Lecture 4 – Infinitesimal symmetries

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In Lecture 3 we have seen that $SO(3)$ acts on $L^2(S^2,\mathbb{C})$ by unitary transformations, and that every $\Psi \in L^2(S^2,\mathbb{C})$ can be written uniquely as a convergent series $\Psi = \sum_{k=0}^{\infty} \Psi_k$ where $\Psi_k \in \mathcal{H}_k(S^2)$ is a <u>spherical harmonic of degree k</u>, the restriction to S^2 of a homogeneous polynomial function f on $\mathbb{R}^3$ which is <u>harmonic</u>, i.e.

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right] f = 0. \tag{1.1}$$

Here $\Delta_{\mathbb{R}^3} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the <u>Laplacian</u>, a linear operator on the vector space $C^\infty(\mathbb{R}^3)$ of smooth (real or complex-valued) functions on $\mathbb{R}^3$. We have defined also the Laplacian Δ_{S^2} on smooth functions on S^2, and we know $\Delta_{S^2} f = -k(1+k) f$ for $f \in \mathcal{H}_k(S^2)$. That is, the spherical harmonics are <u>eigenvectors of the Laplacian</u> on S^2.

Now we return to the symmetries $g \in SO(3)$ of the sphere and Hilbert space $L^2(S^2,\mathbb{C})$ and seek to understand how these interact with the basis of spherical harmonics. Since $SO(3)$ is not abelian we cannot hope to simultaneously diagonalise all the $g \in SO(3)$ (if e.g. there were a dence orthonormal basis of spherical harmonics Y_k^i for $L^2(S^2,\mathbb{C})$ then it cannot be that Y_k^i is an eigenvector for every $g \in SO(3)$, since then the actions of g, h would commute on $L^2(S^2,\mathbb{C})$ hence on $Cts(S^2,\mathbb{C})$ which is false). But the next best thing is true: we will show that with

$$\delta : SO(3)^{op} \longrightarrow U(L^2(S^2,\mathbb{C})) \tag{1.2}$$

denoting the representation $\delta(g) = \hat{C}_g$ of Lemma L3-1, the operator $\delta(g)$ is <u>block diagonal</u>

$$[\delta(g)] = \begin{array}{c|cccc} & \mathcal{H}_0(S^2) & \mathcal{H}_1(S^2) & \mathcal{H}_2(S^2) & \cdots \\ \hline \mathcal{H}_0(S^2) & \blacksquare & 0 & 0 & 0 \\ \mathcal{H}_1(S^2) & 0 & \blacksquare & 0 & 0 \\ \mathcal{H}_2(S^2) & 0 & 0 & \blacksquare & 0 \\ \vdots & & & & \end{array} \tag{1.3}$$

matrix wrt a basis of spherical harmonics

Lemma L4-1 For all $g \in SO(3)$ we have $\delta(g)(\mathcal{H}_k(S^2)) \subseteq \mathcal{H}_k(S^2)$ for all $k \geq 0$.

Proof The following diagram commutes

$$\begin{tikzcd} Cts(\mathbb{R}^3, \mathbb{C}) \arrow[r, "(-)\circ g"] \arrow[d, "(-)|_{S^2}"'] & Cts(\mathbb{R}^3, \mathbb{C}) \arrow[d, "(-)|_{S^2}"] \\ Cts(S^2, \mathbb{C}) \arrow[r, "(-)\circ g"'] & Cts(S^2, \mathbb{C}) \end{tikzcd} \qquad (2.1)$$

and so to prove that $\delta(g)(f) \in \mathcal{H}_k(S^2)$ for any $f \in \mathcal{H}_k(S^2)$ it suffices to prove $P \circ g \in \mathcal{H}_k(3)$ for any $P \in \mathcal{H}_k(3)$ (recall $\mathcal{H}_k(3)$ denotes the space of harmonic polynomials of degree k in three variables), since then if $f \in \mathcal{H}_k(S^2)$ is the restriction of $P \in \mathcal{H}_k(3)$ we have

$$\delta(g)(f) = P|_{S^2} \circ g \overset{(2.1)}{=} (P \circ g)|_{S^2} \in \mathcal{H}_k(S^2). \qquad (2.2).$$

Note that given any 3×3 real matrix A the morphism of $\mathbb{C}$-algebras $\mathbb{C}[x_1, x_2, x_3] \xrightarrow{\varphi_A} \mathbb{C}[x_1, x_2, x_3]$ defined by $\varphi_A(x_i) = \sum_{j=1}^3 A_{ij} x_j$ sends a polynomial $P = P(x_1, x_2, x_3)$ to

$$\varphi_A(P) = P\left(\sum_{j=1}^3 A_{1j} x_j, \sum_{j=1}^3 A_{2j} x_j, \sum_{j=1}^3 A_{3j} x_j\right) \qquad (2.3)$$

and hence as a function on $\mathbb{R}^3$ we have $\varphi_A(P) = P \circ g$ if the matrix of g is A. So to prove that $P \circ g \in \mathcal{H}_k(3)$ whenever $P \in \mathcal{H}_k(3)$ it suffices to show that if A is an orthogonal matrix $A^T A = I$ and $P \in \mathcal{H}_k(3)$ then $\Delta \varphi_A(P) = 0$. But

$$\begin{aligned} \Delta \varphi_A(P) &= \sum_{i=1}^3 \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_i} P\left(\sum_j A_{1j} x_j, \sum_j A_{2j} x_j, \sum_j A_{3j} x_j\right) \\ &= \sum_{i=1}^3 \frac{\partial}{\partial x_i}\left[\varphi_A\left(\frac{\partial P}{\partial x_1}\right) A_{1i} + \varphi_A\left(\frac{\partial P}{\partial x_2}\right) A_{2i} + \varphi_A\left(\frac{\partial P}{\partial x_3}\right) A_{3i}\right] \end{aligned} \qquad (2.4)$$

