Background 2: Category theory

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We assume without comment set theoretic foundations which include sets and classes [FCT]. There is a class of all sets, and of all abelian groups, etc. Category theory is the theory of universal properties, the only sane API for many mathematical constructions.

Def^n A category $\mathscr{C}$ consists of

(1) a class $ob(\mathscr{C})$ whose elements are called objects

(2) for each pair $a, b \in ob(\mathscr{C})$ a set $\mathscr{C}(a,b)$ whose elements are called morphisms from a to b. We write $f : a \longrightarrow b$ to mean $f \in \mathscr{C}(a,b)$.

(3) for each triple $a, b, c \in ob(\mathscr{C})$ a function

$$m_{abc} : \mathscr{C}(b,c) \times \mathscr{C}(a,b) \longrightarrow \mathscr{C}(a,c)$$

called composition. We write $f \circ g$ for $m_{abc}(f, g)$

(4) for each $a \in ob(\mathscr{C})$ a morphism $1_a : a \longrightarrow a$, called the identity

subject to the following axioms:

(1) (associativity) for any objects $a, b, c, d \in ob(\mathscr{C})$ the diagram

$$\begin{tikzcd} \mathscr{C}(c,d) \times \mathscr{C}(b,c) \times \mathscr{C}(a,b) \arrow[r, "m \times 1"] \arrow[d, "1 \times m"'] & \mathscr{C}(b,d) \times \mathscr{C}(a,b) \arrow[d, "m"] \\ \mathscr{C}(c,d) \times \mathscr{C}(a,b) \arrow[r, "m"'] & \mathscr{C}(a,d) \end{tikzcd} \qquad (1.1)$$

commutes. That is, for all $f : a \longrightarrow b$, $g : b \longrightarrow c$, $h : c \longrightarrow d$

$$h \circ (g \circ f) = (h \circ g) \circ f \qquad (1.2)$$

(2) (<u>units</u>) for any objects a, b the diagram ($\mathbb{1}$ is the singleton set $\{*\}$)

$$\begin{tikzcd}
\mathbb{1} \times \mathcal{C}(a,b) \arrow[d, "\iota_b \times 1"'] & \mathcal{C}(a,b) \arrow[l] \arrow[r] \arrow[d, "1"] & \mathcal{C}(a,b) \times \mathbb{1} \arrow[d, "1 \times \iota_a"] \\
\mathcal{C}(b,b) \times \mathcal{C}(a,b) \arrow[r, "m"'] & \mathcal{C}(a,b) & \mathcal{C}(a,b) \times \mathcal{C}(a,a) \arrow[l, "m"]
\end{tikzcd} \tag{2.1}$$

commutes where $\iota_a(*) = 1_a$, $\iota_b(*) = 1_b$. That is, for all $f : a \longrightarrow b$

$$1_b \circ f = f = f \circ 1_a \tag{2.2}$$

<u>Remark</u> Sometimes $\mathcal{C}(a,b)$ is denoted $\mathrm{Hom}_{\mathcal{C}}(a,b)$.

<u>Example B2-1</u> (1) The category <u>Set</u> where the objects are sets, <u>Set</u>(a,b) is the set of functions $a \longrightarrow b$ and composition m is function composition.

(2) The category <u>Ab</u> where the objects are abelian groups, <u>Ab</u>(a,b) is the set of homomorphisms of groups, m is function composition. Similarly <u>Grp</u> denotes the category of all groups.

(3) Given a ring R we obtain a category R<u>Mod</u> of left R-modules (and R-linear maps) and a category <u>Mod</u>R of right R-modules.

(4) Given a field k we write k<u>Vect</u> for k<u>Mod</u>, the category of k-vector spaces and linear maps.

(5) The category with one object $*$ and one morphism $1_* : * \longrightarrow *$.

(6) The category of normed vector spaces over $\mathbb{F}$ and bounded linear maps.

(7) The category of Hilbert spaces and unitary maps.

(8) The category <u>Top</u> of topological spaces and continuous maps.

