Lecture 5 – Angular momentum

12/4/21
updated 26/4/21

Driven to a large degree by physics, our mathematical conception of space has evolved rapidly over the course of the last century. The MHS lectures recapitulate the first part of this evolution, from the study of spaces as sets X of <u>points with structure</u> (e.g. metric or topological spaces) to the study of <u>spaces of functions</u> on X, and finally to the <u>Hilbert space</u> of X (vectors in which are not properly viewed as functions, but rather signals or wavefunctions propagating on X).

In particular to the spheres S^n we may associate function spaces $\mathrm{Cts}(S^n, \mathbb{C})$ and the Hilbert spaces $L^2(S^n, \mathbb{C})$, the cases $n \in \{1,2\}$ of which we have studied carefully.

However this basic picture remains incomplete. Space is not something we perceive directly. It is what a neuroscientist or machine learner would call an <u>inductive bias</u>: a parsimonious set of hypotheses about the structure of the world that efficiently compresses observations [ON, §1]. This is different language for an idea already exposited in Lectures 1 and 2, namely, that given information about a (quantum) system as obtained by one of a set of equivalent observers we can <u>analytically</u> (i.e. without recourse to observation) determine the state according to any other observer in the set. To perform this trick we need to know how to parametrise observers by elements of a Lie group (<u>symmetry</u>) and how this group acts by unitary transformations on Hilbert space (<u>representation</u>). Thus symmetry is revealed as a form of compression.

Now, concepts like "observers" and "measurement" might seem outside the scope of mathematics, but <u>so too is the choice of which objects (or axioms) to study</u>, and considerations like the above strongly suggest that the correct mathematical representative of "the sphere" is not S^n or $\mathrm{Cts}(S^n, \mathbb{C})$ or even $L^2(S^n, \mathbb{C})$, but rather "a Hilbert space for each observer and all translations between them" or what is the same the representation $\mathcal{J}$ of $SO(n+1)$ on $L^2(S^n, \mathbb{C})$ by unitary transformations.

What I'm trying to say is: <u>you have to care</u> about the operators $t_2 \frac{\partial}{\partial t_3} - t_3 \frac{\partial}{\partial t_2}$ of Theorem L4-5!

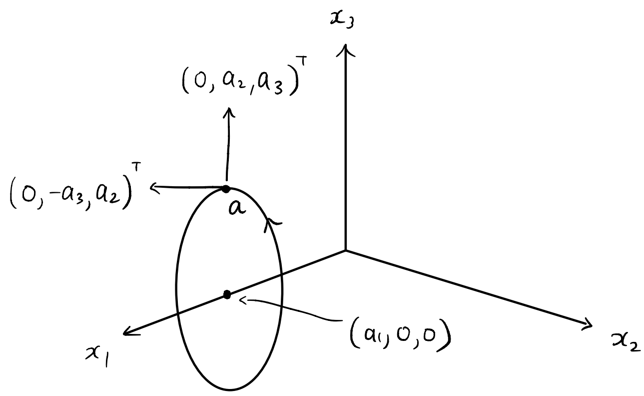
So how should we think about these operators? We focus on $x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2}$ but everything we have to say applies to the general case. Let $f = f(x_1, x_2, x_3)$ be a function with continuous partial derivatives on some open set $U \subseteq \mathbb{R}^3$ and let $a \in \mathbb{R}^3$. Recall that the <u>directional derivative</u> $D_v(f)(a)$ of f at a along a vector $v \in \mathbb{R}^3$ is

$$\lim_{h \to 0} \frac{f(a+hv) - f(a)}{h} = \sum_{i=1}^{n} v_i \frac{\partial f}{\partial x_i}(a) \qquad (2.1)$$

and so given a function f we can interpret $\left[x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2} \right](f)$ as the function which, given $a \in \mathbb{R}^3$ as input, returns the directional derivative

$$a_2 \frac{\partial f}{\partial x_3}(a) - a_3 \frac{\partial f}{\partial x_2}(a) \qquad (2.2)$$

of f at a along the vector $v = (0, -a_3, a_2)$. This has a simple geometric interpretation:



