MAST90068 – Lecture 8

Last lecture we defined a diagram in a category $\mathcal{C}$ of shape G (an oriented graph) to be a morphism of graphs $G \longrightarrow U\mathcal{C}$, or what is the same, a functor $PG \longrightarrow \mathcal{C}$ where PG is the path category.

Example $G = \{ \bullet \}$ a single vertex, no arrows. There is an equivalence of categories

$$[PG, \mathcal{C}] \xrightarrow{E} \mathcal{C}$$

defined on objects by $E(\alpha) = \alpha(\bullet)$ and on morphisms by $E(f) = f_{\bullet}$. The inverse functor $E^{-1} : \mathcal{C} \longrightarrow [PG, \mathcal{C}]$ sends an object C to the diagram assigning $\bullet \longmapsto C$.

The question that motivates today's lecture is: when can a diagram be approximated by an object? Here "approximated" should be understood in the sense of the following examples:

- Let S be a set, $V(S)$ the free abelian group on S. Then $V(S)$ is the "best approximation" to S among abelian groups, in that any function $S \longrightarrow A$, A an abelian group, factors uniquely via a morphism $V(S) \longrightarrow A$ in $\underline{Ab}$

$$\begin{tikzcd} S \arrow[r] \arrow[dr] & V(S) \arrow[d, dashed] \\ & A \end{tikzcd}$$

- If G is an oriented graph, PG is the "best approximation" to G among categories.

In the above examples, to "approximate" sets by abelian groups or graphs by categories, we needed a <u>forgetful</u> functor which says: underlying any group is a set, and underlying any category is a graph.

In what follows G is an oriented graph and $\mathcal{C}$ a category.

<u>Lemma</u> There is a functor $I : \mathcal{C} \longrightarrow [PG, \mathcal{C}]$ sending any object C to the "identity diagram" at C, that is

- for all vertices v of G, $I(C)(v) = C$,
- for all edges e of G, $I(C)(e) = id_C$.

<u>Proof</u> As defined $I(C)$ is a well-defined diagram. If $f : C \to C'$ is a morphism in $\mathcal{C}$ then $I(f) : I(C) \longrightarrow I(C')$ defined by $I(f)_v := f$ is a morphism of diagrams, and it is easily checked this makes I a functor. $\square$

<u>Remark</u> Let $C \in ob(\mathcal{C})$ and $G = \left\{ \begin{tikzcd} & v_1 \arrow[d] \\ v_2 \arrow[r] & v_3 \end{tikzcd} \right\}$. Consider a diagram α in $\mathcal{C}$ of shape G,

$$\begin{tikzcd} C \arrow[rr] \arrow[d] \arrow[drr] & & A_1 \arrow[d, "f"] \\ A_2 \arrow[rr, "g"'] & & A_3 \end{tikzcd} \tag{2.1}$$

A natural transformation $\eta : IC \longrightarrow \alpha$ consists of morphisms $\eta_{v_i} : C \longrightarrow A_i$ such that $f\eta_{v_1} = \eta_{v_3}$, $g\eta_{v_2} = \eta_{v_3}$, i.e. both triangles in (2.1) commute. Note that η is determined by η_{v_1}, η_{v_2} s.t. $f\eta_{v_1} = g\eta_{v_2}$.

Def^N Let $C \in ob(\mathscr{C})$ and $\alpha : PG \to \mathscr{C}$ be a diagram. A <u>cone</u> from C to α is a natural transformation $\eta : IC \longrightarrow \alpha$

Spelled out more concretely:

Def^N A cone from C to a diagram α is a morphism $\eta_v : C \longrightarrow \alpha(v)$ in $\mathscr{C}$ for every vertex v in G, such that for every edge $e : v \to v'$ in G, the diagram

$$\begin{tikzcd} & \alpha(v) \arrow[d, "\alpha(e)"] \\ C \arrow[ur, "\eta_v"] \arrow[r, "\eta_{v'}"'] & \alpha(v') \end{tikzcd}$$

⌜cones on the <u>left</u>⌟

commutes.

