Lecture 2 - Wigner's Theorem

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In the first lecture we explained how observers equivalent to a fixed observer can be viewed as elements of a Lie group and how the "conversion" between measurements of different observers takes the form of an action of that Lie group on the space of possible measurements. We gave the example of the Poincaré group G and measurements $x \in \mathbb{R}^4$ of classical events, where the action $G \times \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ was just matrix multiplication. This is hardly deep, but when we replace the space of measurements $\mathbb{R}^4$ of <u>classical</u> events with the Hilbert space $\mathcal{H}$ of a <u>quantum</u> system a theory of great beauty emerges [7]. As this will be our pathway into the mathematical theory of Lie groups, we will in this lecture give a careful treatment of how we come to believe the class of <u>unitary</u> transformations $\mathcal{H} \to \mathcal{H}$ are the appropriate transformations to represent elements $g \in G$.

We recall briefly some of the axiomatic framework of quantum mechanics [7, §2.1] Associated to a quantum system is a Hilbert space $\mathcal{H}$ of states. A <u>physical state</u> of the system represented by $\mathcal{H}$ is an equivalence class of unit vectors, where two unit vectors φ, ψ are equivalent if there exists $\xi \in U(1)$ (recall this denotes the set of complex numbers z with $|z| = 1$) with $\varphi = \xi \psi$.

<u>Def^n</u> The set of physical states is denoted $S_{\mathcal{H}}$. Elements of $S_{\mathcal{H}}$ are called <u>rays</u> and we denote them by letters like $\mathcal{R}, \mathcal{R}', \mathcal{S}, \mathcal{S}'$.

<u>Def^n</u> A Hilbert space is <u>separable</u> if it contains a countable orthonormal basis, that is, a countable orthonormal set spanning a dense subspace, see [MHS, Theorem L21-10]. Sometimes this is called a <u>complete set</u>.

Most of the typical Hilbert spaces encountered in quantum mechanics are separable (or so I'm told) and I'll restrict to this case so that MHS contains the necessary background to prove Wigner's theorem. The general case is not much harder. So in what follows $\mathcal{H}$ denotes a separable Hilbert space. We write $\langle \psi, \varphi \rangle$ for the pairing and $\bar{\lambda}$ for the conjugate and adopt the physics convention that $\langle \psi, \varphi \rangle$ is linear in φ.

If a system is in a state represented by a ray $\mathcal{R}$, and an experiment is done to test whether it is one of the different states represented by the rays $\mathcal{R}_1, \mathcal{R}_2, \ldots$ which are mutually orthogonal and complete in the sense that we can find an orthonormal dense basis $\{\psi_k\}_{k=1}^{\infty}$ with $\psi_k \in \mathcal{R}_k$ for all $k \geq 1$ (for example if the $\mathcal{R}_k$ represent definite values of one or more observables) then the probability of finding it in the state represented by $\mathcal{R}_n$ is (choose any unit vector $\psi \in \mathcal{R}$)

$$P(\mathcal{R} \longrightarrow \mathcal{R}_n) = |\langle \psi, \psi_n \rangle|^2. \qquad (1.5.1)$$

To quote Weinberg [7] a symmetry transformation is a change in our point of view that does not change the results of possible experiments. If an observer O sees a system in a state represented by a ray $\mathcal{R}$ or $\mathcal{R}_1$ or $\mathcal{R}_2, \ldots$ then an equivalent observer O' who looks at the <u>same</u> system will observe it in a different state $\mathcal{R}', \mathcal{R}'_1, \mathcal{R}'_2, \ldots$ but the two observers must find the same probabilities

$$P(\mathcal{R} \longrightarrow \mathcal{R}_n) = P(\mathcal{R}' \longrightarrow \mathcal{R}'_n). \qquad (1.5.2)$$

The relation between observers is thus a function $Q : S\mathcal{H} \longrightarrow S\mathcal{H}$ which preserves the pairing (1.5.1) between rays. Wigner's theorem shows that every such symmetry arises from a unitary or antiunitary transformation $\mathcal{H} \longrightarrow \mathcal{H}$.

<u>Def^n</u> $\mathbb{C}\mathbb{P}\mathcal{H}$ is the set of equivalence classes for the following equivalence relation on the set $\mathcal{H} \setminus \{0\}$ of nonzero vectors in $\mathcal{H}$

$$\varphi \sim \psi \iff \exists \lambda \in \mathbb{C} \quad \lambda \neq 0 \text{ and } \varphi = \lambda \psi$$

This set (topological space, really) is called the <u>projectivisation</u> of $\mathcal{H}$.

