Lecture 3 – Symmetries of Hilbert space

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We have seen in Lecture 2 how symmetries of a quantum system with Hilbert space $\mathcal{H}$ may be identified with unitary or antiunitary transformations $U : \mathcal{H} \longrightarrow \mathcal{H}$. Now we explore two examples: symmetries of the Hilbert space $L^2(X, \mathbb{C})$ of complex-valued functions on the circle $X = S^1$ and sphere $X = S^2$. We will see that the Lie group $SO(3)$ acts on $L^2(S^2, \mathbb{C})$ by unitary transformations, and in some sense this representation is "universal".

First we briefly recall the definition of $L^2(X, \mathbb{C})$. You have three choices: adopt the definition from [MHS] which does not require measure theory (but you need to know how to complete a normed space), adopt the definition of $L^2(X, \mathbb{C})$ as square-integrable functions modulo some relation (requires measure theory) or wait until I tell you an orthonormal dense basis and adopt that as your definition. All are acceptable.

With $X = S^1$ or $X = S^2$ denoting the unit 1-sphere and 2-sphere

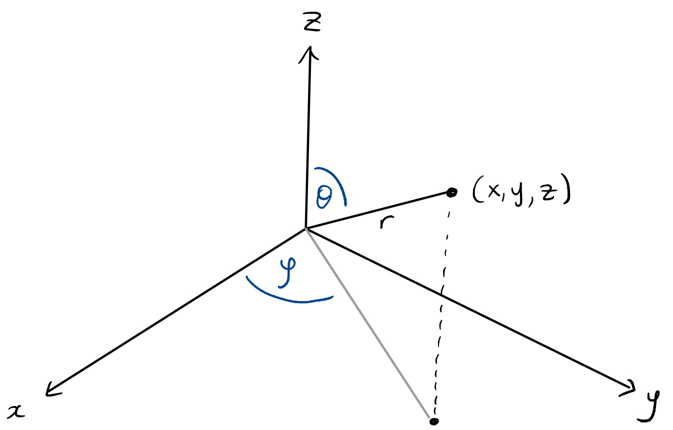
$$S^1 = \{ x \in \mathbb{R}^2 \mid \|x\| = 1 \}, \qquad S^2 = \{ x \in \mathbb{R}^3 \mid \|x\| = 1 \} \tag{1.1}$$

let $Cts(X, \mathbb{C})$ denote the $\mathbb{C}$-vector space of continuous complex-valued functions on X with operations $(\varphi + \psi)(x) = \varphi(x) + \psi(x)$, $(\lambda \varphi)(x) = \lambda \varphi(x)$ and with the norm $\|-\|_2$ defined by

$$\| \varphi \|_2 = \left\{ \int_X |\varphi|^2 \right\}^{1/2} \tag{1.2}$$

where $\int_{S^1}$ is as specified in [MHS, Lecture 17] and $\int_{S^2}$ means integration over the sphere defined as follows. We parametrise S^2 by spherical coordinates, recall

$$\begin{aligned} x_1 &= r \sin\theta \cos\varphi & \quad & 0 \leq \theta \leq \pi,\ 0 \leq \varphi < 2\pi \quad r > 0 \\ x_2 &= r \sin\theta \sin\varphi & & \varphi = \text{azimuthal angle} \\ x_3 &= r \cos\theta & & \theta = \text{polar angle} \end{aligned} \tag{1.3}$$



(2.1)

$$\int_{S^2} \varphi(x)\, dS = \int_0^{2\pi}\int_0^{\pi} \varphi(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)\sin\theta\, d\theta\, d\varphi$$

Exercise L3-1 (if you took MHS) Check that $(S^2, \int_{S^2})$ is an integral pair. You may assume that $\int_{S^2}$ is linear.

By definition $L^2(X,\mathbb{C})$ (which we will sometimes denote simply by $L^2(X)$) is the completion of $(Cts(X,\mathbb{C}), \|-\|_2)$ as a normed space [MHS, L18 p. 21] which means that there is a norm-preserving injective linear map $\iota : Cts(X,\mathbb{C}) \longrightarrow L^2(X,\mathbb{C})$ (that is, we may view all continuous complex-valued functions φ on X as vectors in $L^2(X,\mathbb{C})$, writing $\iota(\varphi)$ simply as φ) and the $\mathbb{C}$-vector space structure and norm on $L^2(X,\mathbb{C})$ can be described as follows

(2.2)

- every vector $\psi \in L^2(X,\mathbb{C})$ is a limit $\psi = \lim_{n\to\infty}\psi_n$ of a sequence of continuous functions $\psi_n \in Cts(X,\mathbb{C})$. (so $Cts(X,\mathbb{C})$ is dense in $L^2(X,\mathbb{C})$)

- if $\psi = \lim_{n\to\infty}\psi_n$, $\varphi = \lim_{n\to\infty}\varphi_n$ with $\psi_n, \varphi_n \in Cts(X,\mathbb{C})$ for all n, then

(a) $\|\psi\| = \lim_{n\to\infty}\|\psi_n\|$ (2.3)

(b) $\varphi + \psi = \lim_{n\to\infty}(\varphi_n + \psi_n)$

(c) $\lambda\varphi = \lim_{n\to\infty}\lambda\varphi_n$ $\forall \lambda \in \mathbb{C}$

The space $L^2(X, \mathbb{C})$ is a Hilbert space [MHS, Theorem L20-13] with pairing between $\Psi = \lim_{n\to\infty} \Psi_n$, $\varphi = \lim_{n\to\infty} \varphi_n$ for $\Psi_n, \varphi_n \in C_\#(X, \mathbb{C})$ given by

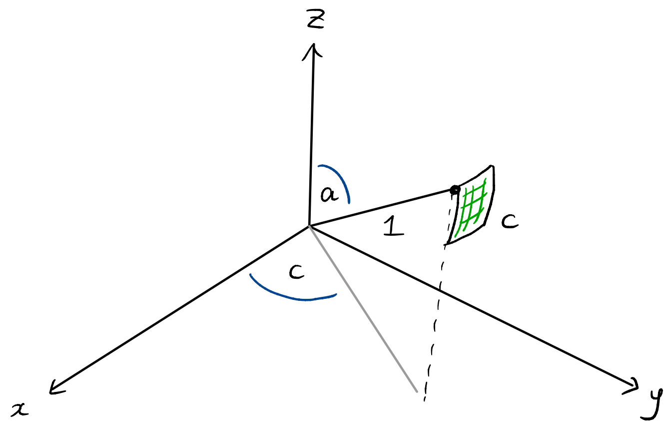
$$\langle \Psi, \varphi \rangle = \lim_{n\to\infty} \langle \Psi_n, \varphi_n \rangle \qquad (3.1)$$

$$= \lim_{n\to\infty} \int_X \overline{\Psi_n}\, \varphi_n$$

(note the conjugation convention differs from [MHS])

It is not appropriate to think of vectors in $L^2(X, \mathbb{C})$ as functions, as given $\Psi \in L^2(X, \mathbb{C})$ the value $\Psi(x) \in \mathbb{C}$ for $x \in X$ is ill-defined, see [MHS, L20, L21]. However the average value over a region is always well-defined:

Exercise L3-2 Borrow ideas from [MHS, Example L20-7] to define for any "spherical rectangle" C defined by $a \leq \theta \leq b$, $c \leq \varphi \leq d$ the quantity $\int_C |\Psi|^2 dS$ for any $\Psi \in L^2(S^2, \mathbb{C})$ (physically, this is interpreted, if $\|\Psi\| = 1$, as the probability of a particle with wavefunction Ψ being found in C).



We write this set C as

$$C[a,b] \times [c,d] = \{ (x,y,z) \mid a \leq \theta \leq b,\ c \leq \varphi \leq d \} \qquad (3.2)$$

<u>Example L3-1</u> The set $\{\frac{1}{\sqrt{2\pi}} e^{in\theta}\}_{n\in\mathbb{Z}}$ is a (countable) orthonormal dense basis for $L^2(S^1,\mathbb{C})$ [MHS, Example L21-3] and the coefficients of an arbitrary vector $\psi \in L^2(S^1,\mathbb{C})$ are the Fourier coefficients.

<u>Def^n</u> The <u>rotation group</u> $SO(n)$ is the group of all linear transformations $f:\mathbb{R}^n \longrightarrow \mathbb{R}^n$ which have determinant 1 and satisfy

$$\langle fx, fy\rangle_{\mathbb{R}^n} = \langle x,y\rangle_{\mathbb{R}^n} \qquad \forall x,y\in\mathbb{R}^n \tag{4.1}$$

where $\langle x,y\rangle_{\mathbb{R}^n} = \sum_{i=1}^n x_i y_i$ is the standard inner product. The group operation is composition.

In [MHS, L3] we showed that $SO(2)$ is precisely the set $\{R_\theta\}_{\theta\in[0,2\pi)}$ of rotations, with $R_\theta R_{\theta'} = R_{\theta+\theta'}$ and $R_\theta = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$. In general we may view $SO(n)$ as a subgroup of all invertible matrices $GL(n,\mathbb{R})$ and in this way (recall Lecture 1) see that the multiplication and inversion on $SO(n)$ are smooth. We will prove later $SO(n)$ is a Lie group.

Next we explain how $SO(n+1)$ acts on the Hilbert space $L^2(S^n,\mathbb{C})$ for $n\in\{1,2\}$.

<u>$SO(n+1)$ acting on $L^2(S^n,\mathbb{C})$</u> $(n\in\{1,2\})$

Let $g\in SO(n+1)$ be given, and observe that $g:\mathbb{R}^{n+1}\longrightarrow\mathbb{R}^{n+1}$ is linear and thus continuous. It follows from (4.1) that g restricts to a continuous map $g: S^n \longrightarrow S^n$. By precomposition we have a $\mathbb{C}$-linear map

$$\begin{aligned} C_g : Cts(S^n,\mathbb{C}) &\longrightarrow Cts(S^n,\mathbb{C}) \\ C_g(\psi) &= \psi\circ g \end{aligned} \tag{4.1}$$

<u>Exercise L3-3</u> Prove that for g as above $\int_{S^n} (\psi \circ g)\, dS = \int_{S^n} \psi\, dS$ for both $n=1$ or $n=2$, using a change-of-variables formula for the Riemann integral.

By the exercise C_g is norm-preserving

$$\| C_g(\psi) \|_2 = \Big\{ \int_{S^n} |\psi \circ g|^2 dS \Big\}^{1/2} = \Big\{ \int_{S^n} |\psi|^2 \circ g \, dS \Big\}^{1/2}$$
$$= \Big\{ \int_{S^n} |\psi|^2 dS \Big\}^{1/2} = \| \psi \|_2, \qquad (5.1)$$

and in particular C_g is bounded $\| C_g \| = 1$ and linear, hence continuous [MHS, Lemma L19-3] and so by the universal property of the completion [MHS, Theorem L18-9] there is a <u>unique</u> continuous linear map $\hat{g}$ making the following diagram commute

$$\begin{array}{ccc} L^2(S^n, \mathbb{C}) & \xrightarrow{\hat{C}_g} & L^2(S^n, \mathbb{C}) \\ \uparrow \iota & & \uparrow \iota \\ \mathrm{Cts}(S^n, \mathbb{C}) & \xrightarrow[C_g]{} & \mathrm{Cts}(S^n, \mathbb{C}) \end{array} \qquad (5.2)$$

By continuity if $\psi = \lim_{n\to\infty} \psi_n$ as in (2.2) then $\hat{C}_g(\psi) = \lim_{n\to\infty} (\psi_n \circ g)$.

