v_{n+1} = v'$$

for each $(e_1,\ldots,e_n) \in \mathrm{Path}_n(v,v') \subseteq \mathrm{Hom}_{PG}(v,v')$. This is a functor since

$$
\begin{aligned}
(Ph)\big((f_1,\dots,f_m)\circ(e_1,\dots,e_n)\big) &= (Ph)\big((e_1,\dots,e_n,f_1,\dots,f_m)\big)\\
&= (h(e_1),\dots,h(e_n),h(f_1),\dots,h(f_m))\\
&= (Ph)(f_1,\dots,f_m)\circ(Ph)(e_1,\dots,e_n).
\end{aligned}
$$

<u>Summary</u> We have constructed a functor $P$.

<u>Unit</u>: there is a morphism of graphs $\eta_G : G \longrightarrow UPG$ natural in $G$, defined by sending $v \longmapsto v$ and an edge $e$ in $G$ to the morphism / edge

$$\eta_G(e : v \longrightarrow v') = (e) \in \mathrm{Path}_1(v,v'). \tag{4.1}$$

This is clearly a morphism of graphs, and it is natural in $G$ since if $h : G \to G'$ is a morphism of graphs, the diagram

$$
\begin{tikzcd}
G \arrow[r, "\eta_G"] \arrow[d, "h"'] & UPG \arrow[d, "UPh"] \\
G' \arrow[r, "\eta_{G'}"'] & UPG'
\end{tikzcd}
\tag{4.2}
$$

commutes, since on vertices

$$UPh(\eta_G(v)) = UPh(v) = h(v) = \eta_{G'}(h(v))$$

and on an edge $e : v \to v'$ in $G$,

$$
\begin{aligned}
UPh(\eta_G(e)) &= UPh((e))\\
&= Ph((e)) = (h(e))\\
&= \eta_{G'}(h(e)).
\end{aligned}
$$

<u>The function $\Phi$</u> Given $G, \mathcal{C}$ we now define

$$\Phi_{G,\mathcal{C}} : \mathrm{Hom}_{\underline{Cat}}(PG, \mathcal{C}) \longrightarrow \mathrm{Hom}_{\underline{Graph}}(G, U\mathcal{C})$$

$$\Phi_{G,\mathcal{C}}(F) := UF \circ \eta_G \qquad (5.1)$$

i.e. $G \xrightarrow{\eta_G} UPG \xrightarrow{UF} U\mathcal{C}$

"sit $G$ inside $PG$ as paths of length 1, evaluate $F$"

We need to show this is a bijection. The naturality (2.1), (2.2) we leave as exercises. We prove bijectivity as a consequence of the following

<u>Claim</u> $\eta_G$ is the universal morphism of graphs from $G$ into the underlying graph of a category. More precisely: if $\mathcal{C}$ is a category and $\alpha : G \longrightarrow U\mathcal{C}$ a morphism of graphs, there is a unique functor $\tilde{\alpha} : PG \longrightarrow \mathcal{C}$ (the <u>factorisation</u>) such that

$$\begin{tikzcd} G \arrow[r, "\alpha"] \arrow[dr, "\eta_G"'] & U\mathcal{C} \\ & UPG \arrow[u, dashed, "U\tilde{\alpha}"'] \end{tikzcd} \qquad (5.2)$$

commutes in <u>Graph</u>.

<u>Proof that $\Phi_{G,\mathcal{C}}$ is a bijection, supposing the Claim</u> surjectivity is clear: given $\alpha : G \to U\mathcal{C}$ we have $\tilde{\alpha}$ s.t. $\Phi_{G,\mathcal{C}}(\tilde{\alpha}) = \alpha$, as $\Phi_{G,\mathcal{C}}(\tilde{\alpha}) = U\tilde{\alpha} \circ \eta_G$ by definition. For injectivity it suffices to note that if there were $\tilde{\alpha}'$ with $\Phi_{G,\mathcal{C}}(\tilde{\alpha}') = \Phi_{G,\mathcal{C}}(\tilde{\alpha}) = \alpha$ then by the uniqueness part of the claim, $\tilde{\alpha}' = \tilde{\alpha}$.