$$= \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left[\sum_{l=1}^{3} \mathcal{S}_A\left(\frac{\partial P}{\partial x_l}\right) A_{li} \right]$$

$$= \sum_{i=1}^{3} \sum_{l=1}^{3} \frac{\partial}{\partial x_i}\left[\mathcal{S}_A\left(\frac{\partial P}{\partial x_l}\right)\right] A_{li}$$

$$= \sum_{i=1}^{3} \sum_{l=1}^{3} \left[\sum_{m=1}^{3} \mathcal{S}_A\left(\frac{\partial^2 P}{\partial x_m \partial x_l}\right) A_{mi} A_{li} \right]$$

$$= \sum_{m=1}^{3} \sum_{l=1}^{3} \mathcal{S}_A\left(\frac{\partial^2 P}{\partial x_m \partial x_l}\right) \left\{ \sum_{i=1}^{3} A_{mi} A^T_{il} \right\}$$

$$= \sum_{m=1}^{3} \sum_{l=1}^{3} \mathcal{S}_A\left(\frac{\partial^2 P}{\partial x_m \partial x_l}\right) \left(AA^T \right)_{ml}$$

$$= \sum_{m=1}^{3} \mathcal{S}_A\left(\frac{\partial^2 P}{\partial x_m^2}\right) = \mathcal{S}_A\left(\sum_{m=1}^{3} \frac{\partial^2 P}{\partial x_m^2}\right)$$

$$= \mathcal{S}_A \Delta P = 0$$

as claimed. It only remains to show $\mathcal{S}_A(P)$ is homogeneous of degree 3, but this is straightforward. $\square$

<u>Exercise L4-1</u> Prove that there is no orthonormal dense basis of $L^2(S^2, \mathbb{C})$ consisting of simultaneous eigenvectors for the set $\{\mathcal{S}(g)\}_{g \in SO(3)}$.

<u>Def</u>

Def^n A <u>representation</u> of a group G on a complex vector space V is a morphism of groups $\rho : G \longrightarrow \mathrm{Aut}_{\mathbb{C}}(V)$ where $\mathrm{Aut}_{\mathbb{C}}(V)$ denotes the group of $\mathbb{C}$-linear bijections from V to itself. A <u>subrepresentation</u> of V is a $\mathbb{C}$-linear subspace $W \subseteq V$ such that $\rho(g)(W) \subseteq W$ for all $g \in G$. In this case the map $g \longmapsto \rho(g)|_W$ makes W a representation of G in its own right.

In this language the lemma says that $\mathcal{H}_k(S^2)$ is a subrepresentation of $L^2(S^2, \mathbb{C})$ for all $k \geqslant 0$. Note that $\mathcal{H}_k(S^2)$ is a finite-dimensional complex Hilbert space and $SO(3)^{op}$ acts on $\mathcal{H}_k(S^2)$ by unitary transformations.

Exercise L4-2 Prove $\dim_{\mathbb{C}} \mathcal{H}_k(S^2) = 2k+1$.

Let us now return to the subject of the first two lectures, and consider two observers at rest relative to each other and with a common origin, at which we imagine is placed a quantum system — say a hydrogen atom — with Hilbert space $\mathcal{H}$. The wave functions $\Psi \in \mathcal{H}$ have a radial component $\Psi = \Psi(r, \theta, \varphi)$ but this can be treated separately, i.e. we can write $\Psi = \gamma(r)\, \eta(\theta, \varphi)$ and treat the angular dependence as a vector $\eta(\theta, \varphi) \in L^2(S^2, \mathbb{C})$. This can be found in any introductory course on quantum mechanics, e.g. [F, Chapter 19]. So we consider $\mathcal{H} = L^2(S^2, \mathbb{C})$ as the Hilbert space of states.

As discussed in Lectures 1 and 2, the two observers O and O' may observe the system to be in different states (i.e. rays $\mathcal{R}, \mathcal{R}' \in S_{\mathcal{H}}$) but as they agree on transition probabilities there is by Wigner's theorem a unitary transformation $U : \mathcal{H} \to \mathcal{H}$ such that if $\Psi \in \mathcal{R}$ then $U(\Psi) \in \mathcal{R}'$, that is, U translates states as seen by O into states as seen by O' (up to a phase factor). Considering such translations between triples O, O', O'' of observers (at rest and with common origin) one argues that these unitary transformations determine a projective unitary representation

$$\rho : SO(3)^{op} \longrightarrow U(\mathcal{H}) \qquad (4.1)$$

where in a projective representation $\rho(g)\rho(h) = e^{i\phi(g,h)}\rho(gh)$ instead of $\rho(g)\rho(h) = \rho(gh)$, see [W, §2.7] for details. Projective representations are important, both in mathematics and physics, but for the moment we assume ρ is a representation in the normal sense.

We should not rush to conclude that $\rho = \delta$ is the representation of $SO(3)$ on $\mathcal{H} = L^2(S^2, \mathbb{C})$ that we have constructed. For all we know there are other, essentially different, representations on $\mathcal{H}$ and it is the physics which decides which is "correct" for our system. Nonetheless let us suppose that δ is in fact the representation translating between our pair of observers: what does the block decomposition of (1.3) and Lemma L4-1 mean in this context?