<u>Lemma L3-1</u> For $n \in \{1,2\}$ and $g \in SO(n+1)$ the linear transformation $\hat{C}_g$ is bijective and unitary, and $\hat{C}_g \hat{C}_h = \hat{C}_{hg}$, $\hat{C}_1 = 1$.

<u>Proof</u> Unitarity follows from Ex L3-3 since for $\varphi, \psi \in L^2(S^n, \mathbb{C})$ written as limits $\psi = \lim_{n\to\infty} \psi_n$, $\varphi = \lim_{n\to\infty} \varphi_n$ as in (2.2) we have by continuity of $\hat{C}_g$ and the definition (3.1) of the pairing

$$
\begin{aligned}
\langle \hat{C}_g(\psi), \hat{C}_g(\varphi) \rangle &= \langle \hat{C}_g(\lim_{n\to\infty} \psi_n), \hat{C}_g(\lim_{n\to\infty} \varphi_n) \rangle \\
&\overset{\hat{C}_g \text{ cts}}{=} \langle \lim_{n\to\infty} \hat{C}_g(\psi_n), \lim_{n\to\infty} \hat{C}_g(\varphi_n) \rangle \\
&\overset{(5.2)\text{ commutes}}{=} \langle \lim_{n\to\infty} C_g(\psi_n), \lim_{n\to\infty} C_g(\varphi_n) \rangle \\
&\overset{(3.1)}{=} \lim_{n\to\infty} \langle \psi_n \circ g, \varphi_n \circ g \rangle \\
&= \lim_{n\to\infty} \int_{S^n} \overline{(\psi_n \circ g)}(\varphi_n \circ g)\, dS \\
&= \lim_{n\to\infty} \int_{S^n} (\overline{\psi_n}\varphi_n)\circ g \; dS \\
&\overset{\text{ExL3-3}}{=} \lim_{n\to\infty} \int_{S^n} \overline{\psi_n}\varphi_n \, dS \\
&= \langle \psi, \varphi \rangle
\end{aligned}
\tag{6.1}
$$

For the second set of claims let $g, h \in SO(n+1)$ be given and observe that since $C_g C_h(\psi) = C_g(\psi \circ h) = (\psi \circ h) \circ g = \psi \circ (hg) = C_{hg}(\psi)$ commutativity of the outer square in

$$
\begin{tikzcd}
L^2(S^n, \mathbb{C}) \arrow[r, "\hat{C}_h"] & L^2(S^n, \mathbb{C}) \arrow[r, "\hat{C}_g"] & L^2(S^n, \mathbb{C}) \\
Cts(S^n, \mathbb{C}) \arrow[u, hook, "\iota"] \arrow[r, "C_h"'] \arrow[rr, bend right, "C_{hg}"'] & Cts(S^n, \mathbb{C}) \arrow[u, hook, "\iota"'] \arrow[r, "C_g"'] & Cts(S^n, \mathbb{C}) \arrow[u, hook, "\iota"']
\end{tikzcd}
\tag{6.2}
$$

implies by the universal property that $\hat{C}_g \circ \hat{C}_h = \hat{C}_{hg}$. Similarly $\hat{C}_1 = 1$. $\square$

<u>Exercise L3-4</u> Prove $\hat{C}_g \hat{C}_h = \hat{C}_{hg}$ using limits.

<u>Defn</u> Given a Hilbert space $\mathcal{H}$ let $U(\mathcal{H})$ denote the group of invertible unitary transformations $\mathcal{H} \longrightarrow \mathcal{H}$ under composition.

This shows that the group $SO(n+1)$ acts on the set $L^2(S^n, \mathbb{C})$ for $n \in \{1,2\}$ by bijective unitary linear transformations. The action is on the right

$$\begin{aligned} L^2(S^n, \mathbb{C}) \times SO(n+1) &\longrightarrow L^2(S^n, \mathbb{C}) \\ (\psi, g) &\longmapsto \hat{C}_g(\psi) \end{aligned} \tag{7.1}$$

Equivalently, there is a homomorphism of groups

$$\begin{aligned} SO(n+1)^{op} &\longrightarrow U(L^2(S^n, \mathbb{C})) \\ g &\longmapsto \hat{C}_g \end{aligned} \tag{7.2}$$

where for a group G the <u>opposite</u> group G^{op} has operation $g * h = hg$. In summary

$$L^2(S^n, \mathbb{C}) \overset{\text{unitary}}{\circlearrowleft} SO(n+1)$$

<u>Example L3-2</u> Set $u_n = \frac{1}{\sqrt{2\pi}} e^{in\theta} \in L^2(S^1, \mathbb{C})$ and note with $g = R_\alpha \in SO(2)$

$$\begin{aligned} \hat{C}_{R_\alpha}(u_n) &= \frac{1}{\sqrt{2\pi}} e^{in(\theta + \alpha)} \\ &= \frac{1}{\sqrt{2\pi}} e^{in\alpha} e^{in\theta} = e^{in\alpha} u_n \end{aligned}$$

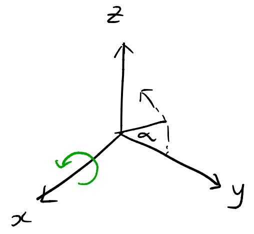
That is, $\{u_n\}_{n \in \mathbb{Z}}$ is an orthonormal dense basis of $L^2(S^1, \mathbb{C})$ consisting of simultaneous eigenvectors for all the $\hat{C}_g$, $g \in SO(2)$.

Basic structure of $SO(3)$

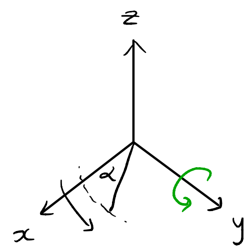
While we more or less understand $SO(2)$ acting on $L^2(S^1;\mathbb{C})$, the case of $SO(3)$ is currently less well-developed. Next we recall the characterisation of $SO(3)$ as a group of rotations.

Def^n (3D rotations) Given $\alpha \in \mathbb{R}$ we define linear transformations

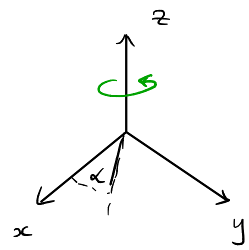
$$R^x_\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}$$



$$R^y_\alpha = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix}$$

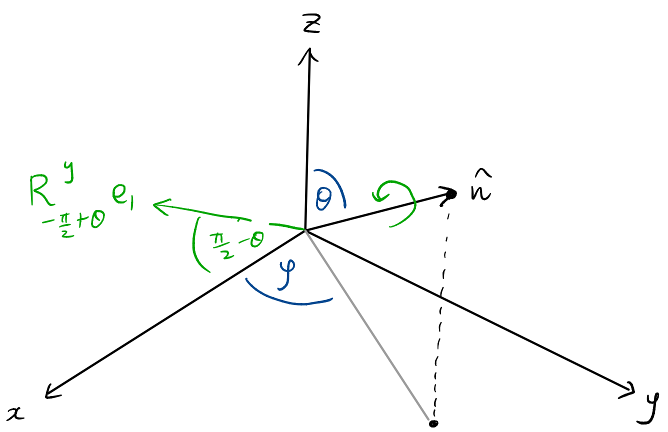


$$R^z_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



Given a unit vector $\hat{n} = R^z_\varphi R^y_{\theta - \frac{\pi}{2}}(e_1)$ for $0 \leq \theta \leq \pi$, $0 \leq \varphi < 2\pi$, we define a linear transformation

$$R^{\hat{n}}_\alpha = R^z_\varphi R^y_{\theta - \frac{\pi}{2}} R^x_\alpha R^y_{\frac{\pi}{2}-\theta} R^z_{-\varphi}$$



Exercise L3-5 Check that $R^{e_1}_\alpha = R^x_\alpha$, $R^{e_2}_\alpha = R^y_\alpha$, $R^{e_3}_\alpha = R^z_\alpha$ and that $R^{\hat{n}}_\alpha \in SO(3)$ for all unit vectors $\hat{n}$.

Exercise L3-6 (i) Prove that the function $S^2 \times [0, 2\pi) \longrightarrow SO(3)$ sending $(\hat{n}, \alpha)$ to $R^{\hat{n}}_\alpha$ is surjective (Hint: characteristic polynomial) and continuous.

(ii) Define $(\hat{n}, \alpha) \sim (\hat{m}, \beta)$ if $R^{\hat{n}}_\alpha = R^{\hat{m}}_\beta$. Give an explicit description of the relation $\sim$ on $S^2 \times [0, 2\pi)$.

Exercise L3-7 Continuing Ex L3-2 let C be as given there with $a<b$ and $c<d$ and consider the restriction map

$$(-)|_C : Cts(S^2, \mathbb{C}) \longrightarrow Cts(C, \mathbb{C}).$$

Prove this is linear and bounded with respect to the L^2-norm $\|-\|_2$ and thus construct a continuous linear extension $(-)|_C : L^2(S^2, \mathbb{C}) \longrightarrow L^2(C, \mathbb{C})$. Use the Riesz representation theorem to prove that $(-)|_C$ admits an adjoint $E : L^2(C, \mathbb{C}) \longrightarrow L^2(S^2, \mathbb{C})$, that is, a continuous linear map satisfying

$$\langle E(\psi), \varphi \rangle = \langle \psi, \varphi|_C \rangle \qquad \forall\, \psi \in L^2(C, \mathbb{C}),\ \varphi \in L^2(S^2, \mathbb{C})$$

Given $\psi \in Cts(C, \mathbb{C})$ give an explicit description of a sequence $\alpha_n \in Cts(S^2, \mathbb{C})$ with $\alpha_n \longrightarrow E(\psi)$ in $L^2(S^2, \mathbb{C})$.

Harmonic polynomials

The analogue of the orthonormal dense basis $\{e^{in\theta}\}_{n\in\mathbb{Z}}$ of $L^2(S^1, \mathbb{C})$ for the sphere are a class of functions known as <u>spherical harmonics</u>. We will construct these functions as restrictions to S^2 of harmonic polynomial functions on $\mathbb{R}^3$. In the following n is an integer $n \geqslant 1$.

<u>Def</u>

Def^n Let $P(n)$ denote the $\mathbb{C}$-vectorspace of <u>polynomials</u> in n variables $x_1, \ldots, x_n$ with complex coefficients. We denote by $P_k(n)$ the subspace of polynomials homogeneous of degree k so $P(n) = \bigoplus_{k \geqslant 0} P_k(n)$.