<u>Proof of Claim</u> Let $\alpha : G \longrightarrow U\mathcal{C}$ be given. We wish to define a functor $\tilde{\alpha} : PG \to \mathcal{C}$ such that (5.2) commutes. This <u>forces us to define</u> for $v$ a vertex of $G$

$$(6.1) \qquad \tilde{\alpha}(v) = \alpha(v) \qquad \left(\text{since we need } U\tilde{\alpha} \circ \eta_G(v) = \alpha(v)\right)$$

and for an edge $e : v \longrightarrow v'$ of $G$

$$(6.2) \qquad \tilde{\alpha}(\underbrace{(e)}_{\text{path of length 1 in } PG}) = \alpha(e) \qquad \left(\text{since we need } U\tilde{\alpha} \circ \eta_G(e) = \alpha(e)\right)$$

To be a functor $\tilde{\alpha}$ must send an empty path at $v \in ob(PG)$ to the identity at $\alpha(v) \in ob(\mathcal{C})$, and for a path of length $n > 1$ to be a functor $\tilde{\alpha}$ must satisfy

$$\begin{aligned} \tilde{\alpha}(e_1, \dots, e_n) &= \tilde{\alpha}\left((e_n) \circ \cdots \circ (e_1)\right) \\ &= \tilde{\alpha}((e_n)) \circ \cdots \circ \tilde{\alpha}((e_1)) \\ &= \alpha(e_n) \circ \cdots \circ \alpha(e_1) \qquad (\text{by } 6.2) \end{aligned}$$

So we adopt this as our <u>def<sup></sup>N</u> of $\tilde{\alpha}$. To show $\tilde{\alpha}$ is a functor it is enough to check that on a pair of sequences

$$\begin{aligned} \tilde{\alpha}\left((f_1, \dots, f_m) \circ (e_1, \dots, e_n)\right) &= \tilde{\alpha}(e_1, \dots, e_n, f_1, \dots, f_m) \\ &= \alpha(f_m) \circ \cdots \circ \alpha(f_1) \circ \alpha(e_n) \circ \cdots \alpha(e_1) \\ &= \left(\alpha(f_m) \circ \cdots \circ \alpha(f_1)\right) \circ \left(\alpha(e_n) \circ \cdots \circ \alpha(e_1)\right) \\ &= \tilde{\alpha}(f_1, \dots, f_m) \circ \tilde{\alpha}(e_1, \dots, e_n). \end{aligned}$$

By construction $U\tilde{\alpha} \circ \eta_G = \alpha$ and by the above we see $\tilde{\alpha}$ is unique, completing the proof of the claim and the Proposition. $\square$

**Ex 1** Prove commutativity of diagrams (2.1) and (2.2).

**Def^N** Let $G$ be an oriented graph and $\mathcal{C}$ a category. A *diagram* in $\mathcal{C}$ of shape $G$ is a morphism of graphs $G \xrightarrow{\alpha} U\mathcal{C}$, or what is the same

- for every vertex $v$ in $G$ an object $\alpha(v)$ of $\mathcal{C}$
- for every edge $e : v \longrightarrow v'$ in $G$ a morphism $\alpha(e) : \alpha(v) \longrightarrow \alpha(v')$ in $\mathcal{C}$.

**Example** $G =$
$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet \arrow[d] & \\ \bullet & \end{tikzcd}$$
a diagram in $\underline{Ab}$ of this shape is
$$\begin{tikzcd} \mathbb{Z} \arrow[r, "3"] \arrow[d, "2"'] & \mathbb{Z} \\ \mathbb{Z} \arrow[d, "5"'] & \\ \mathbb{Z} & \end{tikzcd} \tag{7.1}$$

By the Proposition we may freely identify such a diagram with a functor $PG \to \mathcal{C}$. But this furnishes us automatically with a good notion of *morphisms* of diagrams, namely, natural transformations of the associated functors.