It means that if O observes the state of the system to be $\psi \in \mathcal{H}_k(S^2)$ for some $k \geq 0$ then O' observes it to be $\mathcal{E}(g)(\psi)$ for some $g \in SO(3)$ which is by Lemma L4-1 also in $\mathcal{H}_k(S^2)$. Thus any pair of observers not only agree on transition probabilities but also on the answer to the question "state $\in \mathcal{H}_k(S^2)$?" for any $k \geq 0$. The yes/no answers to this countable set of predicates constitute a stream of bits on which all observers agree

Remark L4-1 The Hamiltonian $H = -\Delta_{S^2}$ acts as a constant on $\mathcal{H}_k(S^2)$ so by the Schrödinger equation $i\hbar \frac{\partial}{\partial t}\psi = H\psi = k(k+1)\psi$ we have $\psi(t) = e^{-i/\hbar\, k(k+1)t}\psi$ which differs only in phase from ψ. Hence the "stream of bits" is stable over time and also invariant under small perturbations in the observer's reference frame. This information is, therefore, by the standard elaborated in Lecture 1, real. I do not know of anything more real than that.

What other questions have "real" answers in this sense? Are there integer or real valued quantities associated to systems like the one considered above which are "real"? Can we classify all such questions and quantities? The theory of Lie algebras provides the answer to these queries.

Generators of rotational symmetry

So far the fact that $SO(3)$ is a space and not just a set has not played any role. Recall that we give $SO(3)$ a topology by identifying it with a space of matrices. We have constructed in Lecture 3 a path $R^{\hat{n}}_{\alpha}$ in $SO(3)$ with parameter α for any unit vector $\hat{n}$ in $\mathbb{R}^3$ and as we continuously vary the group element along this path we can ask if the corresponding unitary transformation on $L^2(S^2, \mathbb{C})$ also varies continuously, or even smoothly.

Exercise L4-3 (i) Prove that $\gamma_{\hat{n}} : \mathbb{R} \longrightarrow SO(3)$ sending α to $R^{\hat{n}}_{\alpha}$ is continuous.

(ii) Prove that $\gamma_{\hat{n}}$ factors uniquely through a continuous map $T_{\hat{n}} : S^1 \longrightarrow SO(3)$.

(ii) [for MHS students] prove that the function $S^2 \longrightarrow \mathcal{L}(SO(3))$ sending $\hat{n}$ to $T_{\hat{n}}$ is continuous where $\mathcal{L}Y$ is the free loop space of Y [MHS, Example L12-1].

To make sense of a continuously or smoothly varying point of $U(L^2(S^2,\mathbb{C}))$ we use that the $\mathcal{E}(g)$ all have a block decomposition (1.3) and we will say a 1-parameter family of such transformations is continuous or smooth if <u>each block</u> varies in that way. Since these blocks are just finite matrices we can easily what continuity and smoothness mean for blocks. First however we need to make some basic remarks about differentiating operators with parameters.

In the following V is a finite-dimensional $\mathbb{C}$-vector space and $\mathrm{End}_{\mathbb{C}}(V)$ denotes the $\mathbb{C}$-vector space of linear operators on V. Given an ordered basis β of V there is an isomorphism of vector spaces

$$\begin{gathered}\mathrm{End}_{\mathbb{C}}(V) \xrightarrow[\cong]{C_\beta} M_d(V) \\ C_\beta(T) = [T]_\beta^\beta\end{gathered} \qquad (6.1)$$

sending an operator to its matrix, where $d = \dim_{\mathbb{C}}(V)$.

<u>Def^n</u> Let $U \subseteq \mathbb{R}^n$ be open. A function $f: U \longrightarrow \mathrm{End}_{\mathbb{C}}(V)$ is <u>smooth</u> if the composite

$$U \xrightarrow{f} \mathrm{End}_{\mathbb{C}}(V) \xrightarrow{C_\beta} M_d(\mathbb{C}) \xrightarrow[\cong]{} \mathbb{C}^{d^2} \qquad (6.2)$$

is smooth, that is, if the entries of the matrix $[f(u)]_\beta^\beta$ are smooth functions of u.

<u>Def^n</u> In the above notation we denote by $C^\infty(U, \mathrm{End}_{\mathbb{C}}(V))$ the set of all smooth functions $U \longrightarrow \mathrm{End}_{\mathbb{C}}(V)$.

We say a function $U \longrightarrow \mathrm{Aut}_{\mathbb{C}}(V)$ is smooth if it is smooth when composed with the inclusion $\mathrm{Aut}_{\mathbb{C}}(V) \hookrightarrow \mathrm{End}_{\mathbb{C}}(V)$. Similarly if $\mathcal{H}$ is a Hilbert space and $W \subseteq \mathbb{R}^n$ is open a function $W \longrightarrow U(\mathcal{H})$ is smooth if it is smooth when composed with $U(\mathcal{H}) \hookrightarrow \mathrm{Aut}_{\mathbb{C}}(\mathcal{H})$

<u>Def^n</u> In the above notation we define for $1 \le i \le n$ a $\mathbb{C}$-linear operator $\frac{\partial}{\partial x_i}$ on $C^\infty(U, \mathrm{End}_{\mathbb{C}}(V))$ to be the following composite

$$\begin{tikzcd} C^\infty(U, \mathrm{End}_{\mathbb{C}}(V)) \arrow[r, "c_\beta \circ (-)"] & C^\infty(U, M_d(\mathbb{C})) \arrow[d, "\frac{\partial}{\partial x_i}"] \\ C^\infty(U, \mathrm{End}_{\mathbb{C}}(V)) & C^\infty(U, M_d(\mathbb{C})) \arrow[l, "c_\beta^{-1} \circ (-)"] \end{tikzcd} \tag{6.5.1}$$

where $\frac{\partial}{\partial x_i}$ acts on matrices of functions entry-wise, that is, if $f : U \longrightarrow M_d(\mathbb{C})$ is identified with a matrix $(f_{jk}(u))$ of functions then the derivative is $\left(\frac{\partial}{\partial x_i}(f_{jk})\right)$.