Example L3-3 $P_0(n) = \mathbb{C}\,1$, $P_1(n) = \mathbb{C}x_1 \oplus \cdots \oplus \mathbb{C}x_n$, $P_2(2) = \mathbb{C}x_1^2 \oplus \mathbb{C}x_1x_2 \oplus \mathbb{C}x_2^2$.

Exercise L3-8 Each $f(x_1, \ldots, x_n) \in P(n)$ determines a function $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ and we let $P'(n) \subseteq Cts(\mathbb{R}^n, \mathbb{C})$ denote the $\mathbb{C}$-linear subspace of such <u>polynomial functions</u>. Prove that the map $P(n) \longrightarrow P'(n)$ sending a polynomial to its function is an isomorphism of $\mathbb{C}$-vector spaces.

Exercise L3-9 Prove that $\dim_{\mathbb{C}} P_k(n) = \binom{n+k-1}{k}$.

Some notation: α, β will stand for <u>multi-indices</u>, that is, elements of $\mathbb{N}^n$ and [of course $0 \in \mathbb{N}$] $|\alpha|$ means $\alpha_1 + \cdots + \alpha_n$, x^α means $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. We write $\sum_\alpha$ for $\sum_{\alpha \in \mathbb{N}^n}$.

Def^n Given $P \in P(n)$ with $P = \sum_\alpha c_\alpha x^\alpha$ we define a $\mathbb{C}$-linear map $\partial(P) : P(n) \longrightarrow P(n)$ by

$$\partial(P) = \sum_\alpha c_\alpha \frac{\partial^k}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \qquad (10.1)$$

Example L3-4 $\partial(x_1^2 + \cdots + x_n^2)$ is the <u>Laplacian</u> $\Delta = \dfrac{\partial^2}{\partial x_1^2} + \cdots + \dfrac{\partial^2}{\partial x_n^2}$.

Given $P = \sum_{|\alpha|=k} c_\alpha x^\alpha$ we denote by $\overline{P}$ the polynomial $\sum_{|\alpha|=k} \overline{c_\alpha} x^\alpha$.

<u>Lemma L3-2</u> For $k \geq 0$ the $\mathbb{C}$-vector space $\mathcal{P}_k(n)$ is a Hilbert space with

$$\langle P, Q \rangle = \left[\partial(Q) \overline{P} \right]_{const} \qquad (11.1)$$

<u>Proof</u> Define a pairing $\langle -,- \rangle'$ on $\mathcal{P}_k(n)$ by

meaning $\alpha_1! \cdots \alpha_n!$

$$\Big\langle \sum_{|\alpha|=k} a_\alpha x^\alpha, \sum_{|\alpha|=k} b_\alpha x^\alpha \Big\rangle' = \sum_{|\alpha|=k} \alpha! \, \overline{a}_\alpha b_\alpha \qquad (11.2)$$

This is just the standard Hilbert space structure on $\mathbb{C}^{\dim \mathcal{P}_k(n)}$ (scaled by factors of $\alpha!$ but these do not change that the pairing defines a Hilbert space). We claim that $\langle P, Q \rangle = \langle P, Q \rangle'$ for all $P, Q \in \mathcal{P}_k(n)$. By construction $\langle -,- \rangle$ is linear in Q and conjugate linear in P, so it suffices to prove this for $P = x^\alpha$ and $Q = x^\beta$. But $\langle x^\alpha, x^\beta \rangle = 0$ if $\alpha - \beta$ has any negative entries, and if $\alpha_i \geq \beta_i$ for $1 \leq i \leq n$ then

$$\begin{aligned}
\langle x^\alpha, x^\beta \rangle &= \left[\frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}} \left(x_1^{\alpha_1} \cdots x_n^{\alpha_n} \right) \right]_{const} \\
&= \left[\frac{\alpha_1!}{(\alpha_1 - \beta_1)!} \cdots \frac{\alpha_n!}{(\alpha_n - \beta_n)!} x_1^{\alpha_1 - \beta_1} \cdots x_n^{\alpha_n - \beta_n} \right]_{const} \\
&= \left[\frac{\alpha!}{(\alpha - \beta)!} x^{\alpha - \beta} \right]_{const} \\
&= \alpha! \, \delta_{\alpha = \beta} = \langle x^\alpha, x^\beta \rangle'. \quad \square
\end{aligned}$$

<u>Exercise L3-10</u> For $P, Q \in \mathcal{P}(n)$ show that $\partial(PQ) = \partial(P) \circ \partial(Q)$ as linear operators.

We sometimes write $\langle -,- \rangle_k$ to indicate the pairing on $\mathcal{P}_k(n)$.

Multiplication by Q is adjoint to operation by $\partial(\bar{Q})$:

Lemma L3-3 Given $P \in \mathcal{P}_k(n)$, $Q \in \mathcal{P}_\ell(n)$, $R \in \mathcal{P}_m(n)$ with $\ell + m = k$

$$\langle P, QR \rangle = \langle \partial(\bar{Q})P, R \rangle \tag{12.1}$$

Proof Using Ex L3-10 and observing $\overline{\partial(Q)P} = \partial(\bar{Q})\bar{P}$

$$\begin{aligned} \langle P, QR \rangle &= \langle P, RQ \rangle \\ &= [\, \partial(RQ)\bar{P} \,]_{const} \\ &= [\, \partial(R)(\partial(Q)\bar{P}) \,]_{const} \\ &= \langle \partial(\bar{Q})P, R \rangle. \quad \square \end{aligned}$$

In particular if $P \in \mathcal{P}_k(n)$ and $R \in \mathcal{P}_{k-2}(n)$ we have

$$\langle P, (x_1^2 + \cdots + x_n^2)R \rangle = \langle \Delta P, R \rangle \tag{12.2}$$

We write $\|x\|^2$ for $x_1^2 + \cdots + x_n^2$.

Def^{n} The space of complex harmonic polynomials of degree k is

$$\mathcal{H}_k(n) = \{\, P \in \mathcal{P}_k(n) \mid \Delta P = 0 \,\} \tag{12.3}$$

Example L3-5 The following polynomials are harmonic when $n = 3$, $(k=0)$ 1, $(k=1)$ x, y, z $(k=2)$ $x^2 - y^2$, xy, yz, yz $(k=3)$ $3y^2z - z^3$, xyz

This is clearly a $\mathbb{C}$-vector subspace, but it is not clear yet what its dimension is, or why we should be particularly concerned with this class of polynomials. But as we will see, these polynomial functions are dense in $L^2(S^2, \mathbb{C})$!

Theorem L3-4 The map $\Delta : \mathcal{P}_k(n) \longrightarrow \mathcal{P}_{k-2}(n)$ is surjective for all $n, k \geq 2$ and

$$\mathcal{P}_k(n) = \mathcal{H}_k(n) \oplus \|x\|^2 \mathcal{P}_{k-2}(n) \tag{13.1}$$

an internal direct sum

Proof Note that if $k<2$ then vacuously $\mathcal{P}_k(n) = \mathcal{H}_k(n)$ so in a sense (13.1) also holds in these cases. The subspace $\mathcal{H}_k(n) = \operatorname{Ker}\Delta$ is closed (in a finite-dimensional Hilbert space every linear subspace is closed) as is $\operatorname{Im}\Delta \subseteq \mathcal{P}_{k-2}(n)$. Hence by [MHS, Lemma L20-7]

$$\begin{aligned} \mathcal{P}_k(n) &= \mathcal{H}_k(n) \oplus \mathcal{H}_k(n)^\perp \\ \mathcal{P}_{k-2}(n) &= \operatorname{Im}\Delta \oplus \operatorname{Im}\Delta^\perp \end{aligned} \tag{13.2}$$

But if $Q \in \operatorname{Im}\Delta^\perp$ then by (12.2)

$$\langle \|x\|^2 Q, \|x\|^2 Q \rangle = \langle \Delta \|x\|^2 Q, Q \rangle = 0 \tag{13.3}$$

hence $\|x\|^2 Q = 0$ in $\mathcal{P}_k(n)$ whence $Q = 0$ in $\mathcal{P}_{k-2}(n)$. So $\operatorname{Im}\Delta^\perp = 0$ and hence $\operatorname{Im}\Delta = \mathcal{P}_{k-2}(n)$, proving that Δ is surjective. We have used that multiplication by $\|x\|^2$ is injective as a map $\mathcal{P}_{k-2}(n) \longrightarrow \mathcal{P}_k(n)$ which is easily checked.

To prove (13.1) we observe that $\|x\|^2 \mathcal{P}_{k-2}(n) \subseteq \mathcal{H}_k(n)^\perp$ by (12.2). and since Δ is surjective

$$\dim \mathcal{P}_k(n) = \dim \mathcal{P}_{k-2}(n) + \dim \mathcal{H}_k(n) \tag{13.4}$$

while from (13.2)

$$\dim \mathcal{P}_k(n) = \dim \mathcal{H}_k(n) + \dim \mathcal{H}_k(n)^\perp. \tag{13.5}$$

Arithmetic gives $\dim \mathcal{P}_{k-2}(n) = \dim \mathcal{H}_k(n)^\perp$ and injectivity of $\|x\|^2(-)$ implies that $\|x\|^2 \mathcal{P}_{k-2}(n) = \mathcal{H}_k(n)^\perp$ as claimed. $\square$

<u>Corollary L3-5</u> Let $k = 2a + b$ where $a, b \in \mathbb{N}$ and $b \in \{0,1\}$. Then for $n, k \geq 2$

$$\mathcal{P}_k(n) = \mathcal{H}_k(n) \oplus \|x\|^2 \mathcal{H}_{k-2}(n) \oplus \cdots \oplus \|x\|^{2a} \mathcal{H}_b(n) \qquad (14.1)$$

<u>Proof</u> By induction on k with n fixed. The base cases are $k = 1$ or $k = 2$ which both follow from the Theorem. In the case $k = 1$ we read (14.1) as $\mathcal{P}_1(n) = \mathcal{H}_1(n)$. Suppose (14.1) holds for all integers $\leq k$ and use the Theorem to write

$$\mathcal{P}_{k+1}(n) = \mathcal{H}_{k+1}(n) \oplus \|x\|^2 \mathcal{P}_{k-1}(n).$$

By hypothesis $\mathcal{P}_{k-1}(n) = \mathcal{H}_{k-1}(n) \oplus \|x\|^2 \mathcal{H}_{k-3}(n) \oplus \cdots \oplus \|x\|^{2a'} \mathcal{H}_{b'}(n)$ where $k-1 = 2a' + b'$ and $b' \in \{0,1\}$. Then it follows

$$\mathcal{P}_{k+1}(n) = \mathcal{H}_{k+1}(n) \oplus \|x\|^2 \mathcal{H}_{k-1}(n) \oplus \cdots \oplus \|x\|^{2(a'+1)} \mathcal{H}_{b'}(n)$$

and $k+1 = 2(a'+1) + b'$. $\square$

Now for the magic! Notice that $\|x\| = 1$ on S^{n-1} so (14.1) says that when you restrict <u>any</u> polynomial function to the sphere it is a sum of harmonic polynomials. More carefully by Exercise L3-8 we have a $\mathbb{C}$-linear map

$$\mathcal{P}_k(n) \hookrightarrow \mathrm{Cts}(\mathbb{R}^n, \mathbb{C}) \xrightarrow{\text{restriction}} \mathrm{Cts}(S^{n-1}, \mathbb{C}) \qquad (14.2)$$

<u>Lemma L3-6</u> The map (14.2) is injective.