**Lemma** With $G, \mathcal{C}$ as above let $\alpha_1, \alpha_2$ diagrams in $\mathcal{C}$ of shape $G$. A family of morphisms $\{\gamma_v : \alpha_1(v) \longrightarrow \alpha_2(v)\}_{v \in V(G)}$ defines a natural transformation $\gamma : \widetilde{\alpha}_1 \longrightarrow \widetilde{\alpha}_2$ of functors $\widetilde{\alpha}_i : PG \to \mathcal{C}$ if and only if for every edge $e : v \to v'$ in $G$,
$$\begin{tikzcd} \alpha_1(v) \arrow[r, "\gamma_v"] \arrow[d, "\alpha(e)"'] & \alpha_2(v) \arrow[d, "\alpha(e)"] \\ \alpha_1(v') \arrow[r, "\gamma_{v'}"'] & \alpha_2(v') \end{tikzcd} \tag{7.2}$$
commutes.

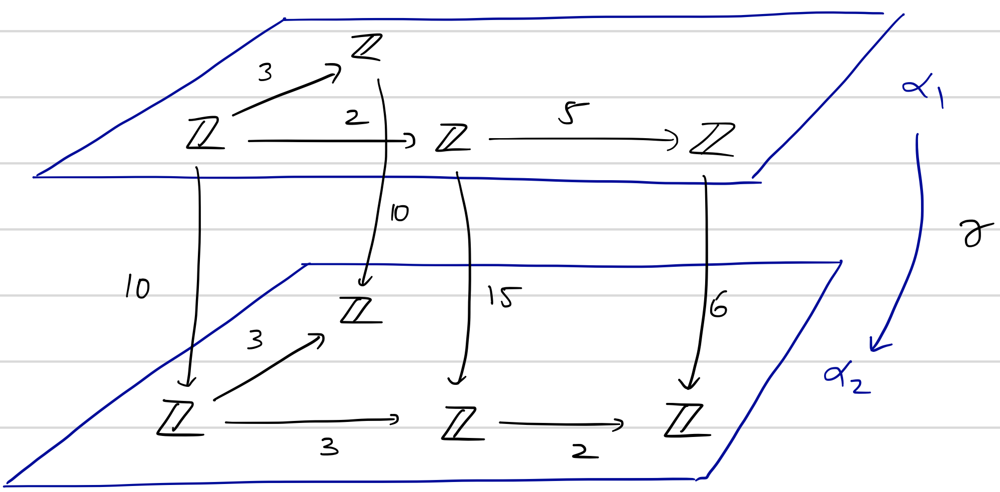
**Ex 2** Prove the Lemma.

**Def$^N$** With $G, \mathcal{C}$ as above, a <u>morphism</u> of diagrams $\alpha_1 \to \alpha_2$ in $\mathcal{C}$ of shape $G$ is a natural transformation $\tilde{\alpha}_1 \to \tilde{\alpha}_2$, i.e. a collection of $\gamma_v$ as in the Lemma, making (7.2) commute for every edge $e$.

**Example** Let $\alpha_1$ be the diagram in (7.1) and let $\alpha_2$ be

$$
\begin{tikzcd}
\mathbb{Z} \arrow[r, "3"] \arrow[d, "3"'] & \mathbb{Z} \\
\mathbb{Z} \arrow[d, "2"'] & \\
\mathbb{Z} &
\end{tikzcd}
$$

Then an example of a morphism $\alpha_1 \xrightarrow{\gamma} \alpha_2$ is




**Def$^N$** The <u>category of diagrams</u> of shape $G$ in $\mathcal{C}$ is $[PG, \mathcal{C}]$.

The question we next occupy ourselves with is: when can a <u>diagram</u> $G \to U\mathcal{C}$ be approximated "universally" by an <u>object</u> of $\mathcal{C}$? This is precisely how limits and colimits are defined.