Exercise L2-1 (i) Prove that there is a bijection $\mathbb{CP}\mathcal{H} \longrightarrow S_{\mathcal{H}}$ sending $[\psi]$ to $[\hat{\psi}]$ where $\hat{\psi} = \frac{1}{\|\psi\|}\psi$. So physical states are points of projective space. Note that $\mathcal{H}$ need not be finite-dimensional.

(ii) Prove that the function $\mathbb{CP}\mathcal{H} \times \mathbb{CP}\mathcal{H} \longrightarrow \mathbb{R}$, $([\varphi],[\psi]) \longmapsto |\langle \hat{\varphi}, \hat{\psi}\rangle|$ is well-defined.

(iii) If you have the background, prove this is also continuous where $\mathbb{CP}\mathcal{H}$ has the quotient topology.

Def^n We define the ray product $(-,-) : S_{\mathcal{H}} \times S_{\mathcal{H}} \longrightarrow \mathbb{R}$ by $(\mathcal{R}, S) = |\langle \varphi, \psi\rangle|$, where $\varphi \in \mathcal{R}$, $\psi \in S$ are arbitrary.

Recall that a function $f : \mathcal{H} \longrightarrow \mathcal{H}$ is antilinear or conjugate linear if $f(\varphi+\varphi') = f(\varphi)+f(\varphi')$ for all $\varphi, \varphi' \in \mathcal{H}$ and $f(\lambda\varphi) = \bar{\lambda} f(\varphi)$ for all $\lambda \in \mathbb{C}$, $\varphi \in \mathcal{H}$. This is the same thing as a linear transformation $\overline{\mathcal{H}} \longrightarrow \mathcal{H}$ see [MHS, L20 p.8]. A linear transformation $f : \mathcal{H} \longrightarrow \mathcal{H}$ is unitary if $\langle f\varphi, f\psi\rangle = \langle \varphi, \psi\rangle$ for all $\varphi, \psi \in \mathcal{H}$. Note that f is bounded and indeed $\|f\| = 1$, so that f is automatically continuous [MHS, Lemma L19-3].

Exercise L2-2 Prove that a unitary transformation is injective, and give a counterexample to show that it isn't necessarily surjective.

An antilinear transformation $f : \mathcal{H} \longrightarrow \mathcal{H}$ is antiunitary if $\langle f\varphi, f\psi\rangle = \overline{\langle \varphi, \psi\rangle}$ for all $\varphi, \psi \in \mathcal{H}$, and one proves such an f is a continuous bijection as above.

Let $\hat{\mathcal{H}} = \{\varphi \in \mathcal{H} \mid \|\varphi\| = 1\}$. We say a function $Q : S_{\mathcal{H}} \longrightarrow S_{\mathcal{H}}$ preserves the ray product if the diagram below commutes:

$$\begin{tikzcd} S_{\mathcal{H}} \times S_{\mathcal{H}} \arrow[rr, "Q \times Q"] \arrow[dr, "|\langle -,-\rangle|"'] & & S_{\mathcal{H}} \times S_{\mathcal{H}} \arrow[dl, "|\langle -,-\rangle|"] \\ & \mathbb{R} & \end{tikzcd}$$

← ray product

<u>Exercise L2-3</u> (i) If $U : \mathcal{H} \longrightarrow \mathcal{H}$ is unitary or antiunitary then there is a commutative diagram

$$\begin{tikzcd} \hat{\mathcal{H}} \arrow[r, "U"] \arrow[d, two heads] & \hat{\mathcal{H}} \arrow[d, two heads] \\ S_{\mathcal{H}} \arrow[r, "Q"'] & S_{\mathcal{H}} \end{tikzcd} \tag{3.1}$$

($\twoheadrightarrow$ denotes the quotient)

where Q preserves the ray product. Prove Q is a bijection if U is.

(ii) Prove that in the situation of (i), Q must be <u>continuous</u> when we give $S_{\mathcal{H}}$ the topology of $\mathbb{C}P\mathcal{H}$.

So bijective unitary and antiunitary transformations $\mathcal{H} \longrightarrow \mathcal{H}$ are the canonical source of ray product-preserving (continuous) bijections $S_{\mathcal{H}} \longrightarrow S_{\mathcal{H}}$ (symmetries of $\mathcal{H}$ yield symmetries of $S_{\mathcal{H}}$). Wigner's theorem asserts that this is the <u>only</u> source.

<u>Theorem</u> (Wigner) Let $\mathcal{H}$ be a separable Hilbert space with $\dim \mathcal{H} > 1$, and let $Q : S_{\mathcal{H}} \longrightarrow S_{\mathcal{H}}$ be a ray product-preserving surjection. Then there exists a bijective unitary or antiunitary transformation $U : \mathcal{H} \longrightarrow \mathcal{H}$ inducing Q in the sense that (3.1) commutes.

we leave the reader to make necessary modifications if $\dim \mathcal{H} < \infty$.

