MAST90068 – Lecture 9

Now we know what limits and colimits are, let us see some examples. To begin with, let us enumerate some examples of limits and colimits that have special names. A good reference here is Borceaux "Handbook of categorical algebra".

Graph G	limit	colimit
$\begin{array}{ccc} & & \bullet \\ & & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array}$	pullback	—
$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \\ \bullet & & \end{array}$	—	pushout
index set J (no edges)	product	coproduct
$\emptyset$ (no vertices, no edges)	terminal object (= empty product)	initial object (= empty coproduct)
$\bullet \rightrightarrows \bullet$	equaliser	coequaliser

<u>Def^N</u> Given a category $\mathcal{C}$ and a set of objects $\{A_j\}_{j\in J}$ a <u>product</u> of the family is a pair $(P, \{\pi_j\}_{j\in J})$ consisting of an object P and morphisms $\pi_j : P \longrightarrow A_j$ such that for every pair $(Q, \{\delta_j\}_{j\in J})$ where $\delta_j : Q \longrightarrow A_j$ there is a unique morphism $\theta : Q \longrightarrow P$ such that $\pi_j \theta = \delta_j$ for all j.

<u>Example</u> If $\mathcal{C} = \underline{\text{Set}}$ then the usual Cartesian product $\prod_{j\in J} A_j$ with the projections $\pi_i((a_j)_j) = a_i$ is a product, with $\theta(q) = (\delta_j(q))$.

<u>Def^N</u> With $\mathcal{C}, \{A_j\}_{j\in J}$ as above a <u>coproduct</u> is a pair $(C, \{\iota_j\}_{j\in J})$ where C is an object and $\iota_j : A_j \longrightarrow C$ are morphisms, such that for every other pair $(Q, \{\delta_j : A_j \longrightarrow Q\}_{j\in J})$ there is a unique morphism $\theta : C \longrightarrow Q$ with $\theta \iota_j = \delta_j$ for all j.

<u>Example</u> (1) If $\mathcal{C} = \underline{\text{Set}}$ then the disjoint union

$$\coprod_{j\in J} A_j = \bigcup_j \{j\} \times A_j$$

with functions $\iota_i : A_i \longrightarrow \coprod_j A_j$ given by $\iota_i(a) = (i, a)$ is a coproduct.

(2) If $\mathcal{C} = \underline{\text{Ab}}$ then $\bigoplus_{j\in J} A_j = \{ (a_j) \in \prod_j A_j \mid \text{only finitely many } a_j \text{ nonzero} \}$ is a coproduct.

assume all the J_α are disjoint

<u>Lemma</u> Let $\mathcal{C}$ be a category, $\{J_\alpha\}_{\alpha\in\Lambda}$ a set of sets, and for each $\alpha\in\Lambda$ suppose $\{A_j\}_{j\in J_\alpha}$ is a set of objects in $\mathcal{C}$. Assuming all involved products exist, there is a canonical isomorphism

$$\prod_{j\in J} A_j \cong \prod_{\alpha\in\Lambda}\prod_{j\in J_\alpha} A_j$$

where $J = \bigcup_\alpha J_\alpha$.

<u>Example</u> Let $(L,\leq)$ be a partially ordered set, viewed as a category.
Given $S\subseteq L$ the product P of the set of objects S would be an object (i.e. $P\in L$) with morphisms $P\longrightarrow s$ for all $s\in S$ (i.e. $P\leq s$ for all s) such that for any other family of morphisms $Q\to s$ (i.e. $Q\leq s$ $\forall s\in S$) there is a unique morphism $Q\longrightarrow P$ (i.e. $Q\leq P$). Since uniqueness is automatic, we conclude

- the <u>product</u> of S is precisely the infimum of S (if exists)

Similarly

- the <u>coproduct</u> of S is precisely the <u>supremum</u> of S (if exists)

<u>Def^n</u> A <u>lattice</u> is a partially ordered sets with binary products and coproducts.

<u>Ex 1</u> Prove that a category with binary products (resp. coproducts) has arbitrary finite products (resp. coproducts).

<u>Ex 2</u> Find a counterexample to the following statement: a category with finite products has an initial object.

Ex 3 Let Rng denote the category of commutative rings (with identity), and suppose we have ring morphisms

$$\begin{tikzcd} S \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & R_1 \\ R_2 & \end{tikzcd}$$

we say R_1, R_2 are S-algebras. Prove that the pushout of this diagram in Rng is the tensor product $R_1 \otimes_S R_2$ with morphisms

$$R_1 \longrightarrow R_1 \otimes_S R_2 \qquad r \longmapsto r \otimes 1$$
$$R_2 \longrightarrow R_1 \otimes_S R_2 \qquad r \longmapsto 1 \otimes r.$$

Example (Initial and terminal object) Often we write $\mathbb{1}$ for a terminal object, $\mathbb{0}$ for initial.

(1) In Set, $\mathbb{0} = \phi$ and $\mathbb{1} = \{\bullet\}$

(2) In Ab, $\mathbb{0} = \mathbb{1} = \{0\}$.