<u>Proof</u> If $P \in \mathcal{P}_k(n)$ then as a polynomial function it is easy to see that for $x \in \mathbb{R}^n$, $\lambda \in \mathbb{C}$ we have $P(\lambda x) = \lambda^k P(x)$ and hence if $P|_{S^{n-1}} = Q|_{S^{n-1}}$ we have for $x \neq 0$ $P(x) = P(\|x\| \cdot \frac{1}{\|x\|} x) = \|x\|^k P(\frac{x}{\|x\|}) = \|x\|^k Q(\frac{x}{\|x\|}) = Q(x)$. If $k = 0$ the claim is vacuous and if $k > 0$ then also $P(0) = Q(0) = 0$ so we're done. $\square$

<u>Exercise L3-11</u> Check that the topology on $\mathcal{P}_k(n)$ associated to the pairing $\langle -,- \rangle$ is just the topology on $\mathbb{C}^{\dim \mathcal{P}_k(n)}$ using the basis $\{x^\alpha\}_{|\alpha|=k}$ and that the map $\mathcal{P}_k(n) \longrightarrow \mathrm{Cts}(\mathbb{R}^n, \mathbb{C})$ is a homeomorphism onto its image.

<u>Def^n</u> A <u>degree k spherical harmonic</u> on S^{n-1} is a continuous function $f: S^{n-1} \longrightarrow \mathbb{C}$ in the image of the map $(k \geqslant 0, n \geqslant 2)$

$$\mathcal{H}_k(n) \hookrightarrow \mathrm{Cts}(\mathbb{R}^n, \mathbb{C}) \xrightarrow{\text{restrict}} \mathrm{Cts}(S^{n-1}, \mathbb{C}),$$

that is, f is a restriction of a harmonic polynomial function of degree k. The subspace of all spherical harmonics of degree k in $\mathrm{Cts}(S^{n-1}, \mathbb{C})$ is denoted $\mathcal{H}_k(S^{n-1})$.

<u>Lemma L3-7</u> The induced map $\mathcal{H}_k(n) \longrightarrow \mathcal{H}_k(S^{n-1})$ is an isomorphism of vector spaces.

<u>Proof</u> It is surjective by definition, and injective by Lemma L3-6. $\square$

<u>Theorem L3-8</u> The $\mathbb{C}$-linear span of $\bigcup_{k \geqslant 0} \mathcal{H}_k(S^{n-1})$ is dense in $\mathrm{Cts}(S^{n-1}, \mathbb{C})$, with respect to the $\|-\|_\infty$ norm.

<u>Proof</u> Let $\mathrm{Poly}(S^{n-1}, \mathbb{C})$ denote the restriction to S^{n-1} of polynomial functions on $\mathbb{R}^n$. By Stone-Weierstrass [MHS, Corollary L16-4] this subset is dense in $\mathrm{Cts}(S^{n-1}, \mathbb{C})$ in the compact-open topology (i.e. the topology from the $\|-\|_\infty$ norm). See [MHS, Lemma L21-1] for the difference between $\mathbb{R}$- and $\mathbb{C}$-valued functions.

But if $f \in \mathrm{Poly}(S^{n-1}, \mathbb{C})$ is the restriction of $P \in \mathcal{P}(n)$ then writing $P = \sum_k P_k$ as a sum of its homogeneous components $P_k \in \mathcal{P}_k(n)$ we can by Corollary L3-5 write P_k (uniquely) as a sum

$$P_k = H_k + \|x\|^2 H_{k-2} + \|x\|^4 H_{k-4} + \cdots + \|x\|^{2a} H_b \qquad (16.1)$$

where H_i are harmonic polynomials of degree i. But realising these as functions on $\mathbb{R}^n$ and restricting to $S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$ leads to

$$P_k|_{S^{n-1}} = H_k|_{S^{n-1}} + H_{k-2}|_{S^{n-1}} + \cdots + H_b|_{S^{n-1}} \qquad (16.2).$$

from which the claim follows. $\square$

Put differently, every continuous $\mathbb{C}$-valued function on S^{n-1} can be arbitrarily well-approximated in the sense of $\|-\|_\infty$ distance by <u>a sum of spherical harmonics</u>. Remarkable! In particular we have found a natural dense subset of the Hilbert space of the sphere:

<u>Corollary L3-9</u> The $\mathbb{C}$-linear span of $\bigcup_{k\geq 0} \mathcal{H}_k(S^2)$ is dense in $L^2(S^2, \mathbb{C})$.

<u>Proof</u> The span is dense in $\mathrm{Cts}(S^2, \mathbb{C})$ with respect also to the $\|-\|_2$-norm (see the technique of [MHS, Lemma L21-2]) and since $\mathrm{Cts}(S^2, \mathbb{C})$ is dense in $L^2(S^2, \mathbb{C})$ by construction this completes the proof. $\square$

<u>Example L3-6</u> Consider the harmonic polynomials $x^2 - y^2$ and xyz of Example L3-5. In spherical coordinates (1.3) with $x_1 = x$, $x_2 = y$, $x_3 = z$

$$(x^2 - y^2)|_{S^2} = \sin^2\theta \cos^2\varphi - \sin^2\theta \sin^2\varphi = \sin^2\theta \cos(2\varphi)$$
$$(xyz)|_{S^2} = \sin^2\theta \sin\varphi \cos\varphi \cos\theta$$

<u>Exercise L3-12</u> Prove that for $k, n \geq 2$ $\dim \mathcal{H}_k(n) = \binom{n+k-1}{k} - \binom{n+k-3}{k-2}$.

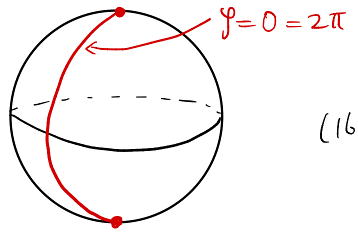
Next we want to argue that $\mathcal{H}_k(S^2)$ is <u>orthogonal</u> to $\mathcal{H}_\ell(S^2)$ if $k \neq \ell$, which we can then use to construct an orthonormal dense basis of spherical harmonics for $L^2(S^2, \mathbb{C})$.

Smooth functions on the sphere

(results and exercises here are labelled "B", as this section can be skipped if you know differential geometry)

Next we want consider a differential operator, the Laplacian, acting on functions on the sphere. That means we need to decide what the derivative of such a function is. Using spherical coordinates this might seem straightforward: we have from (1.3) a surjective continuous map

$$
\begin{aligned}
[0,\pi]\times[0,2\pi) &\xrightarrow{\ j\ } S^2 \\
(\theta,\varphi) &\longmapsto (\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta)
\end{aligned}
$$



(16.1.1)

This of course not injective: $j(0,\varphi)=(0,0,1)$ and $j(\pi,\varphi)=(0,0,-1)$ for all φ, but j is injective when restricted to $(0,\pi)\times(0,2\pi)$, the image of which is the complement $U\subseteq S^2$ of the red line in the above diagram.

<u>Def</u>

Def^{n} A continuous function $f: S^2\to\mathbb{R}$ is <u>j-smooth</u> if

$$
(0,\pi)\times(0,2\pi)\xrightarrow{\ j\ } S^2\xrightarrow{\ f\ }\mathbb{R} \qquad (16.1.2)
$$

is smooth in the usual sense, that is the derivatives $\dfrac{\partial^{a+b}}{\partial\theta^a\partial\varphi^b}(f\circ j)$ exist for all $a,b\geqslant 0$.

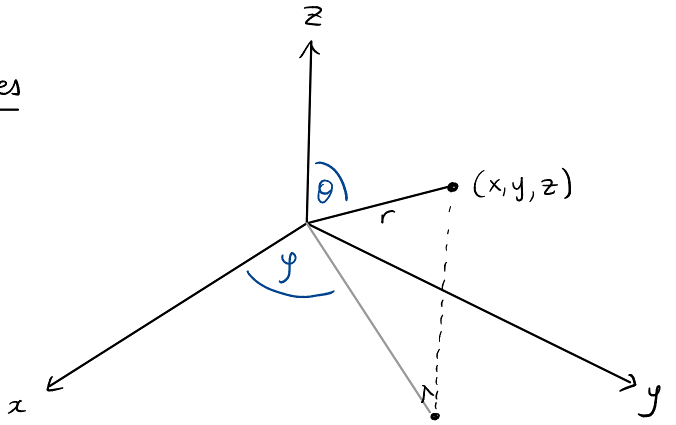
Note that a function can be j-smooth but "behave badly" across the line $\varphi=0$ on the sphere, by having for example

$$
\lim_{h\to 0}\frac{f(\frac{\pi}{2},h)-f(\frac{\pi}{2},0)}{h}\neq\lim_{h\to 0}\frac{f(\frac{\pi}{2},0)-f(\frac{\pi}{2},2\pi-h)}{h}
$$

In this case $\frac{\partial f}{\partial\varphi}$ may exist as a function on $(0,\pi)\times(0,2\pi)$ and thus U, but it may not be periodic in the sense that it extends to a continuous function on all of S^2. This goes to show that <u>smoothness on U is not enough</u> to define smoothness on S^2. The solution is to consider both U and another "coordinate chart" of the same kind.