<u>Proof</u> Let $\{\psi_k\}_{k=1}^{\infty}$ be a countable orthonormal dense basis and let $W \subseteq \mathcal{H}$ denote $\{ \psi \in \mathcal{H} \mid \langle \psi_1, \psi \rangle \neq 0 \}$, which is open. Suppose we can construct a function $U : W \longrightarrow \mathcal{H}$ which is either unitary or antiunitary and induces Q in the sense that whenever $\varphi, \psi \in W$ $\lambda, \mu \in \mathbb{C}$ and $\lambda\varphi + \mu\psi \in W$ then

$$\begin{aligned} U(\lambda\varphi + \mu\psi) &= \lambda U(\varphi) + \mu U(\psi) && \text{(unitary case)} \\ U(\lambda\varphi + \mu\psi) &= \bar{\lambda} U(\varphi) + \bar{\mu} U(\psi) && \text{(anitunitary case)} \end{aligned} \tag{3.2}$$

and for all $\varphi, \psi \in W$

$$\langle U\varphi, U\psi \rangle = \langle \varphi, \psi \rangle \qquad \text{(unitary case)} \tag{4.1}$$
$$\langle U\varphi, U\psi \rangle = \overline{\langle \varphi, \psi \rangle} \qquad \text{(antiunitary case)}$$

and for any $\mathcal{R} \in S_{\mathcal{H}}$ and $\varphi \in \mathcal{R}$, $U(\varphi) \in Q(\mathcal{R})$. We give W the subspace metric, and note that $U : W \to \mathcal{H}$ is uniformly continuous because given $\varphi, \psi \in W$ if we take δ sufficiently small we have

$$\begin{aligned}
\| U\varphi - U\psi \| &= \| U\varphi - U(\varphi + \delta\psi_1) + U(\varphi + \delta\psi_1) - U\psi \| \\
&\leq \| U\varphi - U(\varphi + \delta\psi_1) \| + \| U(\varphi + \delta\psi_1) - U\psi \| \\
&= \| U(\delta\psi_1) \| + \| U(\varphi + \delta\psi_1 - \psi) \| && (4.1) \\
&= \delta \| \psi_1 \| + \| \varphi + \delta\psi_1 - \psi \| \\
&\leq 2\delta \| \psi_1 \| + \| \varphi - \psi \|
\end{aligned}$$

Hence given $\varepsilon > 0$ if we take $\delta < \varepsilon / (2\|\psi_1\| + 1)$ then $\| U\varphi - U\psi \| < \varepsilon$ for all $\varphi, \psi \in W$ as claimed. Hence by the universal property of complete metric spaces [MHS, Lemma L18-4] there is a unique uniformly continuous U^{ext} making

$$\begin{tikzcd}
\mathcal{H} \arrow[r, dashed, "U^{ext}"] & \mathcal{H} \\
W \arrow[u] \arrow[ur, "U"'] &
\end{tikzcd} \tag{4.2}$$

commute. It is straightforward to check linearity and (anti)unitarity of U^{ext} given that this holds on W, completing the proof.

We have now reduced to constructing $U: W \to \mathcal{H}$ which induces Q and is either unitary or antiunitary in the above sense. Let R_k denote the ray containing Ψ_k.

Let us choose arbitrary vectors $\Psi_k' \in Q(R_k)$ for $k \geqslant 1$ and observe that (writing (R, R') for the ray product)

$$
\begin{aligned}
|\langle \Psi_k', \Psi_\ell' \rangle| &= (Q(R_k), Q(R_\ell)) \\
&= (R_k, R_\ell) \\
&= |\langle \Psi_k, \Psi_\ell \rangle| = \delta_{k\ell}
\end{aligned}
$$

so $\{\Psi_k'\}_{k=1}^\infty$ is an orthonormal family. If $\langle \Theta, \Psi_k' \rangle = 0$ for all $k \geqslant 1$ and $\Theta \neq 0$ then $\hat{\Theta} = \frac{1}{\|\Theta\|}\Theta$ belongs to a ray R' and since Q is surjective $R' = Q(R)$ for some ray R and hence $0 = |\langle \hat{\Theta}, \Psi_k' \rangle| = (Q(R), Q(R_k)) = (R, R_k)$ for all $k \geqslant 1$. But choosing any vector $T \in R$ this implies $\langle T, \Psi_k \rangle = 0$ for all $k \geqslant 1$ which is a contradiction [MHS, Theorem L21-10]. This shows that $\{\Psi_k'\}_{k=1}^\infty$ is an orthonormal dense basis in the sense of [MHS, L21 p. 12].