<u>Exercise L4-4</u> (a) Prove that $f : U \longrightarrow \mathrm{End}_{\mathbb{C}}(V)$ is smooth with respect to <u>some</u> basis β iff it is smooth with respect to <u>any</u> basis.

(b) Prove that the operator $\partial/\partial x_i$ of (6.5.1) is independent of the basis β used to define it.

(c) Prove that $C^\infty(U, \mathrm{End}_{\mathbb{C}}(V))$ is a $\mathbb{C}$-algebra where $(fg)(u) = f(u) \circ g(u)$.

(d) Prove that $\frac{\partial}{\partial x_i}$ satisfies the Leibniz rule: that is, for $f, g \in C^\infty(U, \mathrm{End}_{\mathbb{C}}(V))$

$$\frac{\partial}{\partial x_i}(fg) = \frac{\partial}{\partial x_i}(f)g + f\frac{\partial}{\partial x_i}(g).$$

(e) Prove that if $f : U \longrightarrow \mathrm{End}_{\mathbb{C}}(V)$ is smooth then so are the functions $U \longrightarrow \mathbb{C}$ given by $u \longmapsto \mathrm{tr}(f(u))$ and $u \longmapsto \det(f(u))$ (give definitions of these trace and determinant functions and show they are independent of the choice of basis).

Lemma L4-2 For any $\hat{n} \in S^2$ and $k \geq 0$ the function

$$\begin{aligned} \mathbb{R} &\longrightarrow \operatorname{End}_{\mathbb{C}}(\mathcal{H}_k(S^2)) \\ \alpha &\longmapsto \mathcal{Z}(R^{\hat{n}}_{\alpha})\big|_{\mathcal{H}_k(S^2)} \end{aligned}$$

is smooth.

Proof Recall that $\mathcal{P}_k = \mathcal{P}_k(3)$ denotes the space of complex homogeneous polynomials of degree 3, $\mathcal{H}_k \subseteq \mathcal{P}_k$ the subspace of harmonic polynomials. Given $g \in SO(3)$ let A be the matrix of g and $\mathcal{S}_A$ be as in (2.3). Then we define a representation of $SO(3)$ on the complex vectorspace $\mathcal{P}_k$ by

$$\begin{aligned} SO(3)^{op} &\xrightarrow{\ \rho\ } \operatorname{Aut}_{\mathbb{C}}(\mathcal{P}_k) \\ \rho(g) &= \mathcal{S}_A \end{aligned} \tag{7.1}$$

We checked that this restricts to a right action $\mathcal{Z}_k'$ on $\mathcal{H}_k$ in the proof of Lemma L4-1, and it is easy to see that the diagram

$$\begin{tikzcd} & \operatorname{Aut}_{\mathbb{C}}(\mathcal{H}_k) \arrow[d, "\cong"] \\ SO(3)^{op} \arrow[ur, "\mathcal{Z}_k'"] \arrow[r, "\mathcal{Z}_k"'] & \operatorname{Aut}_{\mathbb{C}}(\mathcal{H}_k(S^2)) \end{tikzcd} \tag{7.2}$$

commutes, where the vertical arrow is induced by the isomorphism of vectorspaces $\mathcal{H}_k \cong \mathcal{H}_k(S^2)$ of Lemma L3-7 and $\mathcal{Z}_k$ is the representation induced by $\mathcal{Z}$ of (1.2) and Lemma L4-1. It therefore suffices to prove $\alpha \mapsto \mathcal{Z}_k'(R^{\hat{n}}_{\alpha})$ is smooth. If β' is a $\mathbb{C}$-basis of $\mathcal{H}_k$ extended to a basis β of $\mathcal{P}_k$ then for $g \in SO(3)$

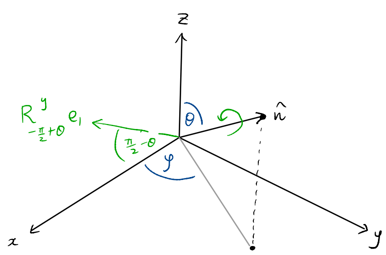
$$[\rho(g)]^{\beta}_{\beta} = \begin{pmatrix} [\delta_k'(g)]^{\beta'}_{\beta'} & 0 \\ 0 & Y \end{pmatrix} \qquad (8.1)$$

for some matrix Y. So it suffices to prove that $\alpha \longmapsto \rho(R^{\hat{n}}_{\alpha})$ is smooth. But if $A^{\alpha} \in M_3(\mathbb{R})$ denotes the matrix of $R^{\hat{n}}_{\alpha}$ then we may take the monomials $\{x^{\beta}\}_{|\beta|=k}$ as our basis for P_k (here $\beta \in \mathbb{N}^3$ and $x^{\beta} = x_1^{\beta_1} x_2^{\beta_2} x_3^{\beta_3}$, $|\beta| = \beta_1 + \beta_2 + \beta_3$) and

$$\begin{aligned} \rho(R^{\hat{n}}_{\alpha})(x^{\beta}) &= \varphi_{A^{\alpha}}(x^{\beta}) \\ &= \prod_{i=1}^{3} \Big[\sum_{j=1}^{3} A^{\alpha}_{ij} x_j \Big]^{\beta_i} \\ &= \sum_{|\gamma|=k} C_{\gamma} x^{\gamma} \end{aligned} \qquad (8.2)$$

where the C_{γ} are some polynomials in $\{A^{\alpha}_{ij}\}_{1 \le i,j \le 3}$. To prove the function $\mathbb{R} \to \mathrm{Aut}_{\mathbb{C}}(P_k)$ sending α to $\rho(R^{\hat{n}}_{\alpha})$ is smooth it therefore suffices to prove that any polynomial function in the $\{A^{\alpha}_{ij}\}_{1 \le i,j \le 3}$ is smooth <u>as a function of α</u>. But since sums, products and scalar multiples of smooth functions are smooth it suffices to prove $\alpha \mapsto A^{\alpha}_{ij} \in \mathbb{R}$ is a smooth function of α for $1 \le i,j \le 3$. But by definition if $\hat{n} = R^{z}_{\varphi} R^{y}_{\theta - \frac{\pi}{2}}(e_1)$ then

