Lecture 1: What is Real?

1/3/2021

Real things have symmetries. This is a mundane but deep principle: the contrapositive is that if a phenomena disappears when you change your coordinate system, or put differently if the phenomena disappears when the measurements are made by a different but "equivalent" observer, then the phenomena is probably not real; it may just be an artifact of your point of view. Indeed this may be a kind of tautology.

But when do two observers count as "equivalent"? What does it mean for the phenomena to "disappear"? In short, the modern answer to the first question is that two observers count as equivalent if their reference frames are related by a <u>Lie group associated to the phenomena</u> itself, and a system of measurements of multiple equivalent observers counts as a "real phenomena" if the measurements <u>transform as a representation</u> of that Lie group. Thus Lie groups and their representations lie at the heart of physics [1,2].

<u>Example</u> The first postulate of special relativity [3] is

> "The laws by which the states of physical systems undergo change are not affected, whether these changes of state be referred to the one or the other of two systems of co-ordinates in uniform translatory motion."

The relevant Lie group is the <u>Poincare group</u> which is the group of continuous maps $\mathbb{R}^4 \longrightarrow \mathbb{R}^4$ generated by translations and the Lorentz transformations $O(3,1) \subseteq GL_4(\mathbb{R})$, which are those linear transformations $F: \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ preserving the Minkowski inner product

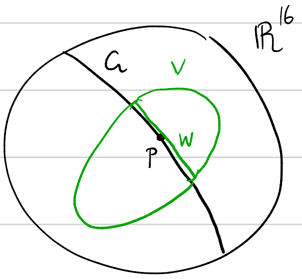
$$\langle \underline{u}, \underline{v} \rangle = -u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4. \tag{1.1}$$

All of this may be derived (as Einstein did) from a short list of postulates which together define when two observers are <u>equivalent</u> from the point of view of special relativity. These ideas are reviewed in MAST30026 Lectures 1–5.

To say that G is a <u>Lie group</u> is simply to say that the product $m : G\times G \longrightarrow G$ and inversion $i : G \longrightarrow G$ defined by $i(g) = g^{-1}$ have continuous derivatives of all orders, that is, they are <u>smooth functions</u>. To give a means of determining which functions defined on G, $G\times G$ are smooth is to define the structure of a <u>smooth manifold</u> on G. We will do this soon in lectures, but for $G = O(3,1) \subseteq M_4(\mathbb{R}) \cong \mathbb{R}^{16}$ with G given its manifold structure as a regular submanifold of $\mathbb{R}^{16}$, we can give an <u>ad hoc definition</u>

We identify $M_4(\mathbb{R}) = \mathbb{R}^{16}$ as a topological space and give G the subspace topology (see Lecture 6 of MAST30026 for the necessary background) with inclusion $j : G \longrightarrow \mathbb{R}^{16}$.

<u>Definition</u> For $U \subseteq G$ open a function $f : U \to \mathbb{R}$ is called <u>smooth</u> if for every $p \in U$ there is an open neighborhood $W \subseteq U$ of p together with $V \subseteq \mathbb{R}^{16}$ open and $g : V \to \mathbb{R}$ smooth (in the usual sense of multivariate calculus) such that $V \cap G = W$ and $g|_W = f|_W$.



$$\begin{tikzcd} V \arrow[rr, bend left, "g"] & & \\ & W \arrow[ul] \arrow[dl] & \mathbb{R} \\ U \arrow[urr, bend right, "f"'] & & \end{tikzcd}$$

Replacing $G \subseteq \mathbb{R}^{16}$ by $G\times G \subseteq \mathbb{R}^{16}\times\mathbb{R}^{16} = \mathbb{R}^{32}$ supplies a definition of a smooth real-valued function on the product $G\times G$ (again, we will return to this more systematically later).