If the desired unitary or antiunitary transformation U exists then by commutativity of (3.1)

$$
\begin{tikzcd}
\Psi_k & \hat{\mathcal{H}} \arrow[r, "U"] \arrow[d, two heads] & \hat{\mathcal{H}} \arrow[d, two heads] & U(\Psi_k) \\
R_k & S_{\mathcal{H}} \arrow[r, "Q"'] & S_{\mathcal{H}} & Q(R_k)
\end{tikzcd}
\tag{4.1}
$$

we must have $U(\Psi_k) = \eta_k \Psi_k'$ for some $\eta_k \in U(1)$.

So the problem of constructing U is precisely the problem of <u>choosing phases</u> η_k such that the assignment $\Psi_k \mapsto \eta_k \Psi_k'$ is either unitary or antiunitary <u>and</u> makes (3.1) commute for <u>every</u> unit vector $\varphi \in \hat{\mathcal{H}}$. The (anti) linearity and (anti) unitary properties are essentially trivial, but making (3.1) commute is not. Note for example that if U is linear and $U(\Psi_k) = \eta_k \Psi_k'$ for $k \geq 1$ then for $k \neq \ell$

$$U\left(\tfrac{1}{\sqrt{2}}(\Psi_k + \Psi_\ell)\right) = \tfrac{1}{\sqrt{2}}\left(\eta_k \Psi_k' + \eta_\ell \Psi_\ell'\right) \qquad (5.1)$$

but with $\varphi = \frac{1}{\sqrt{2}}(\Psi_k + \Psi_\ell)$ it is not clear that with $R_{k\ell}$ the ray containing φ that $\frac{1}{\sqrt{2}}(\eta_k \Psi_k' + \eta_\ell \Psi_\ell')$ belongs to $Q(R_{k\ell})$. Indeed, this <u>cannot</u> be true for <u>arbitrary</u> choices of phases η_k, η_ℓ (why?) so the question is whether it is true for <u>any</u> choice.

We know Q preserves the ray product which implies that for $\varphi' \in Q(R_{k\ell})$ that

$$\begin{aligned} |\langle \varphi', \Psi_i' \rangle| &= (Q(R_{k\ell}), Q(R_i)) \\ &= (R_{k\ell}, R_i) \\ &= \langle \varphi, \Psi_i \rangle \\ &= \tfrac{1}{\sqrt{2}} \delta_{ik} + \tfrac{1}{\sqrt{2}} \delta_{i\ell} \end{aligned} \qquad (5.2)$$

hence $\varphi' = \frac{1}{\sqrt{2}}(\xi_k \Psi_k' + \xi_\ell \Psi_\ell')$ for some $\xi_k, \xi_\ell \in U(1)$. Since φ' was an arbitrary element of the ray $Q(R_{k\ell})$ the phases ξ_k, ξ_ℓ have no particular meaning, but the quotient $\xi_\ell \xi_k^{-1} \in U(1)$ <u>is</u> independent of this choice and depends only on the pair (k, ℓ). We could regard

$$\varphi_{k\ell}' := \tfrac{1}{\sqrt{2}}\left(\Psi_k' + \xi_\ell \xi_k^{-1} \Psi_\ell'\right) \in Q(R_{k\ell}) \qquad (5.3)$$

as a "canonical" choice of representative, since the coefficient of Ψ_k' is 1. But while $R_{k\ell} = R_{\ell k}$ we have $\varphi_{k\ell}' \neq \varphi_{\ell k}'$ in general, which is a problem!

The solution is to fix a special index, say $k=1$, and consider only pairs $(1,\ell)$. As we will see, this corresponds to constucting U only on the open set $W \subseteq \mathcal{H}$ which we have already seen is sufficient. So finally we define

$$U(\psi_1) = \psi_1' \qquad U(\psi_\ell) = \xi_\ell \xi_1^{-1} \psi_\ell' \qquad \ell > 1 \tag{6.1}$$

Note $\{U(\psi_k)\}_{k=1}^{\infty}$ is an orthonormal dense basis and by the above for any $\ell > 1$ that

$$\frac{1}{\sqrt{2}}\left(U(\psi_1) + U(\psi_\ell) \right) \in Q(R_{1\ell}) \tag{6.2}$$

In the earlier notation, we have made the choices $\eta_1 = 1$ and $\eta_\ell = \xi_\ell \xi_1^{-1}$. Note that while $k=1$ is a choice, the complex number $\xi_\ell \xi_1^{-1}$ is determined by Q, the choice of basis $\{\psi_k\}_{k=1}^{\infty}$, and the choices of representatives $\{\psi_k'\}_{k=1}^{\infty}$. We claim that once the pairs $(1,\ell)$ are "sorted out" this resolves all other such combinations, provided they involve the index 1. More precisely:

<u>Claim 1</u> For any sequence of distinct indices $1, i_1, \dots, i_N$ if we set

$$\Phi = \frac{1}{\sqrt{N+1}}\left(\psi_1 + \psi_{i_1} + \cdots + \psi_{i_N} \right) \in R \tag{6.3}$$

then $\frac{1}{\sqrt{N+1}}\left(U(\psi_1) + U(\psi_{i_1}) + \cdots + U(\psi_{i_N}) \right) \in Q(R)$.