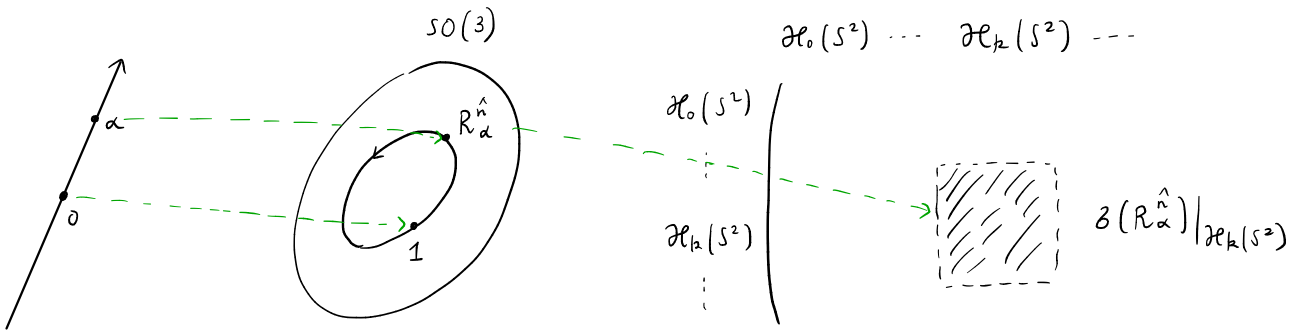
$$R^{\hat{n}}_{\alpha} = R^{z}_{\varphi} R^{y}_{\theta-\frac{\pi}{2}} R^{x}_{\alpha} R^{y}_{\frac{\pi}{2}-\theta} R^{z}_{-\varphi} \qquad (8.3)$$



Hence with S the standard basis of $\mathbb{R}^3$

$$\begin{aligned} A^{\alpha} = [R^{\hat{n}}_{\alpha}]^{S}_{S} &= [R^{z}_{\varphi}]^{S}_{S} [R^{y}_{\theta-\frac{\pi}{2}}]^{S}_{S} [R^{x}_{\alpha}]^{S}_{S} [R^{y}_{\frac{\pi}{2}-\theta}]^{S}_{S} [R^{z}_{-\varphi}]^{S}_{S} \\ &= L \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} L' \end{aligned} \qquad (8.4)$$

for some $L, L' \in M_3(\mathbb{R})$. Thus A^{α}_{ij} is a linear combination of $\cos\alpha, \sin\alpha$, hence smooth. $\square$



(9.1)

For each $k \geq 0$ and $\hat{n} \in S^2$ we have described a smooth path in the space of unitary transformations of $\mathcal{H}_k(S^2)$ to itself, or more concretely in $(2k+1) \times (2k+1)$ matrices. This path is actually a loop since if we increase α by 2π then $R^{\hat n}_{\alpha+2\pi} = R^{\hat n}_\alpha$. By Exercise L3-6 every element of $SO(3)$ is of the form $R^{\hat n}_\alpha$ for some (non-unique) $\hat n, \alpha$, so if we can understand this loop we have completely understood how $SO(3)$ acts on $\mathcal{H}_k(S^2)$ and thus $L^2(S^2; \mathbb{C})$.

To understand the path $\alpha \longmapsto \delta(R^{\hat n}_\alpha)|_{\mathcal{H}_k(S^2)}$ we study the related path $\alpha \longmapsto \rho(R^{\hat n}_\alpha)$ and its derivative. We choose the $\mathbb{C}$-basis $\mathcal{B} = \{x^\beta\}_{|\beta|=k}$ of monomials for P_k and begin with the case $\hat n = (1,0,0)$ of rotation $R^{\hat n}_\alpha = R^x_\alpha$ about the x-axis. The general case is no more difficult. In this case

$$A^\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}$$

and so by (8.2)

$$\frac{d}{d\alpha}\big(\rho(R^x_\alpha)\big)(x^\beta) = \frac{d}{d\alpha}\, \mathcal{G}_{A^\alpha}(x^\beta) \tag{9.2}$$

$$= \frac{d}{d\alpha}\Big[x_1^{\beta_1} (\cos\alpha\, x_2 - \sin\alpha\, x_3)^{\beta_2} (\sin\alpha\, x_2 + \cos\alpha\, x_3)^{\beta_3} \Big]$$

$$= x_1^{\beta_1} \cdot \beta_2 \cdot (\cos\alpha\, x_2 - \sin\alpha\, x_3)^{\beta_2 - 1} \big[-\sin\alpha\, x_2 - \cos\alpha\, x_3 \big] (\sin\alpha\, x_2 + \cos\alpha\, x_3)^{\beta_3}$$

$$+ x_1^{\beta_1} \cdot (\cos\alpha\, x_2 - \sin\alpha\, x_3)^{\beta_2} \cdot \beta_3 \cdot (\sin\alpha\, x_2 + \cos\alpha\, x_3)^{\beta_3 - 1} \big[\cos\alpha\, x_2 - \sin\alpha\, x_3 \big]$$

$$= -\beta_2 x_1^{\beta} (\cos\alpha x_2 - \sin\alpha x_3)^{\beta_2 - 1} (\sin\alpha x_2 + \cos\alpha x_3)^{\beta_3 + 1}$$

$$+ \beta_3 x_1^{\beta} (\cos\alpha x_2 - \sin\alpha x_3)^{\beta_2 + 1} (\sin\alpha x_2 + \cos\alpha x_3)^{\beta_3 - 1}$$

$$= \varphi_{A^\alpha}\left(\left[x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2} \right](x^\beta) \right)$$

We have proven

<u>Lemma L4-3</u> As linear operators on $\mathcal{P}_k$ we have $\frac{d}{d\alpha}\rho(R^x_\alpha) = \rho(R^x_\alpha) \circ \left\{ x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2} \right\}$.