<u>spherical coordinates</u>

$x = r\sin\theta\cos\varphi$
$y = r\sin\theta\sin\varphi$
$z = r\cos\theta$

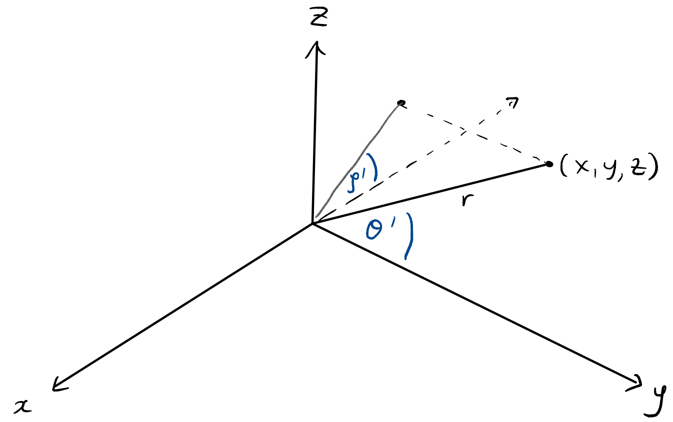


$0 \le \theta \le \pi$
$0 \le \varphi < 2\pi$

(16.2.1)

<u>alt spherical coordinates</u>

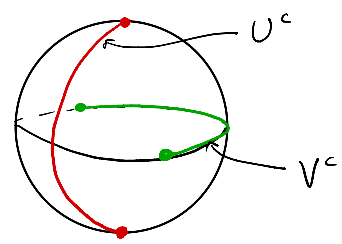
$x = -r\sin\theta'\cos\varphi'$
$y = r\cos\theta'$
$z = r\sin\theta'\sin\varphi'$



$0 \le \theta' \le \pi$
$0 \le \varphi' < 2\pi$

We have two homeomorphisms associated to these coordinates j, j^{alt} as defined below and the open sets $U := Im(j)$, $V = Im(j^{alt})$ cover S^2 (that is, $S^2 = U \cup V$).

$$(0,\pi)\times(0,2\pi) \xrightarrow{j} S^2$$
$$(\theta,\varphi) \longmapsto (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$



$$(0,\pi)\times(0,2\pi) \xrightarrow{j^{alt}} S^2$$
$$(\theta',\varphi') \longmapsto (-\sin\theta'\cos\varphi', \cos\theta', \sin\theta'\sin\varphi') \qquad (16.2.2)$$

We say $f: S^2 \to \mathbb{R}$ is <u>j^{alt}-smooth</u> if $f\circ j^{alt}$ is smooth (i.e. $\frac{\partial^{a+b}}{\partial\theta'^a\partial\varphi'^b}(f\circ j^{alt})$ exist for all $a,b \geq 0$. Finally:

<u>Defn</u> A continuous function $f: S^2 \to \mathbb{R}$ is <u>smooth</u> if both $f\circ j$ and $f\circ j^{alt}$ are smooth functions on $(0,\pi)\times(0,2\pi)$. A complex-valued function on S^2 is smooth if its real and imaginary parts are both smooth.

Note that on the overlap $U \cap V$ we have two sets of "competing" coordinates θ, φ and θ', φ' where we view $\theta = \theta(p)$, $\varphi = \varphi(p)$ as functions of $p \in U$ (resp. θ', φ' and V) using j^{-1} (resp. $(j^{alt})^{-1}$) so that $(\theta(p), \varphi(p)) := j^{-1}(p)$, $(\theta'(p), \varphi'(p)) = (j^{alt})^{-1}(p)$.

How do we express (θ, φ) in terms of (θ', φ') as functions on $U \cap V$?

First recall that

$\arccos : (-1,1) \longrightarrow (0, \pi)$

$$\underset{\theta \qquad \varphi}{(0,\pi)\times(0,2\pi)} \underset{}{\overset{j}{\rightleftarrows}} U$$

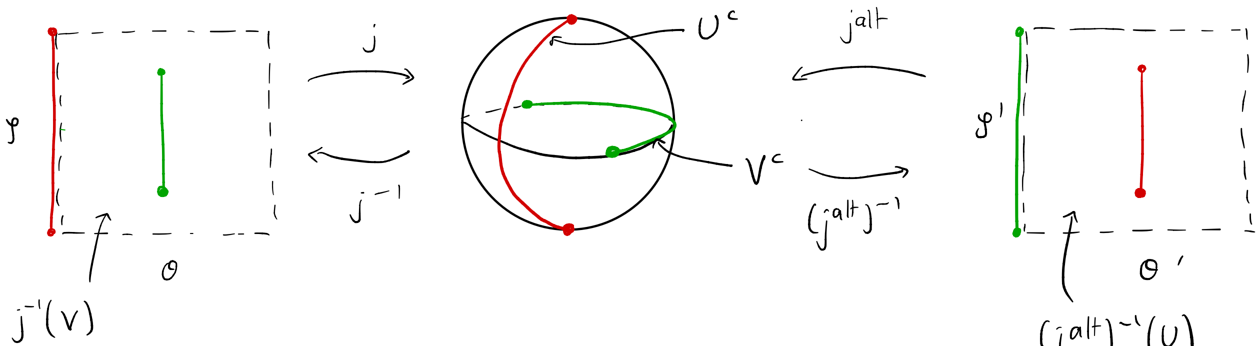
$$j^{-1}(x,y,z) = \big(\arccos(z),\ \arg(x+iy)\big)$$

(16.3.1)

$$\underset{\theta' \qquad \varphi'}{(0,\pi)\times(0,2\pi)} \overset{j^{alt}}{\rightleftarrows} V$$

$$(j^{alt})^{-1}(x,y,z) = \big(\arccos(y),\ \arg(-x+iz)\big)$$

Hence we have (16.3.2)

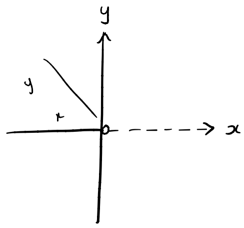


$$\begin{tikzcd} j^{-1}(V) \arrow[r, shift left, "j"] & U\cap V \arrow[l, shift left, "j^{-1}"] \arrow[r, shift right, "(j^{alt})^{-1}"'] & (j^{alt})^{-1}(U) \arrow[l, shift right, "j^{alt}"'] \end{tikzcd}$$

$$(j^{alt})^{-1} j(\theta,\varphi) = \big(\arccos(\sin\theta\sin\varphi),\ \arg(-\sin\theta\cos\varphi + i\cos\theta)\big)$$
$$j^{-1} j^{alt}(\theta',\varphi') = \big(\arccos(\sin\theta'\sin\varphi'),\ \arg(-\sin\theta'\cos\varphi' + i\cos\theta')\big) \qquad (16.3.3)$$

The function $\arg(x+iy)$ can be expressed using the inverse $\arctan : (-\infty,\infty) \longrightarrow (-\frac{\pi}{2},\frac{\pi}{2})$ as a function on $\mathbb{R}^2 \setminus \{(x,0) \mid x \geqslant 0\}$

$$\arg(x+iy) = \begin{cases} \arctan(y/x) & x>0 \\ \frac{\pi}{2} - \arctan(x/y) & y>0 \\ -\frac{\pi}{2} - \arctan(x/y) & y<0 \\ \arctan(y/x)+\pi & x<0, y \geqslant 0 \\ \arctan(y/x) - \pi & x<0, y<0 \end{cases}$$



This is a smooth function since $\arctan$ is, and $\nabla \arg(x+iy) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$. Hence (16.3.3) is two smooth maps, mutually inverse, between $j^{-1}(V)$ and $(j^{alt})^{-1}(U)$. That is, the <u>change of coordinates</u>

$$\theta' = \arccos(\sin\theta \sin\varphi), \quad \varphi' = \arg(-\sin\theta\cos\varphi + i\cos\theta) \qquad (16.4.1)$$

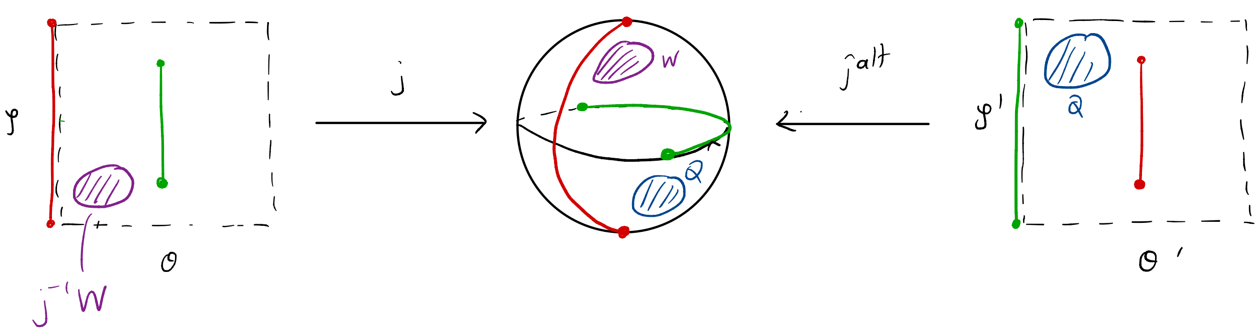
is smooth as a function of θ, φ in the sense that both expressions are smooth on $(0,\pi)\times(0,2\pi)$. These two sets of coordinates give (in principle) two "different" ways to define differentiation on S^2.

<u>**Def^n**</u> For $T \subseteq (0,\pi)\times(0,2\pi)$ open let $C^\infty(T)$ denote the $\mathbb{R}$-linear subspace of $Cts(T,\mathbb{R})$ consisting of smooth functions. For $W \subseteq U$ and $Q \subseteq V$ open we define

$$C^\infty_j(W) = \{ f \in Cts(W,\mathbb{R}) \mid f \circ j \in C^\infty(j^{-1}W) \}$$ smooth according to j

$$C^\infty_{j^{alt}}(Q) = \{ f \in Cts(Q,\mathbb{R}) \mid f \circ j^{alt} \in C^\infty({j^{alt}}^{-1} Q) \}$$ smooth according to j^{alt}

These are $\mathbb{R}$-linear subspaces of $Cts(W,\mathbb{R})$ and $Cts(Q,\mathbb{R})$ respectively.



On the overlap the two coordinate systems agree on which functions are smooth:

Lemma L3B-1 If $W \subseteq U \cap V$ is open then $C^\infty_j(W) = C^\infty_{j^{alt}}(W)$.

Proof Consider the diagram obtained by restricting (16.3.3)

$$\begin{tikzcd}
j^{-1}(W) \arrow[r, shift left, "j"] \arrow[dr, bend right, "f\circ j"'] & W \arrow[l, shift left, "j^{-1}"] \arrow[r, shift right, "(j^{alt})^{-1}"'] \arrow[d, "f"] & (j^{alt})^{-1}(W) \arrow[l, shift right, "j^{alt}"'] \arrow[dl, bend left, "f\circ j^{alt}"] \\
& \mathbb{R} &
\end{tikzcd}$$

Given $f \in Cts(W, \mathbb{R})$ we have (since $(j^{alt})^{-1} \circ j$, $j^{-1} \circ j^{alt}$ are smooth, and by the chain rule the composites of smooth functions between open subsets of $\mathbb{R}^2$ are smooth)

$$\begin{aligned}
f \in C^\infty_j(W) &\iff f \circ j \in C^\infty(j^{-1}W) \\
&\iff (f \circ j) \circ (j^{-1} \circ j^{alt}) \in C^\infty((j^{alt})^{-1}W) \\
&\iff f \circ j^{alt} \in C^\infty((j^{alt})^{-1}W) \\
&\iff f \in C^\infty_{j^{alt}}(W). \quad \square
\end{aligned}$$

So finally we can define which continuous functions count as smooth, on any open subset of S^2, using both coordinate charts in tandem:

Def^n Given $W \subseteq S^2$ open we say a continuous function $f: S^2 \longrightarrow \mathbb{R}$ is <u>smooth</u> if $f|_{U \cap W} \in C^\infty_j(U \cap W)$ and $f|_{V \cap W} \in C^\infty_{j^{alt}}(V \cap W)$. The set of all smooth functions is denoted $C^\infty(W) \subseteq Cts(W, \mathbb{R})$.