(2.3)

We view $a \in \mathbb{R}^3$ as lying on a circle of radius $\sqrt{a_2^2 + a_3^2}$ centered at $(a_1, 0, 0)^T$. A normal vector to this circle at a is given by $(0, a_2, a_3)^T$ (think of ∇f where $f = x_2^2 + x_3^2$) and rotating this by $\frac{\pi}{2}$ anticlockwise in the x_2-x_3 plane gives $(0, -a_3, a_2)^T$. So the directional derivative (2.2) computes for a function f defined in a neighbourhood of a, <u>the rate of change of f along the direction of rotation</u> around the x_1-axis using the right hand rule. The operator $x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2}$ sends each function f to the function which computes this directional derivative at each point $a \in \mathbb{R}^3$.

More generally, with $\hat{n}$ a unit vector and t_1, t_2, t_3 being as in (11.2) of Lecture 4, the operator $t_2 \frac{\partial}{\partial t_3} - t_3 \frac{\partial}{\partial t_2}$ computes a directional derivative tangent to a circle centered on the $\hat{n}$-axis and in the plane orthogonal to that axis, in the direction of right hand rotation.

This interpretation as a directional derivative makes the formula

$$f \circ R^{\hat{n}}_{\alpha} = \exp\left(\alpha\left[t_2 \frac{\partial}{\partial t_3} - t_3 \frac{\partial}{\partial t_2}\right]\right)(f) \qquad (3.1)$$

of Theorem L4-5 look very similar to a <u>Taylor expansion around the sphere</u>. To explain, recall that if we write $S_\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ for the operation $S_\alpha(x) = x + \alpha$ then for a function $g(x)$ with a converging Taylor series expansion in a suitable open neighbourhood (e.g. any polynomial)

$$\begin{aligned} g(x+\alpha) &= g(x) + \frac{\partial g}{\partial x}\alpha + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}\alpha^2 + \cdots \\ &= \sum_{i=0}^{\infty} \frac{1}{i!}\frac{\partial^i g}{\partial x^i}\alpha^i \qquad (3.2) \\ &= \exp\left(\alpha \frac{\partial}{\partial x}\right)(g) \end{aligned}$$

That is, the usual <u>Taylor expansion along a line</u> can be presented as a functional equation

$$g \circ S_\alpha = \exp\left(\alpha \frac{\partial}{\partial x}\right)(g) \qquad (3.3)$$

expressing the action on g of the translation symmetry S_α as the action of an exponential of the differential operator $\alpha \frac{\partial}{\partial x}$. We are familiar with the fundamental role of the Taylor expansion in bridging the gap between the <u>infinitesimal and the finite</u>; now we must learn to view <u>exponentials of differential operators</u> as a more general and powerful tool of the same nature. Next lecture we will develop this insight into a general theory.

<u>Def^n</u> Given a unit vector $\hat{n}$ we call the operator $\mathcal{L}^{\hat{n}} = t_2 \frac{\partial}{\partial t_3} - t_3 \frac{\partial}{\partial t_2}$ (acting on $\mathcal{P}_k$, $\mathcal{H}_k$, or $\mathcal{H}_k(S^2)$ as the case may be) the <u>infinitesimal generator</u> of the symmetry $R^{\hat{n}}_\alpha$ acting on functions via the representation $f \mapsto f \circ R^{\hat{n}}_\alpha$.

Anytime we are presented with a linear operator we are naturally curious about its eigenvectors and eigenvalues. If $g: \mathbb{R} \to \mathbb{C}$ smooth is an eigenvector of $\frac{\partial}{\partial x}$ with eigenvalue λ then

$$\frac{\partial}{\partial x} g = \lambda g \implies g = Ce^{\lambda x} \quad \text{some } C \in \mathbb{R} \tag{4.1}$$