(3) In Rng, $\mathbb{0} = \mathbb{Z}$, $\mathbb{1} = \{0\}$ (i.e. $1=0$)

Example (Equaliser, Coequaliser) Consider $A \overset{f}{\underset{g}{\rightrightarrows}} B$

(1) In Set, Ab, Top, Rng the equaliser is the inclusion

$$\{a \in A \mid f(a) = g(a)\} \hookrightarrow A.$$

(2) In Set the coequaliser is $B/\sim$ where $\sim$ is the equiv. rel^N generated by pair $(f(a), g(a))$ for all $a \in A$.

(3) In $\underline{Ab}$ the coequaliser of f, g is $B \xrightarrow{\pi} B/\mathrm{Im}(f-g)$, π the quotient

<u>Proof</u> Clearly $\pi f(a) = \pi g(a)$ for all a, so $\pi f = \pi g$. If $\mathcal{J} : B \longrightarrow C$ is a morphism with $\mathcal{J} f = \mathcal{J} g$ then $\mathcal{J}(f-g) = 0$ so $\mathcal{J}$ factors uniquely via π. $\square$

<u>Functoriality of limits and colimits</u>

Let $\mathcal{C}$ be a category, G an oriented graph and $\alpha_1, \alpha_2 : PG \longrightarrow \mathcal{C}$ two diagrams, with a morphism $f : \alpha_1 \longrightarrow \alpha_2$. Suppose both α_1, α_2 have limits, written $\eta_1 : I(\lim \alpha_1) \longrightarrow \alpha_1$ and $\eta_2 : I(\lim \alpha_2) \longrightarrow \alpha_2$. Then the morphism $f \circ \eta_1$ in

$$\begin{tikzcd}
I(\lim \alpha_1) \arrow[r, "\eta_1"] \arrow[d, dashed, "I(\lim f)"'] & \alpha_1 \arrow[d, "f"] \\
I(\lim \alpha_2) \arrow[r, "\eta_2"'] & \alpha_2
\end{tikzcd} \qquad (5.1)$$

induces by the universal property of η_2 a <u>unique</u> morphism $\lim \alpha_1 \longrightarrow \lim \alpha_2$ (call it $\lim f$) such that (5.1) commutes.

<u>Proposition</u> The assignment $f \longmapsto \lim f$ is functorial. That is, if $g : \alpha_2 \longrightarrow \alpha_3$ is a morphism of diagrams then $\lim(g \circ f) = \lim(g) \circ \lim(f)$ (assuming $\lim \alpha_3$ also exists) and if $\alpha = \alpha_1 = \alpha_2$ and $f = \mathrm{id}_\alpha$ then $\lim(\mathrm{id}_\alpha) = \mathrm{id}_{\lim \alpha}$.

<u>Proof</u> We prove $\lim(g \circ f) = \lim(g) \circ \lim(f)$ and leave $\lim(\mathrm{id}_\alpha) = \mathrm{id}_{\lim \alpha}$ as an exercise. Consider

$$\begin{tikzcd}
I(\lim \alpha_1) \arrow[r, "\eta_1"] \arrow[d, "I(\lim f)"'] \arrow[dd, bend right=60, "I(\lim(gf))"'] & \alpha_1 \arrow[d, "f"] \\
I(\lim \alpha_2) \arrow[r, "\eta_2"'] \arrow[d, "I(\lim g)"'] & \alpha_2 \arrow[d, "g"] \\
I(\lim \alpha_3) \arrow[r, "\eta_3"'] & \alpha_3
\end{tikzcd}$$

By def^N $\lim(gf) : \lim \alpha_1 \longrightarrow \lim \alpha_3$ is unique s.t. $\eta_3 I(\lim(gf)) = gf\eta_1$.
But we have

$$\begin{aligned}
\eta_3 I(\lim g \circ \lim f) &= \eta_3 I(\lim g) I(\lim f) \\
&= g\eta_2 I(\lim f) \\
&= gf\eta_1
\end{aligned}$$

So we deduce $\lim(gf) = \lim g \circ \lim f$ as claimed. $\square$

<u>Corollary</u> If every diagram of shape G in $\mathcal{C}$ has a limit (and we choose one for each diagram) there is a functor

$$\lim : [PG, \mathcal{C}] \longrightarrow \mathcal{C}$$

- on <u>objects</u>, $\alpha \longmapsto \lim \alpha$
- on <u>morphisms</u>, $f : \alpha_1 \to \alpha_2 \longmapsto \lim f : \lim \alpha_1 \to \lim \alpha_2$.

Ex 4 Check that the bijection of Lecture 8, formula (8.3),

$$\mathrm{Hom}_{\mathscr{C}}(C, \lim \alpha) \cong \mathrm{Hom}_{[PG, \mathscr{C}]}(IC, \alpha) \qquad (7.1)$$

is also natural in α.

Ex 5 Prove that colimits (when they exist) are also functorial, in the sense of the above. Also, show that there is an analogue of (7.1): if α is a diagram with colimit $\varepsilon : \alpha \to \operatorname{colim} \alpha$ there is a bijection

$$\mathrm{Hom}_{\mathscr{C}}(\operatorname{colim} \alpha, C) \cong \mathrm{Hom}_{[PG, \mathscr{C}]}(\alpha, IC) \qquad (7.2)$$

natural in both α, and C.