<u>Definition</u> Let U be either an open subset of G or $G\times G$ and $f : U \to G$ a function. Then f is <u>smooth</u> if every component of $j \circ f : U \longrightarrow \mathbb{R}^{16}$ is smooth, that is, if for $1 \le k \le 16$

$$U \xrightarrow{\ f\ } G \xrightarrow{\ j\ } \mathbb{R}^{16} \xrightarrow{\ \pi_k\ } \mathbb{R}$$

is smooth where π_k denotes the kth projection.

Lemma L1-1 The multiplication $m: G \times G \to G$ and invension $i: G \to G$ are smooth.

For the proof we need Cramer's rule.

Lemma L1-2 (Cramer's rule) Consider an $n \times n$ invertible matrix A over a field k, a vector $b \in \mathbb{R}^n$ and an equation $Ax = b$. The unique solution is

$$x = \frac{1}{\det(A)}\big(\det(A_1), \ldots, \det(A_n)\big)^T \qquad (3.1)$$

where A_i is A with the ith column replaced by b, i.e.

$$A_i = \big(A_{\bullet 1} \cdots A_{\bullet i-1}\ b\ A_{\bullet i+1} \cdots A_{\bullet n}\big). \qquad (3.2)$$

Proof Consider the linear map $T: \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$T(y) = \frac{1}{\det(A)}\big(\det(A_1(y)), \ldots, \det(A_n(y))\big)^T$$

where $A_i(y)$ is A with y substituted in the ith column. This is linear in y by basic properties of determinants: $\det(A_i(y+y')) = \det(A_i(y)) + \det(A_i(y'))$ and $\det(A_i(\lambda y)) = \det(\lambda A_i(y)) = \lambda \det(A_i(y))$. Now observe that

$$T(A_{\bullet i}) = \frac{1}{\det(A)}\big(0, \ldots, \overset{i}{\det(A)}, \ldots, 0\big)$$

$$= (0, \ldots, \overset{i}{1}, \ldots, 0).$$

Hence T is surjective, hence injective (dimension formula), that is T is an isomorphism. Let $T_A: \mathbb{R}^n \to \mathbb{R}^n$ be $T_A(x) = Ax$. Then we have $T T_A(e_i) = T(A_{\bullet i}) = e_i$ for all i so we may conclude $T T_A = 1$, that is, $T = T_A^{-1}$. Then the unique sol^N to (3.1) is $x = A^{-1}b = T(b)$. $\square$

Proof of Lemma L1-1 The multiplication map fits into a commutative diagram

$$\begin{tikzcd}
\mathbb{R}^{32} \arrow[drr] & & \\
\mathbb{R}^{16} \times \mathbb{R}^{16} \arrow[u, "\cong"] \arrow[rr, "M"] & & \mathbb{R}^{16} \\
G \times G \arrow[u, "j \times j"] \arrow[rr, "m"'] & & G \arrow[u, "j"']
\end{tikzcd}$$

where the components of M send two matrices, viewed as column vectors, to some entry of the product, which is a polynomial function hence smooth.
The inversion $i : G \times G \longrightarrow G$ fits into a commutative diagram

$$\begin{tikzcd}
GL_4(\mathbb{R}) \arrow[r, "I"] & \mathbb{R}^{16} \\
G \arrow[u] \arrow[r, "i"'] & G \arrow[u, "j"']
\end{tikzcd}$$

where $GL_4(\mathbb{R}) \subseteq \mathbb{R}^{16}$ is an open set (it is the complement of $\det^{-1}(0)$, a closed set) and I is the function sending an invertible matrix to its inverse. It therefore suffices to show I is smooth, and for this it suffices to show for $1 \leq j \leq 4$ that the function $I_j : GL_4(\mathbb{R}) \longrightarrow \mathbb{R}^4$ sending A to the jth column of A^{-1} is smooth. But then $I_j(A)$ is the unique solution of $A I_j(A) = e_j$ so by Cramer's rule

$$I_j(A) = \frac{1}{\det(A)} \big(\det(A_1), \dots, \det(A_n) \big)^T$$

with A_j defined appropriately. The entries in $I_j(A)$ are rational, hence smooth, functions. $\square$

We have now shown that the operations on $O(3,1)$ are smooth. Technically to show $O(3,1)$ is a Lie group we also have to check that $O(3,1)$ is a regular submanifold of $\mathbb{R}^{16}$ (i.e. that it looks locally like $\mathbb{R}^k \subseteq \mathbb{R}^{16}$ for some k) but we defer this to later.