That is, the space of eigenvectors is one-dimensional and spanned by $e^{\lambda x}$. In this case the RHS of (3.3) reads as

$$\begin{aligned} \exp\left(\alpha \tfrac{\partial}{\partial x}\right)(g) &= \left[\sum_{j=0}^{\infty} \frac{1}{j!} \alpha^j \frac{\partial^j}{\partial x^j} \right](g) \\ &= \sum_{j=0}^{\infty} \frac{1}{j!} \alpha^j \lambda^j g \qquad (4.2) \\ &= e^{\alpha\lambda} g \end{aligned}$$

which of course matches the LHS which is $g \circ S_\alpha = Ce^{\lambda(x+\alpha)} = e^{\alpha\lambda} Ce^{\lambda x} = C^{\alpha\lambda} g$. That is, for an eigenvector of $\frac{\partial}{\partial x}$ the translation symmetry acts by a multiplicative factor. Naturally we are curious about analogous functions on S^2 which are eigenvectors for the infinitesimal generators $\mathcal{L}^{\hat{n}}$. By the same argument as in (4.2)

<u>Lemma L5-1</u> If f is an eigenvector of $\mathcal{L}^{\hat{n}}$ with eigenvalue λ then

$$f \circ R^{\hat{n}}_\alpha = e^{\alpha\lambda} f \qquad \forall \alpha \in \mathbb{R}.$$

A continuous function $f: S^2 \to \mathbb{C}$ is in the kernel of $\mathcal{L}^{\hat{n}}$ if and only if it is rotationally symmetric about the $\hat{n}$-axis, i.e. $f \circ R^{\hat{n}}_\alpha = f$ for all $\alpha \in \mathbb{R}$.

Next we turn to the question: <u>what kind of operator</u> are the $\hat{\gamma}^n$? First recall that a linear operator T on an inner product space $\mathcal{H}$ is called <u>self-adjoint</u> if

$$\langle Tx, y \rangle = \langle x, Ty \rangle \qquad \forall x, y \in \mathcal{H}.$$

<u>Lemma L5-2</u> (Spectral Theorem for Self-Adjoint operators) If $\mathcal{H}$ is a finite-dimensional inner product space and $T: \mathcal{H} \to \mathcal{H}$ is a self-adjoint operator then

(a) all eigenvalues of T are real
(b) eigenvectors corresponding to distinct eigenvectors are orthogonal
(c) if $\mathbb{F} = \mathbb{C}$ then there exists an orthogonal basis of $\mathcal{H}$ consisting of eigenvectors of T.

<u>Proof</u> (a) Suppose $v \in \mathcal{H}$ is nonzero and $Tv = \lambda v$. Then

$$\langle v, Tv \rangle = \langle v, \lambda v \rangle = \lambda \langle v, v \rangle,$$
$$\langle Tv, v \rangle = \langle \lambda v, v \rangle = \bar{\lambda} \langle v, v \rangle.$$

Since T is self-adjoint these are equal and hence as $\langle v, v \rangle \neq 0$, $\lambda = \bar{\lambda}$ is real.

(b) Suppose $v, w \neq 0$ satisfy $Tv = \lambda v$, $Tw = \mu w$ with $\lambda \neq \mu$. Then

$$\langle Tv, w \rangle = \langle \lambda v, w \rangle = \lambda \langle v, w \rangle, \qquad (\text{using (a) } \lambda = \bar{\lambda})$$
$$\langle v, Tw \rangle = \langle v, \mu w \rangle = \mu \langle v, w \rangle.$$

Since $\lambda \neq \mu$ we conclude $\langle v, w \rangle = 0$ and hence v, w are linearly independent.

(c) By induction on $n = \dim \mathcal{H}$. If $n = 0$ (so $\mathcal{H} = 0$) the basis is the empty set. Suppose the statement holds for $n \geq 0$ and $\dim \mathcal{H} = n+1$. Any operator T on a nonzero complex vector space has an eigenvector v_1, say $Tv_0 = \lambda_0 v_0$. Let $\mathcal{H}'$ be $\{v_0\}^\perp$. This has dimension $\leq n$ and we claim $T\mathcal{H}' \subseteq \mathcal{H}'$. To see this note that if $w \in \mathcal{H}'$ then

