Assignment 1 Grading

Grading Scheme 9 Qs, equally weighted, 2 pts per Q, total 18.

LaTeX comments Equations should end with full stops, e.g. let X be a scheme such that

$$H^i(X, \Omega_X) = 0 \quad i > 6.$$

L2, Ex 4 What I meant here is not necessarily surjective, and I'm glad many noticed this.

L2, Ex 5 What is up with all the people writing $X \xrightarrow{xy} Y$? This is a strange (and in my opinion, bad) naming style, as most people will read xy as $x \circ y$. If you insist, write e.g. f_{xy}. But everybody just uses Greek or Roman letters and so should you.

Yoneda uniqueness A few people got this, which was good. Let $F : \mathcal{C}^{op} \to \underline{Set}$ be a functor, $C \in ob(\mathcal{C})$ and

$$\Phi_{C,F} : Nat(h_C, F) \longrightarrow F(C), \quad \Phi_{C,F}(\alpha) = \alpha_C(id_C)$$

the usual Yoneda isomorphism. A natural (in C and F) family of bijections $\pi_{C,F} \circlearrowright Nat(h_C, F)$ will give, upon precomposition, a second natural family of bijections $\Phi \circ \pi$. So we want to know

(1) Can we find such π, s.t.

(2) $(\Phi \circ \pi)_{C,F}(\alpha) = \pi_{C,F}(\alpha)_C(id_C)$ is not $\alpha_C(id_C)$ for some α?

It's hard to work on F because it is generic. But suppose we had a natural iso $\eta^C : h_C \to h_C$ for each $C \in ob(\mathscr{C})$ such that for every $f : C \to C'$ the diagram

$$\begin{tikzcd} h_C \arrow[r, "\eta^C"] \arrow[d, "h_f"'] & h_C \arrow[d, "h_f"] \\ h_{C'} \arrow[r, "\eta^{C'}"'] & h_{C'} \end{tikzcd} \qquad (2.1)$$

commutes. Then the family of bijections

$$\pi_{C,F} : \mathrm{Nat}(h_C, F) \longrightarrow \mathrm{Nat}(h_C, F), \quad \pi_{C,F}(\alpha) = \alpha \circ \eta^C$$

is certainly natural in C, F and $\pi_{C,F}(\alpha)_C(\mathrm{id}_C) = (\alpha \circ \eta^C)_C(\mathrm{id}_C) = \alpha(\eta^C_C(\mathrm{id}_C))$ so as long as $\eta^C_C(\mathrm{id}_C) \neq \mathrm{id}_C$ we have a good shot at (2) as well.

So let us understand what the data of $\{\eta^C \in \mathrm{Nat}(h_C, h_C)\}_{C \in ob(\mathscr{C})}$ plus naturality (2.1) means. We know $\mathrm{Nat}(h_C, h_C) \cong h_C(C) \cong \mathrm{Hom}_{\mathscr{C}}(C, C)$ sending η^C to $\psi_C := \eta^C_C(\mathrm{id}_C)$, and η^C a natural iso $\iff$ ψ_C is an iso. Combining this with (2.1), we have

<u>Lemma</u> There is a bijection between families of natural isos

$$\{\eta^C : h_C \to h_C\}_{C \in ob(\mathscr{C})}, \text{ natural in } C$$

and families $\{\psi_C : C \to C\}_{C \in ob(\mathscr{C})}$ of isos, natural in C, by which we mean that for all $f : C \to C'$ in $\mathscr{C}$, the following commutes:

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\psi_C"'] & C' \arrow[d, "\psi_{C'}"] \\ C \arrow[r, "f"'] & C' \end{tikzcd}.$$

Equivalently, these are the <u>automorphisms</u> of $id_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$.

(The monoid $Nat(id_{\mathcal{C}}, id_{\mathcal{C}})$ is called the <u>center</u> of $\mathcal{C}$. It is a commutative monoid $Z\mathcal{C}$ and the units we denote $Z(\mathcal{C})^{\times}$).

<u>Def^n</u> Given a family of natural isos $\{\psi_c : c \to c\}_{c \in ob(\mathcal{C})}$ as above, define

$$\Phi^{\psi}_{C,F} : Nat(h_C, F) \longrightarrow F(C), \quad \Phi^{\psi}_{C,F}(\alpha) := \alpha_C(\psi_C).$$

This is a natural bijection, by constuction, and

$$\begin{aligned} \forall C \forall F \ \Phi^{\psi}_{C,F} = \Phi_{C,F} \quad &\Longrightarrow \forall C \ \Phi^{\psi}_{C,h_C} = \Phi_{C,h_C} \\ &\Longrightarrow \forall C \ \Phi^{\psi}_{C,h_C}(id_{h_C}) = \Phi_{C,h_C}(id_{h_C}) \\ &\Longrightarrow \forall C \ \psi_C = id_C \end{aligned}$$

<u>Upshot</u> If there is a <u>nontrivial</u> $\{\psi_C\}_C \in Z(\mathcal{C})^{\times}$ then $\Phi^{\psi} \neq \Phi$.

<u>Example</u> Let R be a ring, $\mathcal{C} = R\text{-Mod}$, and $u \in R$ an element which is <u>central</u> and a <u>unit</u>. Then for an R-Module M, define

$$\psi_M : M \longrightarrow M, \quad \psi_M(x) = u \cdot x.$$

This is R-linear (u is central), and an iso (u is a unit in R), and it is natural (clearly). So as long as $u \neq 1$, $\{\psi_M\}_M$ is a nontrivial unit in the center of R-Mod, giving a "essentially different" nat. iso Φ^{ψ}.

<u>Example</u> G an abelian group, $R = \mathbb{Z}G$, $u \in G$, $u \neq e$.