<u>Def^n</u> Let $\mathcal{A}, \mathcal{B}$ be categories. A <u>functor</u> $F : \mathcal{A} \to \mathcal{B}$ is the data of

(1) a function $F : ob(\mathcal{A}) \longrightarrow ob(\mathcal{B})$

(2) for each pair $a, b \in ob(\mathcal{A})$ a function $F_{ab} : \mathcal{A}(a,b) \longrightarrow \mathcal{B}(Fa, Fb)$, where we often write $F(f)$ for $F_{ab}(f)$.

subject to the following axioms:

(1) for any objects $a, b, c \in ob(\mathcal{A})$ the diagram

$$\begin{tikzcd}
\mathcal{A}(b,c) \times \mathcal{A}(a,b) \arrow[r, "m"] \arrow[d, "F_{bc} \times F_{ab}"'] & \mathcal{A}(a,c) \arrow[d, "F_{ac}"] \\
\mathcal{B}(Fb, Fc) \times \mathcal{B}(Fa, Fb) \arrow[r, "m"'] & \mathcal{B}(Fa, Fc)
\end{tikzcd}$$

commutes. That is, for all $f : a \to b$, $g : b \to c$ we have

$$F(g \circ f) = F(g) \circ F(f).$$

(2) for every object $a \in ob(\mathcal{A})$, $F(1_a) = 1_{F(a)}$.

<u>Example B2-2</u> (1) There are <u>forgetful functors</u> $F : \underline{Ab} \longrightarrow \underline{Set}$, $F : \underline{Grp} \longrightarrow \underline{Set}$ etc which send an abelian group (group, etc) to its underlying set, and functions to themselves.

(2) The <u>identity functor</u> $1_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}$ which is the identity on all objects and morphisms.

(3) A <u>subcategory</u> of $\mathcal{A}$ is a functor $F : \mathcal{B} \longrightarrow \mathcal{A}$ which is the inclusion of a subclass $ob(\mathcal{B}) \subseteq ob(\mathcal{A})$ and $\mathcal{B}(a,b) \subseteq \mathcal{A}(a,b)$ for all a, b. (note that $1_a \in \mathcal{B}(a,a)$ if $a \in ob(\mathcal{B})$).

Lemma B2-1 If $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{C}$ are functors then $G \circ F : \mathcal{A} \to \mathcal{C}$ is a functor defined by $a \mapsto G(F(a))$ on objects and $f \mapsto G(F(f))$ on morphisms.

Proof We have $(G \circ F)(1_a) = G(F(1_a)) = G(1_{Fa}) = 1_{G(Fa)} = 1_{(G \circ F)(a)}$ and for composable arrows $f : a \to b$, $g : b \to c$,

$$\begin{aligned} (G \circ F)(g \circ f) &= G(F(g \circ f)) \\ &= G(F(g) \circ F(f)) \\ &= G(F(g)) \circ G(F(f)) = (G \circ F)(g) \circ (G \circ F)(f). \quad \square \end{aligned}$$

Remark Functor composition is associative and $1_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ acts as an identity.

Given a function $f : S \to \mathbb{Z}$ we call $\{ s \in S \mid f(s) \neq 0 \}$ the <u>support</u> of f and say f has <u>finite support</u> if this set is finite. With S any set we consider

$$V(S) = \{ f : S \to \mathbb{Z} \mid f \text{ has finite support} \}$$

This is an abelian group with $(f+g)(s) = f(s) + g(s)$, and it is generated as an abelian group by the functions $\delta_s : S \to \mathbb{Z}$ defined for each $s \in S$ by $\delta_s(t) = 0$ if $s \neq t$ and $\delta_s(s) = 1$. Clearly $f = \sum_{s \in S} f(s) \delta_s$ as elements of $V(S)$. Abusing notation we may write s for δ_s and view $f \in V(S)$ as <u>formal linear combinations</u> of elements of S. We denote by $\iota : S \to V(S)$ the map sending s to $\iota(s) = \delta_s$.

Lemma B2-2 Given any abelian group B, set S and function $h : S \to B$ there is a <u>unique</u> morphism of abelian groups $\tilde{h} : V(S) \to B$ making the following diagram commute

$$\begin{tikzcd} V(S) \arrow[r, dashed, "\tilde{h}"] & B \\ S \arrow[u, "\iota"] \arrow[ur, bend right, "h"'] & \end{tikzcd} \qquad (4.1)$$

<u>Proof</u> We define $\tilde{h}(f) = \sum_{s \in S} f(s)h(s)$. This sum is finite since f has finite support. It is easy to check that $\tilde{h}$ is a morphism of abelian groups, and $\tilde{h} \circ \iota = h$. To see uniqueness suppose $\tilde{h}'$ were another morphism of abelian groups with $\tilde{h}' \circ \iota = \tilde{h} \circ \iota$. Then we have

$$\begin{aligned} \tilde{h}'(f) &= \tilde{h}'\left(\textstyle\sum_{s \in S} f(s)\delta_s\right) \\ &= \textstyle\sum_{s \in S} \tilde{h}'(f(s)\delta_s) \\ &= \textstyle\sum_{s \in S} f(s)\tilde{h}'(\delta_s) \\ &= \textstyle\sum_{s \in S} f(s)h(s) = \tilde{h}(f) \end{aligned} \tag{5.1}$$

so $\tilde{h}$ is unique as claimed. □

The abelian group $V(S)$ (or more correctly the pair $(V(S), \iota)$) is called the <u>free abelian group</u> generated by S. The word "free" summarises the <u>universal property</u> expressed in (4.1). Replacing $\mathbb{Z}$ by any ring R the same method constructs the <u>free R-module</u> on a set S (and if R is a field we call this the <u>free vector space</u>).