$$\langle v_0, Tw\rangle = \langle Tv_0, w\rangle = \langle \lambda_0 v_0, w\rangle = \lambda_0 \langle v_0, w\rangle = 0,$$

hence $Tw \in \mathcal{H}'$. Hence by the inductive hypothesis $\mathcal{H}'$ has an orthogonal basis $v_1, \dots, v_k$ consisting of eigenvectors for T, and since $\mathcal{H} = \operatorname{span}_{\mathbb{C}}\{v_0\} \oplus \mathcal{H}'$ the set $\{v_0, v_1, \dots, v_k\}$ is an orthogonal basis for $\mathcal{H}$ consisting of eigenvectors of T. $\square$

<u>Lemma L5-3</u> (Spectral Theorem for Unitary Operators) If $\mathcal{H}$ is a finite-dimensional inner product space and $U : \mathcal{H} \to \mathcal{H}$ is unitary then

(a) all eigenvalues of U have modulus 1

(b) eigenvectors corresponding to distinct eigenvectors are orthogonal

(c) if $\mathbb{F} = \mathbb{C}$ then there exists an orthogonal basis of $\mathcal{H}$ consisting of eigenvectors of U.

<u>Proof</u> (a) Let $Uv = \lambda v$ with $v \neq 0$. Then

$$\langle v, v\rangle = \langle Uv, Uv\rangle = \langle \lambda v, \lambda v\rangle = \bar{\lambda}\lambda \langle v, v\rangle$$

and since $\langle v, v\rangle$ we have $|\lambda|^2 = \bar{\lambda}\lambda = 1$.

(b) Let $Uv = \lambda v$, $Uw = \mu w$ with $\lambda \neq \mu$ and v, w nonzero. Then

$$\langle v, w\rangle = \langle Uv, Uw\rangle = \bar{\lambda}\mu \langle v, w\rangle.$$

If $\langle v, w\rangle \neq 0$ then $\bar{\lambda}\mu = 1$ and so multiplying both sides by λ, we have $\mu = \lambda\bar{\lambda}\mu = \lambda$, a contradiction. Hence $\langle v, w\rangle = 0$ as claimed.

(c) As in the proof of Lemma L5-2 suppose $\mathcal{H} \neq 0$ and let v_0 be an eigenvector of U with eigenvalue λ_0. It suffices to show with $\mathcal{H}' = \{v_0\}^{\perp}$ that $U\mathcal{H}' \subseteq \mathcal{H}'$. If $w \in \mathcal{H}'$ then $\langle v_0, Uw\rangle = \langle UU^{-1}v_0, Uw\rangle = \langle U^{-1}v_0, w\rangle$.

But from $Uv_0 = \lambda v_0$ we obtain $U^{-1}(v_0) = \lambda^{-1}v_0$ so $\langle v_0, Uw\rangle = \bar{\lambda}^{-1}\langle v_0, w\rangle = 0$. $\square$

note by (a) that $\lambda \neq 0$

Let $(\mathcal{H}, \langle , \rangle)$ be a finite-dimensional complex inner product space. We can find an orthonormal basis $\beta = (v_1, \dots, v_n)$ (possibly $n = 0$) and this induces an isomorphism of complex vector spaces

$$\begin{aligned} \mathcal{H} &\xrightarrow{\;c_\beta\;} \mathbb{C}^n \\ c_\beta\left(\textstyle\sum_{i=1}^n a_i v_i\right) &= (a_1, \dots, a_n) \end{aligned} \tag{7.1}$$