Def^N A <u>cocone</u> from α to an object C is a natural transformation $\varepsilon : \alpha \longrightarrow IC$. This is a family of morphisms $\varepsilon_v : \alpha(v) \longrightarrow C$ for every vertex v, such that for every edge $e : v \to v'$ the diagram

$$\begin{tikzcd} \alpha(v) \arrow[d, "\alpha(e)"'] \arrow[dr, "\varepsilon_v"] & \\ \alpha(v') \arrow[r, "\varepsilon_{v'}"'] & C \end{tikzcd}$$

⌜cocones on the <u>right</u>⌟

commutes.

Convention Let us write natural transformations $IC \to \alpha$ (i.e. cones) as simply $C \to \alpha$, where this will not cause confusion, and similarly for cocones.

If for a fixed diagram α there is a <u>universal</u> cone into α (i.e. the "best approximation to α by an object, on the left") we call it the <u>limit</u> of the diagram. The universal <u>cocone</u> (if it exists) is the <u>colimit</u> of α

<u>Def^N</u> Let $C \in ob(\mathcal{C})$ and $\alpha : PG \to \mathcal{C}$ be a diagram. A cone $\eta : IC \to \alpha$ is a <u>limit</u> if for any other cone $IC' \xrightarrow{\theta} \alpha$ there is a unique morphism $\tilde{\theta} : C' \to C$ making the diagram

$$\begin{tikzcd}
IC \arrow[rr, "\eta"] & & \alpha \\
& IC' \arrow[ul, dashed, "I\tilde{\theta}"] \arrow[ur, "\theta"'] &
\end{tikzcd}$$

commute in $[PG, \mathcal{C}]$.

Spelled out more concretely:

<u>Def^N</u> A cone $\eta : IC \to \alpha$ is a <u>limit</u> if for every cone θ from C' to α,

$$\begin{tikzcd}
& & \alpha(v) \arrow[d, "\alpha(e)"] \\
C' \arrow[r, dashed, "\tilde{\theta}"] \arrow[urr, bend left, "\theta_v"] \arrow[drr, bend right, "\theta_{v'}"'] & C \arrow[ur, "\eta_v"] \arrow[dr, "\eta_{v'}"'] & \\
& & \alpha(v')
\end{tikzcd}$$

there is a unique morphism $\tilde{\theta} : C' \to C$ such that for every vertex v, we have $\eta_v \circ \tilde{\theta} = \theta_v$.

we call this the <u>universal property</u> of the limit

<u>Note</u> Not every diagram has a limit (i.e. can be approximated by an object).

<u>Example</u> In the situation of (2.1), the limit is a pair of morphisms $C \to A_1$, $C \to A_2$ such that for any $C' \to A_1$, $C' \to A_2$ making the square commute, there is unique $C' \to C$ s.t. $C' \to C \to A_1 = C' \to A_1$ and $C' \to C \to A_2 = C' \to A_2$

$$\begin{tikzcd}
C' \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& C \arrow[r] \arrow[d] & A_1 \arrow[d] \\
& A_2 \arrow[r] & A_3
\end{tikzcd}$$

That is, the limit is precisely the <u>pullback</u>.

<u>Lemma</u> If a diagram α has a limit, it is unique up to unique isomorphism. That is, if $\eta, \eta' : C \to \alpha$ are both limits, there is a unique morphism $\psi : C' \to C$ such that

$$\begin{tikzcd}
C' \arrow[d, "\psi"'] \arrow[dr, "\eta'_v"] & \\
C \arrow[r, "\eta_v"'] & \alpha(v)
\end{tikzcd}$$

commutes for all v, and this ψ is an isomorphism.