Given a polynomial $P \in \mathcal{P}_k$ this shows that as functions on $\mathbb{R}^3$

$$\frac{d}{d\alpha}\left(\rho(R^x_\alpha)\right)(P) = \left\{ x_2 \frac{\partial}{\partial x_3}(P) - x_3 \frac{\partial}{\partial x_2}(P) \right\} \circ R^x_\alpha \qquad (10.1)$$

Now $\frac{d}{d\alpha}\rho(R^x_\alpha)$ is by construction a smooth function $\mathbb{R} \longrightarrow \mathrm{End}_{\mathbb{C}}(\mathcal{P}_k)$ which we may evaluate at $\alpha = 0$ to yield an operator $\frac{d}{d\alpha}(\rho(R^x_\alpha))|_{\alpha=0}$ on $\mathcal{P}_k$, which by (10.1) is

$$P \longmapsto x_2 \frac{\partial}{\partial x_3}(P) - x_3 \frac{\partial}{\partial x_2}(P) \qquad (10.2)$$

That is, as $\mathbb{C}$-linear operators on $\mathcal{P}_k$

$$\frac{d}{d\alpha}\left(\rho(R^x_\alpha)\right)\Big|_{\alpha=0} = x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2} \qquad (10.3)$$

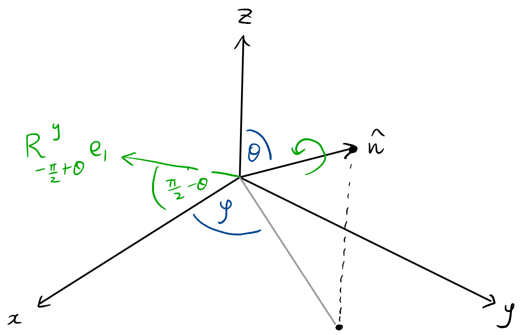
The appearance of the operator $x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2}$ as the <u>infinitesimal</u> version of the symmetry $\rho(R^x_\alpha)$ of the $\mathbb{C}$-vector space $\mathcal{P}_k$ is our first hint of Lie algebras. One way of reading (10.3) is that for α small we have a good approximation (of operators)

$$\rho(R^x_\alpha) \approx 1 + \alpha \cdot \left\{ x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2} \right\} \qquad (10.4)$$

Next we consider the general case where $\hat{n} \in S^2$ and with the notation of (8.3)

$$R^{\hat{n}}_{\alpha} = R^z_{\varphi} R^y_{\theta-\frac{\pi}{2}} R^x_{\alpha} R^y_{\frac{\pi}{2}-\theta} R^z_{-\varphi}. \qquad (11.1)$$

The relevant differential operator is now $t_2 \frac{\partial}{\partial t_3} - t_3 \frac{\partial}{\partial t_2}$ where t_1, t_2, t_3 are the coordinates with the same orientation as x_1, x_2, x_3 but rotated to have $\hat{n}$ as the x-axis:



$$T^{\hat{n}} := [R^y_{\frac{\pi}{2}-\theta} R^z_{-\varphi}]^S_S$$

$$t_i = \mathcal{J}_{T^{\hat{n}}}(x_i) = \sum_{j=1}^{3} T^{\hat{n}}_{ij} x_j \qquad (11.2)$$

<u>Lemma L4-4</u> As linear operators on $\mathcal{P}_k$ we have $\frac{d}{d\alpha}\rho(R^{\hat{n}}_{\alpha}) = \rho(R^{\hat{n}}_{\alpha}) \circ \{ t_2 \frac{\partial}{\partial t_3} - t_3 \frac{\partial}{\partial t_2} \}$

<u>Proof</u> Since ρ is a representation and $\frac{d}{d\alpha}$ satisfies the Leibniz rule (Ex L4-4)

$$\frac{d}{d\alpha}\rho(R^{\hat{n}}_{\alpha}) = \frac{d}{d\alpha}\Big(\rho(R^z_{\varphi} R^y_{\theta-\frac{\pi}{2}} R^x_{\alpha} R^y_{\frac{\pi}{2}-\theta} R^z_{-\varphi})\Big)$$

$$= \frac{d}{d\alpha}\Big(\rho(R^y_{\frac{\pi}{2}-\theta} R^z_{-\varphi})\rho(R^x_{\alpha})\rho(R^z_{\varphi} R^y_{\theta-\frac{\pi}{2}})\Big)$$

$$= \rho(R^y_{\frac{\pi}{2}-\theta} R^z_{-\varphi}) \frac{d}{d\alpha}\rho(R^x_{\alpha})\, \rho(R^z_{\varphi} R^y_{\theta-\frac{\pi}{2}})$$

Lemma L4-3

$$= \rho(R^y_{\frac{\pi}{2}-\theta} R^z_{-\varphi})\rho(R^x_{\alpha})\{ x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2}\}\rho(R^z_{\varphi} R^y_{\theta-\frac{\pi}{2}})$$

$$= \rho(R^x_{\alpha} R^y_{\frac{\pi}{2}-\theta} R^z_{-\varphi})\rho(R^z_{\varphi} R^y_{\theta-\frac{\pi}{2}})\rho(R^y_{\frac{\pi}{2}-\theta} R^z_{-\varphi})\{ x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2}\}\rho(R^z_{\varphi} R^y_{\theta-\frac{\pi}{2}})$$

$$= \rho(R^{\hat{n}}_{\alpha})\, \mathcal{J}_{T^{\hat{n}}} \{ x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2}\} \mathcal{J}_{T^{\hat{n}}}^{-1}$$