<u>Proof of claim 1</u> Take an arbitrary unit vector $\psi = \sum_{k=1}^{\infty} c_k \psi_k$ with $c_1 \neq 0$ (i.e. $\psi \in W$) and any $\psi' \in Q(R)$ where $\psi \in R$. Writing $\psi' = \sum_{k=1}^{\infty} c_k' U(\psi_k)$ we have for $k \geq 1$

$$\begin{aligned} |c_k'| = |\langle U(\psi_k), \psi' \rangle| &= (Q(R_k), Q(R)) \\ &= (R_k, R) = |\langle \psi_k, \psi \rangle| = |c_k| \end{aligned} \tag{6.4}$$

Now since for $k \neq 1$ we have $\frac{1}{\sqrt{2}}(U(\Psi_1) + U(\Psi_k)) \in Q(\mathcal{R}_{1k})$ we have

$$
\begin{aligned}
|C_1' + C_k'| &= \sqrt{2}\,|\langle \Psi', \tfrac{1}{\sqrt{2}}(U(\Psi_1) + U(\Psi_k))\rangle| \\
&= \sqrt{2}\,(Q(\mathcal{R}), Q(\mathcal{R}_{1k})) \\
&= \sqrt{2}\,|\langle \Psi, \tfrac{1}{\sqrt{2}}(\Psi_1 + \Psi_k)\rangle| \\
&= |C_1 + C_k|.
\end{aligned}
\tag{7.1}
$$

Combining these two equations gives (if $C_k \neq 0$)

$$
\begin{aligned}
|1 + C_1/C_k| &= |C_1 + C_k| / |C_k| \\
&= |C_1' + C_k'| / |C_k'| \\
&= |1 + C_1'/C_k'|
\end{aligned}
\tag{7.2}
$$

since we also have $|C_1/C_k| = |C_1'/C_k'|$ we can set $C_1/C_k = a + ib$ and $C_1'/C_k' = c + id$ and deduce $a^2 + b^2 = c^2 + d^2$, $(1+a)^2 + b^2 = (1+c)^2 + d^2$ from which we obtain for all $k \neq 1$

$$
\begin{aligned}
Re(C_k/C_1) &= Re(C_k'/C_1') \\
Im(C_k/C_1) &= \pm Im(C_k'/C_1')
\end{aligned}
\tag{7.3}
$$

and therefore for each $k \neq 1$ at least one of the following holds (perhaps both)

$$
\begin{aligned}
&① \quad C_k'/C_1' = C_k/C_1 \\
&② \quad C_k'/C_1' = \overline{(C_k/C_1)}
\end{aligned}
\tag{7.4}
$$

Now apply the above to Φ in the claim, with $\Phi \in \mathcal{R}$, and let $\Phi' \in Q(\mathcal{R})$. From ①, ② we see that $C_k'/C_1' = C_k/C_1 = 1$ for $k \in \{i_1, \dots, i_N\}$ so there is $\xi \in U(1)$ with $\Phi' = \frac{1}{\sqrt{N+1}}(\xi U(\Psi_1) + \xi U(\Psi_{i_1}) + \dots + \xi U(\Psi_{i_N}))$ as claimed. □

<u>Claim 2</u> For any unit vector $\psi = \sum_{k=1}^{\infty} c_k \psi_k$ with $c_1 \neq 0$ with $\psi \in R$ we have $\sum_{k=1}^{\infty} c_k U(\psi_k) \in Q(R)$ or $\sum_{k=1}^{\infty} \overline{c_k} U(\psi_k) \in Q(R)$.

<u>Proof of Claim 2</u> From the proof of Claim 1 we know that for every $k \neq 1$ one of ① or ② in (7.4) holds (possibly both). We claim further that one (or both) of conditions ⓘ, ⓘⓘ below hold (that is, either ① or ② holds <u>consistently</u>)

$$\text{(i)}\ \forall k \neq 1 \quad c'_k / c'_1 = c_k / c_1$$

$$\text{(ii)}\ \forall k \neq 1 \quad c'_k / c'_1 = \overline{(c_k / c_1)} \tag{8.1}$$