Exercise L3B−1 (a) Prove that $C^\infty(W) \subseteq \mathrm{Cts}(X,\mathbb{R})$ is a $\mathbb{R}$-linear subspace

(b) Prove that if $W \subseteq U$ then $C^\infty(W) = C^\infty_j(W)$ and if $W \subseteq V$ then $C^\infty(W) = C^\infty_{j^{alt}}(W)$.

The next exercise shows that $C^\infty(-)$ is a <u>sheaf</u> on S^2 (we will discuss sheaves later).

Exercise L3B−2 (a) Prove that if $W' \subseteq W$ is open and $f \in C^\infty(W)$ then $f|_{W'} \in C^\infty(W')$.

(b) Prove that if $W \subseteq S^2$ is open and $\{W_\alpha\}_{\alpha\in\Lambda}$ is an open cover of W (that is, for every $\alpha \in \Lambda$ W_α is an open subset of W and $\bigcup_{\alpha\in\Lambda} W_\alpha = W$) and $\{f_\alpha\}_{\alpha\in\Lambda}$ is a family of functions $f_\alpha \in C^\infty(W_\alpha)$ such that $f_\alpha|_{W_\alpha\cap W_\beta} = f_\beta|_{W_\alpha \cap W_\beta}$ for all $\alpha,\beta\in\Lambda$ then there exists a <u>unique</u> $f \in C^\infty(W)$ such that $f|_{W_\alpha} = f_\alpha$ for all $\alpha \in \Lambda$.

(<u>Note</u>: the empty cover is a cover of ϕ and yields $C^\infty(\phi) = \{*\}$).

<u>Differential operators on the sphere</u>

Now that we have defined the sheaf of smooth functions $W \mapsto C^\infty(W)$ on S^2 we can define operators $\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\theta'}, \frac{\partial}{\partial\varphi'}$. These obviously depend on the chosen coordinates but certain combinations (such as the Laplacian) do not. Recall $U = \mathrm{Im}(j)$, $V = \mathrm{Im}(j^{alt})$.

<u>Def</u>^{n} Given $W \subseteq U$ we define an $\mathbb{R}$-linear operator $\frac{\partial}{\partial\theta} : C^\infty(W) \longrightarrow C^\infty(W)$ by

$$C^\infty(W) \xrightarrow[(-)\circ j]{\cong} C^\infty(j^{-1}W) \xrightarrow[\frac{\partial}{\partial\theta}]{} C^\infty(j^{-1}W) \xrightarrow[(-)\circ j^{-1}]{\cong} C^\infty(W)$$

where the outer maps are $f \mapsto f\circ j$ and $g \mapsto g \circ j^{-1}$ respectively. Similarly we define $\partial/\partial\varphi$ and using j^{alt}, V in place of j, U we define $\frac{\partial}{\partial\theta'}, \frac{\partial}{\partial\varphi'}$ on $C^\infty(W)$ for any open subset $W \subseteq V$.

How can we extend these operators to arbitrary open subsets $W \subseteq S^2$?

Lemma L3B-2 If $W \subseteq U \cap V$ is open then for $f \in C^\infty(W)$ we have

$$\frac{\partial f}{\partial \theta} = \frac{-\cos\theta \sin\varphi}{\sqrt{1-\sin^2\theta\sin^2\varphi}}\frac{\partial f}{\partial \theta'} + \frac{\cos\varphi}{\cot^2\theta + \cos^2\varphi}\frac{\partial f}{\partial \varphi'}$$

$$\frac{\partial f}{\partial \varphi} = \frac{-\sin\theta\cos\varphi}{\sqrt{1-\sin^2\theta\sin^2\varphi}}\frac{\partial f}{\partial \theta'} - \frac{\cot\theta\sin\varphi}{\cos^2\varphi + \cot^2\theta}\frac{\partial f}{\partial \varphi'} \qquad (16.7.1)$$

as elements of $C^\infty(W)$.

Proof By definition $\frac{\partial}{\partial\theta} f$ is the top row in the following commutative diagram

$$\begin{tikzcd}
C^\infty(W) \arrow[r, "(-)\circ j"] \arrow[d, "(-)\circ j^{alt}"'] & C^\infty(j^{-1}W) \arrow[r, "\frac{\partial}{\partial\theta}"] & C^\infty(j^{-1}W) \arrow[r, "(-)\circ j^{-1}"] \arrow[dr, "(-)\circ j^{-1} j^{alt}"'] & C^\infty(W) \\
C^\infty(j^{alt-1}W) \arrow[ur, "(-)\circ j^{alt-1} j"'] \arrow[rrr, bend right, "\Psi"] & & & C^\infty(j^{alt-1}W) \arrow[u, "(-)\circ j^{alt-1}"']
\end{tikzcd} \qquad (16.7.2)$$

where by (16.3.3) the map Ψ is defined for $g = g(\theta', \varphi') \in C^\infty(j^{alt-1}W)$ by

$$\Psi(g) = \frac{\partial}{\partial\theta}\left[g \circ j^{alt-1} j \right] \circ j^{-1} j^{alt}$$

$$= \frac{\partial}{\partial\theta}\Big[g\Big(\overbrace{\arccos(\sin\theta\sin\varphi)}^{\theta'}, \overbrace{\arg(-\sin\theta\cos\varphi + i\cos\theta)}^{\varphi'} \Big)\Big] \circ j^{-1} j^{alt}$$

chain rule
$$= \left[\frac{\partial g}{\partial \theta'}\frac{\partial \theta'}{\partial\theta} + \frac{\partial g}{\partial\varphi'}\frac{\partial\varphi'}{\partial\theta} \right] \circ j^{-1} j^{alt} \qquad \frac{\partial}{\partial u}\arccos(u) = \frac{-1}{\sqrt{1-u^2}}$$

Now with $u = \sin\theta\sin\varphi$, and $a = -\sin\theta\cos\varphi$, $b = \cos\theta$

$$\frac{\partial\theta'}{\partial\theta} = \frac{-1}{\sqrt{1-u^2}}\frac{\partial}{\partial\theta}(u) = \frac{-\cos\theta\sin\varphi}{\sqrt{1-\sin^2\theta\sin^2\varphi}} \qquad (16.7.3)$$

$$\frac{\partial \varphi'}{\partial \theta} = \frac{\partial}{\partial a}\arg(a+ib)\frac{\partial a}{\partial \theta} + \frac{\partial}{\partial b}\arg(a+ib)\frac{\partial b}{\partial \theta}$$

$0<\theta<\pi \quad \therefore \sin\theta \neq 0$

$$= \frac{-b}{a^2+b^2}\left[-\cos\theta\cos\varphi\right] + \frac{a}{a^2+b^2}\left[-\sin\theta\right]$$

$$= \frac{\cos^2\theta\cos\varphi}{\cos^2\theta+\sin^2\theta\cos^2\varphi} + \frac{\sin^2\theta\cos\varphi}{\cos^2\theta+\sin^2\theta\cos^2\varphi} = \frac{\cos\varphi}{\cot^2\theta+\cos^2\varphi}$$

Hence

$$\Psi(g) = \left[\frac{-\cos\theta\sin\varphi}{\sqrt{1-\sin^2\theta\sin^2\varphi}}\frac{\partial g}{\partial \theta'} + \frac{\cos\varphi}{\cot^2\theta+\cos^2\varphi}\frac{\partial g}{\partial \varphi'}\right]\circ j^{-1}j^{\text{alt}}$$

and so referring to (16.7.2), given $f: W \longrightarrow \mathbb{R}$ smooth and writing $f(\theta',\varphi')$ for $f\circ j^{\text{alt}}$

$$\frac{\partial f}{\partial \theta} = \Psi(f(\theta',\varphi'))\circ j^{\text{alt}^{-1}}$$

$$= \frac{-\cos\theta\sin\varphi}{\sqrt{1-\sin^2\theta\sin^2\varphi}}\frac{\partial f}{\partial \theta'} + \frac{\cos\varphi}{\cot^2\theta+\cos^2\varphi}\frac{\partial f}{\partial \varphi'}$$

as claimed.

Let Φ fit in a diagram like (16.7.2) but with $\frac{\partial}{\partial \varphi}$ replacing $\frac{\partial}{\partial \theta}$. Then

$$\Phi(g) = \frac{\partial}{\partial \varphi}\left[g\circ j^{\text{alt}^{-1}}j\right]\circ j^{-1}j^{\text{alt}}$$

$$= \frac{\partial}{\partial \varphi}\left[g\left(\overbrace{\arccos(\sin\theta\sin\varphi)}^{\theta'}, \overbrace{\arg(-\sin\theta\cos\varphi+i\cos\theta)}^{\varphi'}\right)\right]\circ j^{-1}j^{\text{alt}}$$

chain rule

$$= \left[\frac{\partial g}{\partial \theta'}\frac{\partial \theta'}{\partial \varphi} + \frac{\partial g}{\partial \varphi'}\frac{\partial \varphi'}{\partial \varphi}\right]\circ j^{-1}j^{\text{alt}}$$

Now with $u = \sin\theta\sin\varphi$ and $a = -\sin\theta\cos\varphi$, $b=\cos\theta$

$$\frac{\partial \theta'}{\partial \varphi} = \frac{\partial}{\partial u}\arccos(u)\frac{\partial u}{\partial \varphi} = \frac{-1}{\sqrt{1-u^2}}\sin\theta\cos\varphi = \frac{-\sin\theta\cos\varphi}{\sqrt{1-\sin^2\theta\sin^2\varphi}}$$

$$\begin{aligned}\frac{\partial \varphi'}{\partial \varphi} &= \frac{\partial}{\partial a}\arg(a+ib)\frac{\partial a}{\partial \varphi} + \frac{\partial}{\partial b}\arg(a+ib)\frac{\partial b}{\partial \varphi} \\ &= \frac{-b}{a^2+b^2}\sin\theta\sin\varphi + \frac{a}{a^2+b^2}\cdot 0 \\ &= \frac{-\sin\theta\cos\theta\sin\varphi}{\sin^2\theta\cos^2\varphi+\cos^2\theta} = \frac{-\cot\theta\sin\varphi}{\cos^2\varphi+\cot^2\theta}\end{aligned}$$

Then we derive the second formula in (16.7.1) as before. $\square$

<u>Exercise L3B-3</u> (a) Prove that for $W \subseteq S^2$ open the subset $C^\infty(W) \subseteq \mathrm{Cts}(W, \mathbb{R})$ is closed under multiplication and hence (in combination with ExL3B-1) that $C^\infty(W)$ is an $\mathbb{R}$-subalgebra.

(b) Let $\mathrm{Hom}_{\mathbb{R}}(V_1, V_2)$ denote the $\mathbb{R}$-vector space of linear transformations between vector spaces V_1, V_2. Prove that for $W \subseteq S^2$ open with $\mathcal{J}(W) = \mathrm{Hom}_{\mathbb{R}}(C^\infty(W), C^\infty(W))$ the map

multiplication in $C^\infty(W)$

$$a : C^\infty(W) \times \mathcal{J}(W) \longrightarrow \mathcal{J}(W),\quad a(f, \varphi)(g) = f \cdot \varphi(g)$$

makes $\mathcal{J}(W)$ into a left $C^\infty(W)$-module.