This is an isomorphism of <u>inner product spaces</u> if we define the pairing $\langle , \rangle$ on the standard basis $e_1, \dots, e_n$ of $\mathbb{C}^n$ by $\langle e_i, e_j \rangle = \langle v_i, v_j \rangle = \delta_{ij}$ so that $(\mathbb{C}^n, \langle , \rangle)$ is the standard inner product $\langle a, b \rangle = \sum_{i=1}^n \overline{a_i} b_i$ where $a, b \in \mathbb{C}^n$. If T is a linear operator on $\mathcal{H}$ then there is a unique linear operator T' on $\mathbb{C}^n$ making the diagram

$$\begin{tikzcd} \mathcal{H} \arrow[r, "c_\beta"] \arrow[d, "T"'] & \mathbb{C}^n \arrow[d, "T'"] \\ \mathcal{H} \arrow[r, "c_\beta"'] & \mathbb{C}^n \end{tikzcd} \tag{7.2}$$

commute (namely $T' = c_\beta \circ T \circ c_\beta^{-1}$) and T is self-adjoint or unitary iff. T' is so. In particular note that $(\mathcal{H}, \|-\|)$ with the norm induced by $\langle , \rangle$ is isomorphic as a normed (hence metric) space to $(\mathbb{C}^n, \|-\|)$ and hence $(\mathcal{H}, \langle , \rangle)$ is a Hilbert space.

<u>Exercise L5-1</u> Let $(\mathcal{H}, \langle , \rangle)$ be a finite-dimensional complex inner product space. We say a function $f : \mathcal{H} \longrightarrow \mathbb{C}$ is <u>continuous</u> (resp. <u>smooth</u>) if for every ordered orthonormal basis β the function $\mathbb{R}^{2n} = \mathbb{C}^n \longrightarrow \mathbb{C} = \mathbb{R}^2$ given by $f \circ c_\beta^{-1}$ is continuous (resp. smooth).

(i) Prove f is continuous (resp. smooth) if this condition holds for <u>any</u> β.

(ii) Prove that for any $v \in \mathcal{H}$ the functions $\langle v, - \rangle, \langle -, v \rangle : \mathcal{H} \to \mathbb{C}$ are both smooth.

<u>Lemma L5-4</u> Let $\mathcal{H}$ be a finite-dimensional complex inner product space and $T : \mathcal{H} \to \mathcal{H}$ a linear operator. Then T is self-adjoint if and only if $e^{i\alpha T}$ is unitary for every $\alpha \in \mathbb{R}$. Moreover every unitary operator on $\mathcal{H}$ is of the form e^{iT} for some self-adjoint operator T.

<u>Proof</u> Suppose T is self-adjoint. Then using Ex L5-1

$$
\begin{aligned}
\langle e^{iT}x, e^{iT}y \rangle &= \langle \lim_{n\to\infty} \sum_{j=0}^{n} \tfrac{1}{j!} i^j T^j x, e^{iT}y \rangle \\
&\overset{\langle -, v\rangle \text{ cts}}{=} \lim_{n\to\infty} \langle \textstyle\sum_{j=0}^{n} \tfrac{1}{j!} i^j T^j x, e^{iT}y \rangle \\
&= \lim_{n\to\infty} \textstyle\sum_{j=0}^{n} \tfrac{1}{j!} (-i)^j \langle T^j x, e^{iT}y \rangle \\
&\overset{T \text{ self-adjoint}}{=} \lim_{n\to\infty} \textstyle\sum_{j=0}^{n} \tfrac{1}{j!} (-i)^j \langle x, T^j e^{iT}y \rangle \\
&= \lim_{n\to\infty} \langle x, \textstyle\sum_{j=0}^{n} \tfrac{1}{j!} (-i)^j T^j e^{iT}y \rangle \\
&\overset{\langle v, -\rangle \text{ cts}}{=} \langle x, \lim_{n\to\infty} \textstyle\sum_{j=0}^{n} \tfrac{1}{j!} (-iT)^j e^{iT}y \rangle \\
&= \langle x, e^{-iT} e^{iT} y \rangle \\
&= \langle x, y \rangle
\end{aligned}
\tag{8.1}
$$

see Lemma B1-9

so e^{iT} is unitary. Now suppose that $e^{i\alpha T}$ is unitary for all $\alpha \in \mathbb{R}$ so that

$$\langle x, e^{i\alpha T} y \rangle = \langle e^{-i\alpha T} x, y \rangle \qquad \forall x, y \in \mathcal{H} \tag{8.2}$$