To prove this let $l \neq k$ be both different to 1. Then $\frac{1}{\sqrt{3}}(\psi_1 + \psi_k + \psi_l) \in S$ for some ray S and by Claim 1 we have $\frac{1}{\sqrt{3}}(U(\psi_1) + U(\psi_k) + U(\psi_l)) \in Q(S)$. Hence if ψ is as in the statement of Claim 2 with $\psi' = \sum_{k=1}^{\infty} c'_k U(\psi_k) \in Q(R)$ chosen arbitrarily,

$$\begin{aligned} |c'_1 + c'_k + c'_l| &= \sqrt{3}\, |\langle \psi', \tfrac{1}{\sqrt{3}}(U(\psi_1) + U(\psi_k) + U(\psi_l)) \rangle| \\ &= \sqrt{3}\,(Q(R), Q(S)) \\ &= \sqrt{3}\,(R, S) \\ &= |c_1 + c_k + c_l| \end{aligned} \tag{8.2}$$

Dividing through by $|c_1| = |c'_1|$ we find that

$$\left| 1 + c'_k/c'_1 + c'_l/c'_1 \right| = \left| 1 + c_k/c_1 + c_l/c_1 \right| \tag{8.3}$$

Suppose for a contradiction that neither ⓘ nor ⓘⓘ applies to ψ. Then without loss of generality both c_k/c_1, c_l/c_1 are complex and $c_k/c_1 = c'_k/c'_1$, $c'_l / c'_1 = \overline{c_l/c_1}$. Then by (8.3)

$$\left| 1 + {c_k}/{c_1} + {\overline{c_\ell}}/{\overline{c_1}} \right| = \left| 1 + {c_k}/{c_1} + {c_\ell}/{c_1} \right|$$

which implies $\operatorname{Im}(c_k/c_1)\operatorname{Im}(c_\ell/c_1) = 0$ which contradicts the hypothesis that both ratios are complex. This completes the proof that either (i) applies in which case

$$\psi' = \sum_{k=1}^{\infty} c_k' U(\psi_k) = \sum_{k=1}^{\infty} c_1' \frac{c_k'}{c_1'} U(\psi_k) = \sum_{k=1}^{\infty} c_1' \frac{c_k}{c_1} U(\psi_k) = \frac{c_1'}{c_1} \sum_{k=1}^{\infty} c_k U(\psi_k)$$

or (ii) applies in which case

$$\psi' = \sum_{k=1}^{\infty} c_k' U(\psi_k) = \sum_{k=1}^{\infty} c_1' \frac{c_k'}{c_1'} U(\psi_k) = \sum_{k=1}^{\infty} c_1' \overline{\frac{c_k}{c_1}}\, U(\psi_k) = \frac{c_1'}{\overline{c_1}} \sum_{k=1}^{\infty} \overline{c_k}\, U(\psi_k)$$

as claimed. $\square$

We say that a ray R is <u>real</u> if it contains a vector ψ such that for all $k \geq 1$ the coefficient $\langle \psi_k, \psi \rangle$ is real. Note that a ray R is real if and only if for every $\psi = \sum_{k=1}^{\infty} c_k \psi_k \in R$ and distinct indices $k \neq \ell$ we have that $\overline{c_k} c_\ell$ is real (in fact it suffices to have $\overline{c_k} c_\ell$ real for any fixed k and all ℓ).

Given a ray R and $\psi = \sum_{k=1}^{\infty} c_k \psi_k \in R$ with $c_1 \neq 0$ note that <u>both</u> $\sum_{k=1}^{\infty} c_k U(\psi_k)$ and $\sum_{k=1}^{\infty} \overline{c_k} U(\psi_k)$ belong to $Q(R)$ if R is real, and moreover the converse holds: if both of these vectors belong to $Q(R)$ and we set $c_k = r_k e^{i\theta_k}$ for $r_k \in \mathbb{R}$ then there exists $\xi \in U(1)$ with $c_k = \xi \overline{c_k}$ for all $k \geq 1$ and hence $\overline{c_k} c_\ell = \xi^{-1} c_k \xi \overline{c_\ell} = c_k \overline{c_\ell}$ is real, hence the ray R is real.

<u>Def^n</u> We call a non-real ray R <u>normal</u> if for any (hence every) $\psi = \sum_{k=1}^{\infty} c_k \psi_k \in R$ we have $c_1 \neq 0$ and $\sum_{k=1}^{\infty} c_k U(\psi_k) \in Q(R)$.

<u>Def^n</u> We call a non-real ray R <u>conjugate</u> if for any (hence every) $\psi = \sum_{k=1}^{\infty} c_k \psi_k \in R$ we have $c_1 \neq 0$ and $\sum_{k=1}^{\infty} \overline{c_k} U(\psi_k) \in Q(R)$.

By Claim 2 and the above discussion each non-real ray R with $\Psi \in R$ satisfying $\langle \Psi, \Psi \rangle \neq 0$ is either normal or conjugate, and not both.