<u>Exercise B2-1</u> Prove that if k is a field and S is a set, then $\{\delta_s\}_{s \in S}$ is a basis of the free vector space $V(S)$.

<u>Lemma B2-3</u> Given a function $f : S \longrightarrow S'$ between sets, let $V(f)$ denote the unique morphism of abelian groups making

$$\begin{tikzcd} V(S) \arrow[r, dashed, "V(f)"] & V(S') \\ S \arrow[u, "\iota"] \arrow[r, "f"'] & S' \arrow[u, "\iota"'] \end{tikzcd} \tag{5.2}$$

commute. Then $V : \underline{\mathrm{Set}} \longrightarrow \underline{\mathrm{Ab}}$ is a functor, called the <u>free functor</u>.

<u>Proof</u> If $f = 1_S$ then $1_{V(S)}$ makes (5.2) commute and thus $V(1_S) = 1_{V(S)}$ by uniqueness. If $g : S' \longrightarrow S''$ then commutativity of

$$\begin{tikzcd}
V(S) \arrow[r, "V(f)"] & V(S') \arrow[r, "V(g)"] & V(S'') \\
S \arrow[u, "\iota"] \arrow[r, "f"'] & S' \arrow[u, "\iota"'] \arrow[r, "g"'] & S'' \arrow[u, "\iota"']
\end{tikzcd} \qquad (6.1)$$

means that $\iota \circ (g \circ f) = (V(g) \circ V(f)) \circ \iota$ and so by uniqueness $V(g) \circ V(f) = V(g \circ f)$. $\square$

Given abelian groups A, B, T a function $\varphi : A \times B \longrightarrow T$ is <u>bilinear</u> if $\varphi(a + a', b) = \varphi(a, b) + \varphi(a', b)$ and $\varphi(a, b + b') = \varphi(a, b) + \varphi(a, b')$ for all $a, a' \in A$ and $b, b' \in B$.

<u>Lemma B2-4</u> Let A, B be abelian groups. There exists a pair (T, φ) consisting of an abelian group T and bilinear map $\varphi : A \times B \longrightarrow T$ which is universal, in the following sense: if (Q, ψ) is any such pair, consisting of an abelian group Q and $\psi : A \times B \longrightarrow Q$ bilinear, there is a unique morphism of abelian groups $\tilde{\psi}$ making

$$\begin{tikzcd}
A \times B \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & T \arrow[d, dashed, "\tilde{\psi}"] \\
& Q
\end{tikzcd}$$

commute.

<u>Proof</u> Let $V(A \times B)$ denote the free abelian group on the set $A \times B$, and $\iota : A \times B \rightarrow V(A \times B)$ the canonical map. We will define a subgroup $N \subseteq V(A \times B)$ and prove that the composite $A \times B \xrightarrow{\iota} V(A \times B) \longrightarrow V(A \times B)/N =: T$ is the required universal bilinear map φ.

Let N be the subgroup generated by the elements

$$\iota(a+a',b)-\iota(a,b)-\iota(a',b) \qquad a,a'\in A \;\; b\in B \qquad (7.1)$$
$$\iota(a,b+b')-\iota(a,b)-\iota(a,b') \qquad a\in A \;\; b,b'\in B$$

Set $T := V(A\times B)/N$ as above and let φ be $\pi\circ\iota$ where $\pi : V(A\times B)\longrightarrow T$ is the quotient. Then

$$\begin{aligned}\varphi(a+a',b) &= \pi(\iota(a+a',b))\\ &= \pi(\iota(a,b)+\iota(a',b)) \qquad (7.2)\\ &= \pi(\iota(a,b))+\pi(\iota(a',b))\\ &= \varphi(a,b)+\varphi(a',b)\end{aligned}$$

and similarly $\varphi(a,b+b')=\varphi(a,b)+\varphi(a,b')$ so φ is bilinear. To prove the universal property suppose $\psi : A\times B\longrightarrow Q$ is bilinear. Then by the universal property of the free abelian group we have a unique morphism of groups $\psi' : V(A\times B)\longrightarrow Q$ making