For now let us return to the question "what is real?" and complete the first telling of the story that will serve as a motivating thread for the class. This brings us to Wigner's theorem, following Weinberg [2, §1.2].

Observers As discussed above, the Lorentz group $O(3,1)$ is the group of all linear transformations $T: \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ such that $\langle Tx, Ty \rangle = \langle x, y \rangle$ for all $x, y \in \mathbb{R}^4$ where $\langle , \rangle$ is (1.1).

Def^n The Poincaré group is the group of functions $T(\Lambda, a): \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ defined by $T(\Lambda, a)(x) = \Lambda x + a$ where $\Lambda \in O(3,1)$ and $a \in \mathbb{R}^4$. The group operation is composition.

For our purposes an observer is a person or device which makes entries in a private copy of $\mathbb{R}^4$ detailing events (t, x, y, z) which occur around them. Observers are related by the following rules (as a simple first approximation)

Suppose that an observer A records a series of events corresponding to the constant linear motion of another observer B. Then A can use the following procedure to predict how B will measure any event that A itself measures:

- Determine from B's motion an appropriate element $T(\Lambda, a)$ of the Poincaré group
- Given an event $x \in \mathbb{R}^4$ as measured by A, then A predicts that B will measure $T(\Lambda, a)(x)$

Special Relativity says that A's predictions of B's measurements will be correct.

We will prove later that the Poincaré group is a Lie group. So the above rules give our first example of "equivalent observers" being related by a Lie group (in fact we may, once a "standard" observer is chosen, identify <u>observers</u> with <u>Lie group elements</u>) and their measurements being related by an action of this Lie group, in this case the action ($G =$ Poincaré group)

$$\begin{aligned} G \times \mathbb{R}^4 &\xrightarrow{\varphi} \mathbb{R}^4 \\ (A, x) &\longmapsto Ax \end{aligned} \tag{6.1}$$

From a physical point of view it seems natural to not only require φ to be a <u>continuous</u> function of its inputs (small variations in the observer B's relative position or motion lead to small variations in their predicted measurements) but also <u>smooth</u>. Arguably if this were not the case then the phenomena being measured could not be measured by macroscopic observers at all, since every observer is itself fluctuating across time (see MAST30026 L1 p. 9 for more discussion).

If G is a Lie group then a <u>representation</u> of G is a vector space V together with a smooth map $G \times V \longrightarrow V$ satisfying the obvious axioms. Right now it is not clear what we mean by "smooth" but this will be defined carefully later. The general principle of relativistic quantum mechanics is that associated to each particle is a representation of the Poincaré group in this sense on the Hilbert space of particle states. Thus one starting point for an introduction to Quantum Field Theory (QFT) is the study of the representation theory of this Lie group [2]. As we will see this study is much assisted by the use of <u>Lie algebras</u>, which exhibit the "infinitesimal generators" of Lie groups.

<u>Next lecture</u> we will look at the Lie group $SO(2)$ acting on the Hilbert space $L^2(S^1)$.

References

[1] H. B. Callen "Thermodynamics and an introduction to thermostatistics" 2nd edition, John Wiley & Sons, 1985.

[2] S. Weinberg "The Quantum Theory of Fields: Volume 1" Cambridge University Press 1995.

[3] A. Einstein "Zur Elektrodynamik bewegter Körper" Annalen der Physik 17, 891–921, 1905.