<u>Proof</u> Such a ψ exists by the universal property of $\eta : C \to \alpha$, and likewise there is a unique $\psi' : C \to C'$ with $\eta'_v \circ \psi' = \eta_v$ for all v. We claim $\psi \circ \psi' = 1_C$ and symmetrically $\psi' \circ \psi = 1_{C'}$. Observe that for any v,

$$
\begin{aligned}
(5.1) \qquad & \eta_v \circ (\psi \circ \psi') \\
&= (\eta_v \circ \psi) \circ \psi' \\
&= \eta'_v \circ \psi' \\
&= \eta_v \\
&= \eta_v \circ 1_C
\end{aligned}
\qquad
\begin{tikzcd}
C' \arrow[dr, "\eta'_v"] \arrow[d, bend right, swap, "\psi'", leftarrow] \arrow[d, bend left, "\psi"] & \\
C \arrow[r, "\eta_v"] & \alpha(v)
\end{tikzcd}
$$

But $\eta_v : C \to \alpha(v)$ is a cone, so by the universal property of the limit $C \xrightarrow{\eta} \alpha$, there is a <u>unique</u> morphism $C \to C$ s.t. $C \to C \to \alpha(v)$ equals η_v for all v. That is, 1_C is <u>unique</u> with this property. From (5.1) we conclude $\psi \circ \psi' = 1_C$. $\square$

We are therefore justified in saying <u>the</u> limit of α (if it exists), and denoting it $\lim \alpha$.

<u>Def^N</u> Let J be a set, viewed as an oriented graph with no edges. A diagram $J \to \mathscr{C}$ is just a collection of objects $\{A_j\}_{j \in J}$ indexed by J. The limit of this diagram is called the <u>product</u> and written $\prod_{j \in J} A_j$ (if it exists).

That is, the product consists of an object C, morphisms $\pi_j : C \to A_j$ for each j, which is a universal family in the sense that if $\{\rho_j : C' \to A_j\}_{j \in J}$ is another collection of morphisms there is a <u>unique</u> morphism $\tilde{\rho} : C' \to C$ such that

$$
\begin{tikzcd}
 & & A_j \\
C' \arrow[urr, bend left, "\rho_j"] \arrow[r, swap, "\tilde{\rho}"] & C \arrow[ur, swap, "\pi_j"] &
\end{tikzcd}
$$

commutes for all j.

<u>Def^N</u> Let $C \in ob(\mathcal{C})$ and $\alpha : PG \to \mathcal{C}$ be a diagram. A cocone $\varepsilon : \alpha \to IC$ is a <u>colimit</u> if for any other cocone $\alpha \xrightarrow{\theta} IC'$ there is a unique morphism $\tilde{\theta} : C \to C'$ making the diagram

$$\begin{tikzcd}
IC \arrow[dr, dashed, "I\tilde{\theta}"'] & & \alpha \arrow[ll, "\varepsilon"'] \arrow[dl, "\theta"] \\
& IC' &
\end{tikzcd}$$

commute in $[PG, \mathcal{C}]$.

That is, a cocone $\varepsilon : \alpha \to IC$ is a colimit if for every cocone $\theta : \alpha \to IC'$

$$\begin{tikzcd}
\alpha(v) \arrow[d, "\alpha(e)"'] \arrow[dr, "\varepsilon_v"] \arrow[drr, bend left, "\theta_v"] & & \\
\alpha(v') \arrow[r, "\varepsilon_{v'}"'] \arrow[rr, bend right, "\theta_{v'}"'] & C \arrow[r, dashed, "\tilde{\theta}"] & C'
\end{tikzcd}$$

there is a unique morphism $\tilde{\theta} : C \to C'$ such that for every vertex v, $\tilde{\theta} \circ \varepsilon_v = \theta_v$.

<u>Ex 1</u> Given an oriented graph G let G^{op} denote the graph with the same vertices but all arrows reversed. Prove this is a functor $(-)^{op} : \underline{Graph} \to \underline{Graph}$. Since $(-)^{op} \circ (-)^{op} = id$ this is an <u>isomorphism</u> of categories.

<u>Ex 2</u> We defined earlier the opposite category $\mathcal{C}^{op}$ and for $F : \mathcal{C} \to \mathcal{D}$ the functor $F^{op} : \mathcal{C}^{op} \to \mathcal{D}^{op}$. Check this is a functor $(-)^{op} : \underline{Cat} \to \underline{Cat}$, which again is an isomorphism since $(-)^{op} \circ (-)^{op} = id$.