<u>Claim 3</u> Suppose R, S are non-real rays with R normal and S conjugate. If $\Phi = \sum_{k=1}^{\infty} C_k \Psi_k \in R$ and $\Psi = \sum_{k=1}^{\infty} D_k \Psi_k \in S$ then

$$\sum_{k,l=1}^{\infty} \mathrm{Im}(D_k \overline{D_l}) \, \mathrm{Im}(C_k \overline{C_l}) = 0.$$

<u>Proof</u> Using standard theory of Hilbert spaces (see e.g. [MHS, Theorem L21-10]

$$\left| \sum_{k=1}^{\infty} D_k C_k \right|^2 = \left| \left\langle \sum_{k=1}^{\infty} \overline{D_k} \, U(\Psi_k), \sum_{l=1}^{\infty} C_l \, U(\Psi_l) \right\rangle \right|$$
$$= (Q(S), Q(R)) = (S, R) = \left| \sum_{k=1}^{\infty} \overline{D_k} C_k \right|^2$$

Since the two series converge absolutely (see e.g. [MHS, Lemma L21-5]) so do their conjugates, and we can expand the square of the absolute value as the product of the series with its conjugate

$$\sum_{a \geq 0} \sum_{k+l=a} D_k C_k \overline{D_l} \, \overline{C_l} = \sum_{a \geq 0} \sum_{k+l=a} \overline{D_k} C_k D_l \overline{C_l}$$

which implies

$$\sum_{a \geq 0} \sum_{k+l=a} C_k \overline{C_l} \left(D_k \overline{D_l} - \overline{D_k} D_l \right) = 0$$

and thus $\sum_{k,l} \mathrm{Im}(D_k \overline{D_l}) \, \mathrm{Im}(C_k \overline{C_l}) = 0$ as claimed. $\square$

Note that $(\mathcal{R}_1, \mathcal{R}) \neq 0$ if and only if for all $\Psi \in \mathcal{R}$ we have $\langle \Psi_1, \Psi \rangle \neq 0$.

<u>Claim 4</u> If $\mathcal{R}, S$ are non-real rays with $(\mathcal{R}_1, \mathcal{R}) \neq 0$, $(\mathcal{R}_1, S) \neq 0$ then either both are normal or both are conjugate.

<u>Proof of Claim 4</u> Suppose for a contradiction that $\mathcal{R} \neq S$ exist with neither both $\mathcal{R}, S$ normal nor both conjugate. Let us say $\mathcal{R}$ is normal (hence not conjugate) and S is conjugate (hence not normal). We let $\Phi \in \mathcal{R}$, $\Upsilon \in S$ be as in the statement of Claim 3 and produce $\Omega = \sum_{k=1}^{\infty} E_k \Psi_k$ with $E_1 \neq 0$ such that

$$\sum_{k,\ell} \mathrm{Im}(D_k \overline{D_\ell})\, \mathrm{Im}(E_k \overline{E_\ell}) \neq 0$$

$$\sum_{k,\ell} \mathrm{Im}(C_k \overline{C_\ell})\, \mathrm{Im}(E_k \overline{E_\ell}) \neq 0 \tag{10.2}$$

which is a contradiction, since the ray $\mathcal{T} \ni \Omega$ is either conjugate or normal (Claim 2).

Since $\mathcal{R}, S$ are non-real, there is an index $\ell \neq 1$ such that $C_1 \overline{C_\ell}$ is complex. If we can choose ℓ such that $D_1 \overline{D_\ell}$ is also complex then with $\Omega = \frac{1}{\sqrt{2}}(\Psi_1 + i\Psi_\ell)$ both sums in (10.2) are nonzero as required.

If we <u>cannot</u> so choose, let $\ell \neq n$ be indices such that $C_1 \overline{C_\ell}$, $D_1 \overline{D_n}$ are complex but $C_1 \overline{C_n}$, $D_1 \overline{D_\ell}$ are real (otherwise we are in the previous case). Set

$$\Omega = \frac{1}{\sqrt{2}}(i\Psi_1 + \Psi_\ell + \Psi_n)$$

in which case the above sums are nonzero. $\square$

So either <u>every non-real ray is normal</u> or <u>every non-real ray is conjugate</u> (restricting attention to those rays with a nonzero ray product with $\mathcal{R}_1$).