With the notation of smooth functions acting on operators from ExL3B-3 (16.7.1) can be written as an equality of elements of $\mathcal{J}(W)$

$$\begin{aligned}\frac{\partial}{\partial \theta} &= \frac{-\cos\theta\sin\varphi}{\sqrt{1-\sin^2\theta\sin^2\varphi}}\frac{\partial}{\partial\theta'} + \frac{\cos\varphi}{\cot^2\theta+\cos^2\varphi}\frac{\partial}{\partial\varphi'} \\ \frac{\partial}{\partial \varphi} &= \frac{-\sin\theta\cos\varphi}{\sqrt{1-\sin^2\theta\sin^2\varphi}}\frac{\partial}{\partial\theta'} - \frac{\cot\theta\sin\varphi}{\cos^2\varphi+\cot^2\theta}\frac{\partial}{\partial\varphi'}\end{aligned} \tag{16.9.1}$$

The Laplacian

We are now prepared to define the Laplacian Δ_{S^2} on $C^\infty(W)$ for $W \subseteq S^2$ open. We define it first on coordinate patches and then argue that these operators "glue" together. For $W \subseteq U$ and $Q \subseteq V$, $f \in C^\infty(W)$ and $g \in C^\infty(Q)$ we define

$$\Delta^U_{S^2}(f) := \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2} \qquad (17.1)$$

$$\Delta^V_{S^2}(g) := \frac{1}{\sin\theta'}\frac{\partial}{\partial\theta'}\left(\sin\theta'\frac{\partial g}{\partial\theta'}\right) + \frac{1}{\sin^2\theta'}\frac{\partial^2 g}{\partial\varphi'^2} \qquad (17.2)$$

These operators $\Delta^U_{S^2} : C^\infty(W) \longrightarrow C^\infty(W)$ and $\Delta^V_{S^2} : C^\infty(Q) \longrightarrow C^\infty(Q)$ are linear.

Lemma L3B-3 For each open subset $W \subseteq S^2$ there is a unique $\mathbb{R}$-linear operator $\Delta_{S^2} : C^\infty(W) \longrightarrow C^\infty(W)$ such that the following diagrams commute

$$\begin{tikzcd} C^\infty(W) \arrow[r, "\Delta_{S^2}"] \arrow[d, "(-)|_{U\cap W}"'] & C^\infty(W) \arrow[d, "(-)|_{U\cap W}"] \\ C^\infty(U\cap W) \arrow[r, "\Delta^U_{S^2}"'] & C^\infty(U\cap W) \end{tikzcd} \qquad \begin{tikzcd} C^\infty(W) \arrow[r, "\Delta_{S^2}"] \arrow[d, "(-)|_{V\cap W}"'] & C^\infty(W) \arrow[d, "(-)|_{V\cap W}"] \\ C^\infty(V\cap W) \arrow[r, "\Delta^V_{S^2}"'] & C^\infty(V\cap W) \end{tikzcd} \qquad (17.3)$$

Proof Uniqueness and linearity follow respectively from the sheaf condition (Ex L3B-2 (b)) and the linearity of $\Delta^U_{S^2}, \Delta^V_{S^2}$ so it suffices to show that for $f \in C^\infty(W)$ we have

$$\Delta^U_{S^2}(f|_{U\cap W})\big|_{U\cap V\cap W} = \Delta^V_{S^2}(f|_{V\cap W})\big|_{U\cap V\cap W} \qquad (17.4)$$

If this holds then $\Delta^U_{S^2}(f|_{U\cap W}) \in C^\infty(U\cap W)$ and $\Delta^V_{S^2}(f|_{V\cap W}) \in C^\infty(V\cap W)$ satisfy the compatibility condition of Ex L3B-2 (b) for the cover $\{U\cap W, V\cap W\}$ of W and hence $\Delta_{S^2}(f) \in C^\infty(W)$ with the desired properties exists. To prove (17.4) it suffices to prove $\Delta^U_{S^2}(g) = \Delta^V_{S^2}(g)$ where $g = f|_{U\cap V\cap W}$. More generally, let $T \subseteq U\cap V$ be open. Then for $g \in C^\infty(T)$ using Lemma L3B-2

We write $a = \dfrac{-\cos\theta\sin\varphi}{\sqrt{1-\sin^2\theta\sin^2\varphi}}$, $b = \dfrac{\cos\varphi}{\cot^2\theta+\cos^2\varphi}$, $c = \dfrac{-\sin\theta\cos\varphi}{\sqrt{1-\sin^2\theta\sin^2\varphi}}$, $d = \dfrac{-\cot\theta\sin\varphi}{\cos^2\varphi+\cot^2\theta}$

so that

$$\frac{\partial}{\partial\theta} = a\frac{\partial}{\partial\theta'} + b\frac{\partial}{\partial\varphi'}, \quad \frac{\partial}{\partial\varphi} = c\frac{\partial}{\partial\theta'} + d\frac{\partial}{\partial\varphi'} \tag{17.1.1}$$

Then we compute

$$\begin{aligned}
\Delta^{\vee}_{S^2}(g) &= \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial g}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 g}{\partial\varphi^2} \\
&= \frac{1}{\sin\theta}\left[\cos\theta\frac{\partial g}{\partial\theta} + \sin\theta\frac{\partial^2 g}{\partial\theta^2}\right] + \frac{1}{\sin^2\theta}\frac{\partial^2 g}{\partial\varphi^2} \qquad (17.1.2)\\
&= \cot\theta\left[a\frac{\partial g}{\partial\theta'} + b\frac{\partial g}{\partial\varphi'}\right] + \frac{\partial}{\partial\theta}\left[a\frac{\partial g}{\partial\theta'} + b\frac{\partial g}{\partial\varphi'}\right] + \frac{1}{\sin^2\theta}\frac{\partial}{\partial\varphi}\left[c\frac{\partial g}{\partial\theta'} + d\frac{\partial g}{\partial\varphi'}\right] \\
&= a\cot\theta\frac{\partial g}{\partial\theta'} + b\cot\theta\frac{\partial g}{\partial\varphi'} + \frac{\partial a}{\partial\theta}\frac{\partial g}{\partial\theta'} + a\frac{\partial}{\partial\theta}\frac{\partial g}{\partial\theta'} + \frac{\partial b}{\partial\theta}\frac{\partial g}{\partial\varphi'} + b\frac{\partial}{\partial\theta}\frac{\partial g}{\partial\varphi'} \\
&\quad + \frac{1}{\sin^2\theta}\frac{\partial c}{\partial\varphi}\frac{\partial g}{\partial\theta'} + \frac{1}{\sin^2\theta}c\frac{\partial}{\partial\varphi}\frac{\partial g}{\partial\theta'} + \frac{1}{\sin^2\theta}\frac{\partial d}{\partial\varphi}\frac{\partial g}{\partial\varphi'} + \frac{1}{\sin^2\theta}d\frac{\partial}{\partial\varphi}\frac{\partial g}{\partial\varphi'}
\end{aligned}$$

Now

$$\begin{aligned}
\frac{\partial}{\partial\theta}\frac{\partial g}{\partial\theta'} &= a\frac{\partial^2 g}{\partial\theta'^2} + b\frac{\partial^2 g}{\partial\varphi'\partial\theta'}, \quad \frac{\partial}{\partial\theta}\frac{\partial g}{\partial\varphi'} = a\frac{\partial^2 g}{\partial\theta'\partial\varphi'} + b\frac{\partial^2 g}{\partial\varphi'^2} \\
\frac{\partial}{\partial\varphi}\frac{\partial g}{\partial\theta'} &= c\frac{\partial^2 g}{\partial\theta'^2} + d\frac{\partial^2 g}{\partial\varphi'\partial\theta'}, \quad \frac{\partial}{\partial\varphi}\frac{\partial g}{\partial\varphi'} = c\frac{\partial^2 g}{\partial\theta'\partial\varphi'} + d\frac{\partial^2 g}{\partial\varphi'^2}
\end{aligned} \tag{17.1.3}$$

Hence

$$\begin{aligned}
\Delta^{\vee}_{S^2}(g) &= \left\{a\cot\theta + \frac{\partial a}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial c}{\partial\varphi}\right\}\frac{\partial g}{\partial\theta'} + \left\{b\cot\theta + \frac{\partial b}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial d}{\partial\varphi}\right\}\frac{\partial g}{\partial\varphi'} \\
&\quad + \left\{a^2 + \frac{c^2}{\sin^2\theta}\right\}\frac{\partial^2 g}{\partial\theta'^2} + \left\{b^2 + \frac{d^2}{\sin^2\theta}\right\}\frac{\partial^2 g}{\partial\varphi'^2} \qquad (17.1.4)\\
&\quad + \left\{ab + ab + \frac{cd}{\sin^2\theta} + \frac{cd}{\sin^2\theta}\right\}\frac{\partial^2 g}{\partial\theta'\partial\varphi'}
\end{aligned}$$

$$a = \frac{-\cos\theta\sin\varphi}{\sqrt{1-\sin^2\theta\sin^2\varphi}}, \quad b = \frac{\cos\varphi}{\cot^2\theta+\cos^2\varphi}, \quad c = \frac{-\sin\theta\cos\varphi}{\sqrt{1-\sin^2\theta\sin^2\varphi}}, \quad d = \frac{-\cot\theta\sin\varphi}{\cos^2\varphi+\cot^2\theta}$$

One checks that
$$a^2 + \frac{c^2}{\sin^2\theta} = 1, \quad b^2 + \frac{d^2}{\sin^2\theta} = \frac{1}{\sin^2\theta'}$$

$$a\cot\theta + \frac{\partial a}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial c}{\partial\varphi} = \cot\theta'$$

$$b\cot\theta + \frac{\partial b}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial d}{\partial\varphi} = 0$$

$$ab + \frac{cd}{\sin^2\theta} = 0$$

Hence

$$\Delta^{U}_{S^2}(g) = \frac{\partial^2 g}{\partial\theta'^2} + \cot\theta'\frac{\partial g}{\partial\theta'} + \frac{1}{\sin^2\theta'}\frac{\partial^2 g}{\partial\varphi'^2} = \Delta^{V}_{S^2}(g). \ \square$$

<u>Exercise L3B-4</u> If $W' \subseteq W \subseteq S^2$ are open prove that

$$\begin{array}{ccc} C^\infty(W) & \xrightarrow{\Delta_{S^2}} & C^\infty(W) \\ {\scriptstyle (-)|_{W'}}\downarrow & & \downarrow{\scriptstyle (-)|_{W'}} \\ C^\infty(W') & \xrightarrow[\Delta_{S^2}]{} & C^\infty(W') \end{array}$$

commutes. We say Δ_{S^2} is a morphism of sheaves.