Now observe that on a monomial t^β

$$\varphi_{T^{\hat{n}}} \circ \left\{ x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2} \right\} \circ \varphi_{T^{\hat{n}}}^{-1} (t^\beta)$$

$$= \varphi_{T^{\hat{n}}} \left(x_2 \frac{\partial}{\partial x_3} (x^\beta) - x_3 \frac{\partial}{\partial x_2} (x^\beta) \right) \qquad (12.1)$$

$$= \varphi_{T^{\hat{n}}} \left(\beta_3 x_2 x_1^{\beta_1} x_2^{\beta_2} x_3^{\beta_3 - 1} - \beta_2 x_3 x_1^{\beta_1} x_2^{\beta_2 - 1} x_3^{\beta_3} \right)$$

$$= \beta_3 t_2 t_1^{\beta_1} t_2^{\beta_2} t_3^{\beta_3 - 1} - \beta_2 t_3 t_1^{\beta_1} t_2^{\beta_2 - 1} t_3^{\beta_3}$$

$$= \left(t_2 \frac{\partial}{\partial t_3} - t_3 \frac{\partial}{\partial t_2} \right) (t^\beta)$$

Since the t^β form a $\mathbb{C}$-basis for P_k we conclude that

$$\frac{d}{d\alpha} \rho(R^{\hat{n}}_\alpha) = \rho(R^{\hat{n}}_\alpha) \circ \left\{ t_2 \frac{\partial}{\partial t_3} - t_3 \frac{\partial}{\partial t_2} \right\}$$

as claimed. □

You might notice that Lemma L4-4 looks similar to a familiar <u>differential equation</u> $\frac{d}{dx} f(x) = \lambda f(x)$ with initial condition $f(0) = 1$ and unique solution $f(x) = e^{\lambda x}$. Indeed it <u>is</u> a system of differential equations and has a unique solution for this reason, by Picard's theorem [MHS, Lecture 15]. We are therefore led to conclude that the action $\rho(R^{\hat{n}}_\alpha)$ is a <u>matrix exponential</u> as stated in the theorem below.

With $d = \dim_{\mathbb{C}} P_k$ we give $\mathrm{End}_{\mathbb{C}}(P_k)$ the metric induced by the isomorphism $\mathrm{End}_{\mathbb{C}}(P_k) \cong \mathbb{C}^{d^2}$ of (6.1) for any basis β and the standard metric on $\mathbb{C}^{d^2}$. Given $\varphi \in \mathrm{End}_{\mathbb{C}}(P_k)$ the series $\exp(\varphi) := \sum_{i \geq 0} \frac{1}{i!} \varphi^i$ converges in this metric. You may have seen this proved elsewhere, but it also follows from the proof of the next theorem.

<u>Theorem L4-5</u> As operators on $\mathcal{P}_k$ we have $\rho(R^{\hat{n}}_\alpha) = \exp(\alpha[t_2 \frac{\partial}{\partial t_3} - t_3 \frac{\partial}{\partial t_2}])$.

<u>Proof</u> Let M denote the matrix of the operator $t_2 \frac{\partial}{\partial t_3} - t_3 \frac{\partial}{\partial t_2}$ with respect to some ordered basis of $\mathcal{P}_k$ (we don't care which). Then with X a $d \times d$ matrix of unknown functions of α the ordinary differential equation

$$\frac{d}{d\alpha} X = XM \qquad (13.1)$$

is actually a <u>system</u> of d^2 differential equations

$$\frac{d}{d\alpha} X_{ij}(\alpha) = \sum_{\ell=1}^{d} X_{i\ell}(\alpha) M_{\ell j}. \qquad (13.2)$$

If we add the initial condition that $X(0) = I_d$ then we may apply Picard's theorem in the form stated in [MHS, Ex L15-1] (see Tutorial 8 of the 2020 class for a full solution) to see that the initial value problem has a <u>unique</u> solution on $[-\delta, \delta] \subseteq \mathbb{R}$ for some δ. As in [MHS, Example L15-1] this may be extended to all of $\mathbb{R}$, that is, there is a unique sol^N on $\mathbb{R}$ and moreover it is the fixed point of the iteration on matrices of functions

$$X^{(n+1)}(\alpha) = 1 + \int_0^\alpha X^{(n)}(\gamma) M \, d\gamma \qquad (13.3)$$

starting with $X^{(0)} = 1$. It is easy to see that $X^{(1)} = 1 + \alpha M$,

$$\begin{aligned} X^{(2)}(\alpha) &= 1 + \int_0^\alpha (1 + \gamma M) M \, d\gamma \\ &= 1 + \alpha M + \tfrac{1}{2}\alpha^2 M^2 \qquad (13.4) \end{aligned}$$

and by induction $X^{(n)}(\alpha) = \sum_{i=0}^{n} \frac{1}{n!} \alpha^n M^n$. Hence the unique solution, which is $\lim_{n \to \infty} X^{(n)}$, is the convergent series $\exp(\alpha M)$ as claimed. Now by Lemma L4-4, $\rho(R^{\hat{n}}_\alpha)$ is <u>also</u> a solution of this IVP, hence by uniqueness $\rho(R^{\hat{n}}_\alpha) = \exp(\alpha M)$. □

Remark Hence for any polynomial P

$$\rho(R^{\hat{n}}_{\alpha})(P) = P + \alpha\left[t_2\frac{\partial}{\partial t_3} - t_3\frac{\partial}{\partial t_2}\right](P) + \frac{1}{2}\alpha^2\left[t_2\frac{\partial}{\partial t_3} - t_3\frac{\partial}{\partial t_2}\right]^2(P) + \cdots$$

Exercise L4-5 In the proof of Theorem L4-5 prove that the hypotheses for Picard's theorem are satisfied, following the approach in [MHS, Tutorial 8 2020].