$$\begin{tikzcd} V(A\times B) \arrow[r, "\psi'"] & Q \\ A\times B \arrow[u, "\iota"] \arrow[ur, bend right, "\psi"'] & \end{tikzcd}$$

commute, and

$$\begin{aligned}\psi'(\iota(a+a',b)-\iota(a,b)-\iota(a',b)) &= \psi'(\iota(a+a',b))-\psi'(\iota(a,b))-\psi'(\iota(a',b))\\ &= \psi(a+a',b)-\psi(a,b)-\psi'(a',b)\\ &= 0\end{aligned}$$

similarly $\psi'(\iota(a,b+b')-\iota(a,b)-\iota(a,b'))=0$ so $\psi'(N)=0$ and hence there is a unique morphism of abelian groups $\tilde{\psi} : T\longrightarrow Q$ with $\tilde{\psi}\circ\pi=\psi'$. Thus

$$\tilde{\psi} \circ \varphi = \tilde{\psi} \circ \pi \circ \iota = \psi' \circ \iota = \psi$$

To see that $\tilde{\psi}$ is unique with this property suppose κ were a morphism of abelian groups with $\kappa \circ \varphi = \psi$ so $(\kappa \circ \pi) \circ \iota = \psi$. By the universal property of the free vector space $\kappa \circ \pi = \psi'$ and thus $\kappa \circ \pi = \psi' = \tilde{\psi} \circ \pi$. Since π is surjective $\kappa = \tilde{\psi}$. $\square$

<u>Def^n</u> A morphism $f : a \to b$ in a category $\mathcal{A}$ is an <u>isomorphism</u> if there exists $g : b \to a$ with $f \circ g = 1_b$, $g \circ f = 1_a$.

Any two pairs $(T, \varphi), (T', \varphi')$ with the same universal property are (uniquely) isomorphic:

<u>Lemma B2-5</u> In the above situation, suppose (T', φ') also possesses the universal property. By the respective universal properties there are unique morphisms of abelian groups δ, ϵ as in the diagram below

$$\begin{tikzcd}
 & T \arrow[d, bend right, "\delta"'] \\
A \times B \arrow[ur, "\varphi"] \arrow[dr, "\varphi'"'] & \\
 & T' \arrow[u, bend right, "\epsilon"']
\end{tikzcd}$$

satisfying $\delta\varphi = \varphi'$ and $\epsilon\varphi' = \varphi$, and these morphisms are mutually inverse, $\epsilon\delta = 1_T$, $\delta\epsilon = 1_{T'}$.

<u>Proof</u> The stated morphisms δ, ϵ exist by hypothesis. Note that $\delta \circ \epsilon : T' \to T'$ satisfies $\delta \circ \epsilon \circ \varphi' = \delta \circ \varphi = \varphi'$ and so by the universal property of (T', φ') we have $\delta \circ \epsilon = 1_{T'}$. Similarly $\epsilon \circ \delta = 1_T$. $\square$

<u>Def^n</u> Given abelian groups A, B we define $A \otimes B = V(A \times B)/N$ as above and call this the <u>tensor product</u> of A, B. The equivalence class $[\iota(a,b)] \in A \otimes B$ is denoted $a \otimes b$.

Exercise B2-2 (i) Prove that $a\otimes 0 = 0\otimes b = 0$ in $A\otimes B$ for all $a\in A$, $b\in B$

(ii) Prove that $\{a\otimes b\}_{a\in A, b\in B}$ generates $A\otimes B$ as an abelian group but give an example of A, B for which $A\times B \longrightarrow A\otimes B$, $(a,b)\longmapsto a\otimes b$ is not surjective (not all elements of $A\otimes B$ are "pure" tensors)

(iii) Given morphisms of abelian groups $\alpha : A \longrightarrow A'$, $\beta : B\longrightarrow B'$ prove that there is a well-defined morphism of abelian groups $\alpha\otimes\beta : A\otimes B\longrightarrow A'\otimes B'$ with $(\alpha\otimes\beta)(a\otimes b) = \alpha(a)\otimes\beta(b)$. Prove that $1_A\otimes 1_B = 1_{A\otimes B}$ and if $\alpha' : A'\longrightarrow A''$, $\beta' : B'\longrightarrow B''$ are morphisms of abelian groups then

$$(\alpha'\otimes\beta')\circ(\alpha\otimes\beta) = (\alpha'\circ\alpha)\otimes(\beta'\circ\beta).$$

References

[FCT] Foundations for Category Theory, therisingsea.org.

[B] Borceux, "Handbook of categorical algebra".