The operator Δ_{S^2} (or really the family of operators on $C^\infty(W)$ for each open $W \subseteq S^2$) is called the <u>Laplacian</u>. The Laplacian is defined intrinsically (i.e. without choosing coordinates) on any Riemannian manifold (on $\mathbb{R}^n$ it is $\Delta_{\mathbb{R}^n} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$) but for this course we will make do with the above construction by glueing operators over an open cover of S^2.

<u>Exercise L3-13</u> Prove that for a smooth function f on $\mathbb{R}^3$ the following holds on $\mathbb{R}^3 \setminus \{0\}$

$$\Delta_{\mathbb{R}^3} f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\Delta_{S^2} f \qquad (17.3).$$

Lemma L3-10 For every harmonic polynomial $P \in \mathcal{H}_k(3)$ the restriction $f = P|_{S^2} \in \mathcal{H}_k(S^2)$ is an eigenvector of Δ_{S^2} with eigenvalue $-k(1+k)$.

Proof By direct calculation using (17.3). We have for $r > 0$ and $x \in S^2$ an expression for P in spherical coordinates $P(r, \theta, \varphi) = r^k f(\theta, \varphi)$ and hence

$$0 = \Delta P = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial P}{\partial r}\right) + \frac{1}{r^2}\Delta_{S^2} P$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \cdot k \cdot r^{k-1} f\right) + \frac{1}{r^2} r^k \Delta_{S^2} f$$

$$= \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^{1+k} f\right) + r^{k-2}\Delta_{S^2} f$$

$$= k(1+k) r^{k-2} f + r^{k-2}\Delta_{S^2} f$$

so $\Delta_{S^2} f = -k(1+k) f$ as claimed. $\square$

Def^n A continuous function $f : S^2 \longrightarrow \mathbb{C}$ is smooth if $Re(f), Im(f) \in C^\infty(S^2)$ and we denote by $C^\infty(S^2, \mathbb{C}) \subseteq Cts(S^2, \mathbb{C})$ the $\mathbb{C}$-linear subspace of all smooth functions.

The Laplacian extends to a $\mathbb{C}$-linear operator on $C^\infty(S^2, \mathbb{C})$ by acting separately on the real and imaginary parts : $Re(\Delta_{S^2} f) = \Delta_{S^2} Re(f)$, $Im(\Delta_{S^2} f) = \Delta_{S^2} Im(f)$.

Lemma L3-11 Given $f, g \in C^\infty(S^2, \mathbb{C})$ we have

$$\int_{S^2} \overline{\Delta_{S^2} f}\, g \, dS = \int_{S^2} \bar{f}\, \Delta_{S^2} g \, dS \qquad (18.1)$$

We will return to this proof momentarily, but first we want to examine its consequences. In the following we write $a_{k,n}$ for $\dim_{\mathbb{C}} \mathcal{H}_k(n)$.

<u>Theorem L3-12</u> *We have a direct sum decomposition*

$$L^2(S^2;\mathbb{C}) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k(S^2) \qquad (19.1)$$

in the sense that the summands are closed, pairwise orthogonal, and every $f \in L^2(S^2;\mathbb{C})$ *can be written* <u>*uniquely*</u> *as a converging series*

$$f = \sum_{k=0}^{\infty} f_k \qquad f_k \in \mathcal{H}_k(S^2). \qquad (19.2)$$

<u>Proof</u> Let $k \neq l$ be given, and let $f \in \mathcal{H}_k(S^2)$, $g \in \mathcal{H}_l(S^2)$. Then by Lemma L3-10 we have $\Delta_{S^2} f = -k(1+k)f$, $\Delta_{S^2} g = -l(1+l)g$ and hence by Lemma L3-11

$$\begin{aligned} \langle f, g \rangle &= \langle -\frac{1}{k(1+k)} \Delta_{S^2} f, g \rangle \\ &= -\frac{1}{k(1+k)} \langle f, \Delta_{S^2} g \rangle \qquad (19.3) \\ &= \frac{l(1+l)}{k(1+k)} \langle f, g \rangle. \end{aligned}$$

Suppose $\langle f, g\rangle \neq 0$ then $l(1+l) = k(1+k)$ which implies $l = k$ a contradiction (the function $x + x^2$ is increasing for $x > 0$), hence $\langle f, g \rangle = 0$ as claimed. The subspaces $\mathcal{H}_k(S^2)$ are finite-dimensional, hence closed.

Now let $\{Y_k^1, \ldots, Y_k^{a_k}\}$ be an orthonormal basis for $\mathcal{H}_k(S^2)$, produced say by the Gram-Schmidt process. Since $\langle Y_k^i, Y_l^j \rangle = 0$ whenever $k \neq l$ the set $\beta = \{ Y_k^i \}_{k \geq 0, 1 \leq i \leq a_k}$ is a countable orthonormal set. Moreover $\bigcup_{k \geq 0} \mathcal{H}_k(S^2) \subseteq \operatorname{span}_{\mathbb{C}} \beta$ so by Corollary L3-9 the set β is an orthonormal dense basis for $L^2(S^2;\mathbb{C})$. The statement of (19.2) now follows from [MHS, Theorem L21-10] with $f_k = \sum_{i=1}^{a_k} \langle Y_k^i, f \rangle Y_k^i$. $\square$

Proof of Lemma L3-11 Writing $f = f^{Re} + i f^{Im}$, $g = g^{Re} + i g^{Im}$ suppose (18.1) holds for real-valued functions f, g. Then

$$\langle \overline{\Delta_{S^2} f}, g \rangle = \langle \overline{\Delta_{S^2} f^{Re} + i \Delta_{S^2} f^{Im}}, g^{Re} + i g^{Im} \rangle$$

$$= \langle \Delta_{S^2} f^{Re} - i \Delta_{S^2} f^{Im}, g^{Re} + i g^{Im} \rangle$$

$$= \langle \Delta_{S^2} f^{Re}, g^{Re} \rangle + i \langle \Delta_{S^2} f^{Re}, g^{Im} \rangle + i \langle \Delta_{S^2} f^{Im}, g^{Re} \rangle - \langle \Delta_{S^2} f^{Im}, g^{Im} \rangle \tag{20.1}$$

$$= \langle f^{Re}, \Delta_{S^2} g^{Re} \rangle + i \langle f^{Re}, \Delta_{S^2} g^{Im} \rangle + i \langle f^{Im}, \Delta_{S^2} g^{Re} \rangle - \langle f^{Im}, \Delta_{S^2} g^{Im} \rangle$$

$$= \langle f^{Re}, \Delta_{S^2} g \rangle + \langle -i f^{Im}, \Delta_{S^2} g \rangle$$

$$= \langle \bar{f}, \Delta_{S^2} g \rangle$$

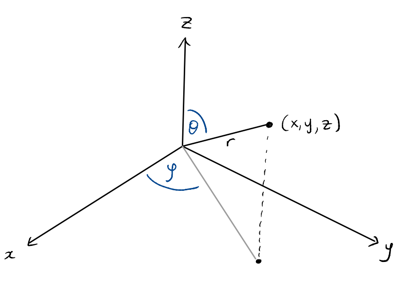
So we may assume f, g real. We need to show

note that the way the integral is defined we need only use one coordinate chart

$$\int_0^{2\pi} \int_0^{\pi} \left\{ \frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial f}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2 f}{\partial\varphi^2} \right\} g \sin\theta \, d\theta \, d\varphi \tag{20.2}$$

$$= \int_0^{2\pi} \int_0^{\pi} f \left\{ \frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial g}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2 g}{\partial\varphi^2} \right\} \sin\theta \, d\theta \, d\varphi.$$

No problem!



Cancelling the factor of $\sin\theta$ and reordering the integral gives for the RHS

$$\int_0^{2\pi}\int_0^{\pi} f\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial g}{\partial\theta}\right)d\theta\,d\varphi + \int_0^{\pi}\frac{1}{\sin\theta}\int_0^{2\pi} f\,\frac{\partial^2 g}{\partial\varphi^2}\,d\varphi\,d\theta \qquad (*)$$

and for the LHS

$$\int_0^{2\pi}\int_0^{\pi} g\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial f}{\partial\theta}\right)d\theta\,d\varphi + \int_0^{\pi}\frac{1}{\sin\theta}\int_0^{2\pi} g\,\frac{\partial^2 f}{\partial\varphi^2}\,d\theta\,d\varphi \qquad (**)$$

Set $h = \frac{\partial g}{\partial\varphi}$. Then via integration by parts we "move all derivatives onto f":

$$\begin{aligned}\int_0^{2\pi} f\,\frac{\partial^2 g}{\partial\varphi^2}\,d\varphi &= \int_0^{2\pi} f\,\frac{\partial h}{\partial\varphi}\,d\varphi \\ &= \left[fh\right]_0^{2\pi} - \int_0^{2\pi} h\,\frac{\partial f}{\partial\varphi}\,d\varphi \\ &= -\int_0^{2\pi} h\,\frac{\partial f}{\partial\varphi}\,d\varphi\end{aligned} \qquad (21.1)$$

Because $f = f(\theta,\varphi)$ is periodic in both variables by definition. In particular as a function of θ, $f(\theta,2\pi) - f(\theta,0) \equiv 0$. This "periodicity + integration by parts trick" is everywhere (see e.g. [MHS, Lemma L21-3]) and with a more subtle property replacing periodicity it is a central idea in differential geometry. Anyway, continuing (21.1) and again using integration by parts

$$\begin{aligned} &= -\int_0^{2\pi}\frac{\partial g}{\partial\varphi}\frac{\partial f}{\partial\varphi}\,d\varphi \qquad (21.2)\\ &= -\left\{\left[g\,\frac{\partial f}{\partial\varphi}\right]_0^{2\pi} - \int_0^{2\pi} g\,\frac{\partial^2 f}{\partial\varphi^2}\,d\varphi\right\} \\ &= \int_0^{2\pi} g\,\frac{\partial^2 f}{\partial\varphi^2}\,d\varphi\end{aligned}$$

Hence the second summands in (*), (**) agree. Now with $h = \sin\theta \frac{\partial g}{\partial \theta}$

$$\int_0^\pi f \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial g}{\partial\theta}\right) d\theta = \int_0^\pi f \frac{\partial h}{\partial\theta}\, d\theta$$

$$= \left[fh\right]_0^\pi - \int_0^\pi \frac{\partial f}{\partial\theta} h\, d\theta \tag{22.1}$$

$$= -\int_0^\pi \frac{\partial f}{\partial\theta} \sin\theta \frac{\partial g}{\partial\theta}\, d\theta$$

$$= \left[\frac{\partial f}{\partial\theta}\sin\theta\, g\right]_0^\pi + \int_0^\pi \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial f}{\partial\theta}\right) g\, d\theta$$

This time the vanishing of the $[-]_0^\pi$ terms is because $\sin(0) = \sin(\pi) = 0$ (there is no reason to assume $f(0) = f(\pi)$). This proves (*) = (**) and completes the proof. $\square$

References

[MHS] Metric and Hilbert spaces MAST30026