We now define U on unit vectors $\Psi \in W$, that is, unit vectors with $\langle \Psi_1, \Psi \rangle \neq 0$.
If every non-real ray is normal and $\Psi = \sum_{k=1}^{\infty} C_k \Psi_k$ is a unit vector in a ray $\mathcal{R}$ with $(\mathcal{R}_1, \mathcal{R}) \neq 0$ (not assumed non-real) then we define

$$U(\Psi) := \sum_{k=1}^{\infty} C_k U(\Psi_k) \qquad (11.3)$$

This series converges by [MHS, Lemma L21-5] and clearly agrees with the earlier definition of U on the Ψ_k. If every non-real ray is conjugate then we define

$$U(\Psi) := \sum_{k=1}^{\infty} \overline{C_k}\, U(\Psi_k). \qquad (11.4)$$

By definition in either case $U(\Psi) \in Q(\mathcal{R})$. It remains to define U on <u>all</u> vectors.

We obviously set $U(0) = 0$ and if $\Psi \in W$ is nonzero we define $U(\Psi) := \|\Psi\|\, U(\Psi / \|\Psi\|)$.
It remains to prove that U is (anti) linear in the sense of (3.2) <u>on W</u> and (anti) unitary in the sense of (4.1) on W. We prove that if every non-real ray is normal then $U : W \to \mathcal{H}$ is linear and unitary in this sense, and we leave the argument that U is antilinear and antiunitary in the case where every non-real ray is conjugate to the reader.

Suppose every non-real ray is normal, that $\Phi, \Psi \in W$ and $\lambda, \mu \in \mathbb{C}$ are such that $\lambda \Phi + \mu \Psi \in W$. Then with $L = \|\lambda \Phi + \mu \Psi\| \neq 0$, and

$$\hat{\Phi} = \sum_{k=1}^{\infty} C_k \Psi_k, \quad \hat{\Psi} = \sum_{k=1}^{\infty} D_k \Psi_k,$$

we have

$$
\begin{aligned}
U(\lambda\varphi+\mu\psi) &= L\,U\left(\tfrac{\lambda}{L}\varphi+\tfrac{\mu}{L}\psi\right)\\
&= L\,U\left(\textstyle\sum_{k=1}^{\infty}\left(\tfrac{\lambda}{L}C_k+\tfrac{\mu}{L}D_k\right)\psi_k\right)\\
&= L\textstyle\sum_{k=1}^{\infty}\left(\tfrac{\lambda}{L}C_k+\tfrac{\mu}{L}D_k\right)U(\psi_k)\\
&= \textstyle\sum_{k=1}^{\infty}\lambda C_k U(\psi_k)+\sum_{k=1}^{\infty}\mu D_k U(\psi_k)\\
&= \lambda U(\varphi)+\mu U(\psi)
\end{aligned}
$$

and

$$
\begin{aligned}
\langle U(\varphi), U(\psi)\rangle &= \|\varphi\|\cdot\|\psi\|\,\langle U(\hat\varphi), U(\hat\psi)\rangle\\
&= \|\varphi\|\,\|\psi\|\,\left\langle \textstyle\sum_{k=1}^{\infty}C_k U(\psi_k), \sum_{l=1}^{\infty}D_k U(\psi_l)\right\rangle\\
&= \|\varphi\|\,\|\psi\|\textstyle\sum_{k=1}^{\infty}\overline{C_k}\,D_k\\
&= \langle\varphi,\psi\rangle.
\end{aligned}
$$

This suffices, by what we said at the beginning of the proof, to show that there exists an (anti)unitary $U:\mathcal{H}\longrightarrow\mathcal{H}$ inducing Q. This map is injective by Ex L2-2 and surjective since Q is surjective. $\square$

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<u>Exercise L2-4</u> Check that U^{ext} of (4.2) is either linear and unitary or antilinear and antiunitary, using the construction of U^{ext} in [MHS, L18]. Prove also that U^{ext} is a bijection.

<u>Exercise L2-5</u> Why is it <u>not</u> suspicious that Q is not assumed continuous in Wigner's theorem?

<u>Exercise L2-6</u> Given Q is the U in Wigner's theorem unique? If not, what is the relationship between two unitary or antiunitary transformations $U_1, U_2 : \mathcal{H} \longrightarrow \mathcal{H}$ inducing the same function $S_{\mathcal{H}} \longrightarrow S_{\mathcal{H}}$?

<u>Exercise L2-7</u> What is the correct statement of Wigner's theorem if $\dim \mathcal{H} = 1$?

<u>Exercise L2-8</u> As a consequence of Wigner's theorem we see that a <u>surjective</u> ray product preserving function $Q : S_{\mathcal{H}} \longrightarrow S_{\mathcal{H}}$ is necessarily also injective. Give a direct proof (you may recycle an appropriate part of the argument from the proof of the theorem, but make this a <u>minimal</u> part).

<u>Exercise L2-9</u> In the situation of the Theorem prove that $W = \{ \psi \in \mathcal{H} \mid \langle \psi_1, \psi \rangle \neq 0 \}$ is a dense subset of $\mathcal{H}$.

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