Ex 3 Check that the diagram of functors

$$\begin{tikzcd} \underline{\text{Graph}} \arrow[r, "(-)^{op}"] \arrow[d, "P"'] & \underline{\text{Graph}} \arrow[d, "P"] \\ \underline{\text{Cat}} \arrow[r, "(-)^{op}"'] & \underline{\text{Cat}} \end{tikzcd}$$

commutes, in particular $(PG)^{op} = PG^{op}$.

Ex 4 Prove that for an oriented graph G and category $\mathcal{C}$, there is an isomorphism

$$[PG, \mathcal{C}] \cong [P(G^{op}), \mathcal{C}^{op}].$$

That is, diagrams in $\mathcal{C}$ of shape G $=$ diagrams in $\mathcal{C}^{op}$ of shape G^{op} (hopefully this is obvious!). Given a diagram $\alpha : PG \longrightarrow \mathcal{C}$ and corresponding diagram $\alpha^{op} : PG^{op} \longrightarrow \mathcal{C}^{op}$ prove that there is a bijection for any $C \in \mathcal{C}$

$$\mathrm{Hom}_{[PG, \mathcal{C}]}(IC, \alpha) \cong \mathrm{Hom}_{[PG^{op}, \mathcal{C}^{op}]}(\alpha^{op}, IC)$$

natural in both C and α. (i.e. cones $C \to \alpha$ = cocones $\alpha^{op} \to C$). Prove that this bijection identifies limits on the left with colimits on the right.

Ex 5 Prove colimits are unique up to unique isomorphism.

Recall (1) $F : \underline{\mathrm{Ab}} \longrightarrow \underline{\mathrm{Set}}$ forgetful, the best approximation to a set S by an abelian group was $V(S)$, with natural bijections

$$\mathrm{Hom}_{\underline{\mathrm{Ab}}}(V(S), B) \xrightarrow{\cong} \mathrm{Hom}_{\underline{\mathrm{Set}}}(S, F(B)) \qquad (8.1)$$

(2) $U : \underline{\mathrm{Cat}} \longrightarrow \underline{\mathrm{Graph}}$ forgetful, the best approximation to a graph G by a category is the path category PG, with natural bijections

$$\mathrm{Hom}_{\underline{\mathrm{Cat}}}(PG, \mathscr{C}) \xrightarrow{\cong} \mathrm{Hom}_{\underline{\mathrm{Graph}}}(G, U\mathscr{C}) \qquad (8.2)$$

Lemma Let G be an oriented graph, $\mathscr{C}$ a category and $\alpha : PG \rightarrow \mathscr{C}$ a diagram. For any cone $\eta : IC \longrightarrow \alpha$ there is a function

$$\Phi : \mathrm{Hom}_{\mathscr{C}}(C', C) \longrightarrow \mathrm{Hom}_{[PG, \mathscr{C}]}(IC', \alpha)$$

$$\Phi(f) := \eta \circ If$$

which is natural in C', α, and Φ is a bijection if and only if η is the limit of α.

Ex 6 Prove the Lemma.

That is, there is a natural bijection

$$\mathrm{Hom}_{\mathscr{C}}(C', \lim \alpha) \cong \mathrm{Hom}_{[PG, \mathscr{C}]}(IC', \alpha) \qquad (8.3)$$

(compare and contrast to (8.1), (8.2)).

Ex 7 Check that the usual product in Set is a limit in this sense, and thus is also the categorical product.

Ex 8 What is the product in Ab and Top?

Ex 9 Is there always a product in SSet?

Ex 10 What is the limit of the diagram (on the graph $\overbrace{\cdots \to \bullet \to \bullet \to \bullet}^{\text{vertices indexed by } \mathbb{N}}$)

$$\cdots \longrightarrow \mathbb{R}[x]/(x^3) \xrightarrow{\pi} \mathbb{R}[x]/(x^2) \xrightarrow{\pi} \mathbb{R}[x]/(x)$$

in the category Ab, or commutative rings, where $\pi(\overline{f(x)}) = \overline{f(x)}$ is the quotient map at each edge.