Exercise L4-6 What trigonometric identity explains the suspicious looking formula

$$1 = \rho(R^{\hat{n}}_{2\pi}) = \exp\left(2\pi\left[t_2\frac{\partial}{\partial t_3} - t_3\frac{\partial}{\partial t_2}\right]\right).$$

Exercise L4-7 Prove that if $P \in \mathcal{P}_k$ is harmonic then so is $\left[t_2\frac{\partial}{\partial t_3} - t_3\frac{\partial}{\partial t_2}\right](P)$ where t_1, t_2, t_3 are the coordinates associated as above to arbitrary $\hat{n}$.

Exercise L4-8 Write formulas for $\mathfrak{z}(R^x_\alpha)$, $\mathfrak{z}(R^y_\alpha)$, $\mathfrak{z}(R^z_\alpha)$ as operators on $\mathcal{H}_k(S^2)$, as matrix exponentials of differential operators in spherical coordinates. By inventing your own spherical coordinates write a general formula for $\mathfrak{z}(R^{\hat{n}}_\alpha)$ as an exponential.

Exercise L4-9* Find explicit formulas for $\operatorname{tr}\mathfrak{z}(R^{\hat{n}}_\alpha)$, $\det\mathfrak{z}(R^{\hat{n}}_\alpha)$ on $\mathcal{H}_k(S^2)$ (cf Ex L4-4 (e)).

Exercise L4-10 In Lecture 3 p 16.5 we defined what it means for a function $f: U \longrightarrow \mathbb{C}$, where $U \subseteq S^2$ is open, to be smooth. Using this, give a definition for what it means for a function $f: U \longrightarrow \operatorname{End}_{\mathbb{C}}(V)$ to be smooth, where $U \subseteq S^2$ is open and V is a finite-dimensional $\mathbb{C}$-vector space. According to this definition prove that the function $S^2 \longrightarrow \operatorname{End}_{\mathbb{C}}(\mathcal{H}_k(S^2))$ sending $\hat{n}$ to $\mathfrak{z}(R^{\hat{n}}_\alpha)|_{\mathcal{H}_k(S^2)}$ is smooth, where $\alpha \in \mathbb{R}$ is arbitrary and fixed.

In conclusion, we have for $\hat{n} \in S^2$ and $\alpha \in \mathbb{R}$ a commutative diagram

$$
\begin{tikzcd}
\mathcal{H}_k(S^2) \arrow[r, "\cong"] \arrow[d, "\mathcal{S}(R^{\hat{n}}_\alpha)"'] & \mathcal{H}_k \arrow[r, "\mathrm{inc}"] \arrow[d] & P_k \arrow[d, "\rho(R^{\hat{n}}_\alpha)"] \\
\mathcal{H}_k(S^2) \arrow[r, "\cong"'] & \mathcal{H}_k \arrow[r, "\mathrm{inc}"'] & P_k
\end{tikzcd}
\tag{15.1}
$$

The operator $t_2 \frac{\partial}{\partial t_3} - t_3 \frac{\partial}{\partial t_2}$ on P_k restricts to $\mathcal{H}_k$ and thus to spherical harmonics, and when viewed as an operator on $\mathcal{H}_k(S^2)$ we have by Theorem L4-5

$$\mathcal{S}(R^{\hat{n}}_\alpha) = \exp\Big(\alpha\Big[t_2 \frac{\partial}{\partial t_3} - t_3 \frac{\partial}{\partial t_2}\Big]\Big) \tag{15.2}$$

While the action $\mathcal{S}(R^{\hat{n}}_\alpha)(\Psi) = \Psi \circ R^{\hat{n}}_\alpha$ is quite explicit on a spherical harmonic, the form (15.2) of the action is much more useful, since it presents the unitary transformation $\mathcal{S}(R^{\hat{n}}_\alpha)$ of $\mathcal{H}_k(S^2)$ as the exponential of an <u>infinitesimal symmetry</u>, namely the differential operator $t_2 \frac{\partial}{\partial t_3} - t_3 \frac{\partial}{\partial t_2}$, in the sense that

$$\mathcal{S}(R^{\hat{n}}_\alpha) = 1 + \alpha\Big[t_2 \frac{\partial}{\partial t_3} - t_3 \frac{\partial}{\partial t_2}\Big] + O(\alpha^2) \tag{15.3}$$

With this in hand we are one step closer to "completely" understanding the representation $\mathcal{S}$ of $SO(3)$ on $L^2(S^2, \mathbb{C})$. To proceed further we will develop the abstract theory of these infinitesimal symmetries.

<u>Exercise L4-11</u> Give orthonormal bases for $\mathcal{H}_0(S^2)$, $\mathcal{H}_1(S^2)$, $\mathcal{H}_2(S^2)$, $\mathcal{H}_3(S^2)$ and compute the matrix of $\mathcal{S}(R^{x}_\alpha)$ in each of these bases.

Exercise L4-12 With the notation of (11.2) prove

(i) if $\hat{n} = (0,1,0)$ then $t_1 = x_2$, $t_2 = -x_1$, $t_3 = x_3$

(ii) if $\hat{n} = (0,0,1)$ then $t_1 = x_3$, $t_2 = x_2$, $t_3 = -x_1$.

References

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