Differentiating both sides with respect to α (if you like, reduce via the previous page to the case of $\mathbb{C}^n$ and the standard pairing to verify the following)

$$\frac{d}{d\alpha}\langle x, e^{i\alpha T}y\rangle = \frac{d}{d\alpha}\langle e^{-i\alpha T}x, y\rangle$$

$$\therefore \quad \langle x, \frac{d}{d\alpha}e^{i\alpha T}y\rangle = \langle \frac{d}{d\alpha}e^{-i\alpha T}x, y\rangle \tag{9.1}$$

$$\therefore \quad \langle x, iTe^{i\alpha T}y\rangle = \langle -iTe^{-i\alpha T}x, y\rangle$$

Evaluating at $\alpha = 0$ yields

$$\langle x, iTy\rangle = \langle -iTx, y\rangle$$

and hence $\langle x, Ty\rangle = \langle Tx, y\rangle$ so that T is self-adjoint.

Finally suppose $U: \mathcal{H} \to \mathcal{H}$ is unitary and let $v_1, \ldots, v_n$ be an orthonormal basis for $\mathcal{H}$ consisting of eigenvectors for U (Lemma L5-3) with say $Uv_k = \lambda_k v_k$. Then $|\lambda_k| = 1$ by Lemma L5-3 (i) say $\lambda_k = e^{i\theta_k}$. Let $T: \mathcal{H} \to \mathcal{H}$ be the linear transformation $T(v_k) = \theta_k v_k$. Then T is certainly self-adjoint as

$$\begin{aligned} &\langle T(\textstyle\sum_{k=1}^n a_k v_k), \sum_{k=1}^n b_k v_k\rangle \\ = &\langle \textstyle\sum_{k=1}^n a_k \theta_k v_k, \sum_{k=1}^n b_k v_k\rangle \\ = &\textstyle\sum_{k=1}^n \bar{a}_k \theta_k b_k \\ = &\langle \textstyle\sum_{k=1}^n a_k v_k, T(\sum_{k=1}^n b_k v_k)\rangle \end{aligned} \tag{9.2}$$

and $e^{iT}(v_k) = e^{i\theta_k}v_k$ by the same argument as in (4.2), so $e^{iT} = U$. $\square$

The way to think about this is that every self-adjoint operator T (an <u>observable</u> in the language of quantum mechanics) <u>generates</u> a 1-parameter family $\{e^{i\alpha T}\}_{\alpha\in\mathbb{R}}$ of unitary transformations (symmetries of Hilbert space). In fact we will show later that any reasonable 1-parameter family of symmetries is so generated.

<u>Lemma L5-5</u> For $k \geq 0$ the operator $i\mathcal{T}^{\hat{n}}$ on $\mathcal{H}_k(S^2)$ is self-adjoint. Hence the eigenvalues of $\mathcal{T}^{\hat{n}}$ are all pure imaginary.

<u>Proof</u> By Theorem L4-5, $\delta(R^{\hat{n}}_\alpha) = \exp(\alpha\mathcal{T}^{\hat{n}}) = \exp(i[-i\alpha\mathcal{T}^{\hat{n}}])$ and since this operator is unitary it follows from Lemma L5-4 that $-i\alpha\mathcal{T}^{\hat{n}}$ is self-adjoint, and taking $\alpha \neq 0$ gives the first claim. For the second, any eigenvector v of $\mathcal{T}^{\hat{n}}$ with eigenvalue λ is an eigenvector of $i\mathcal{T}^{\hat{n}}$ with eigenvalue $i\lambda$, and since by Lemma L5-2 we have that $i\lambda$ is real it follows that $\mathrm{Re}(\lambda) = 0$. $\square$

To say that $i\mathcal{T}^{\hat{n}}$ is self-adjoint is to say that $\mathcal{T}^{\hat{n}}$ is <u>skew-self-adjoint</u>, i.e.

$$\langle \mathcal{T}^{\hat{n}} x, y\rangle = -\langle x, \mathcal{T}^{\hat{n}} y\rangle \qquad \forall x,y \in \mathcal{H}$$

<u>Def^n</u> We write $L_{\hat{n}} = -i\mathcal{T}^{\hat{n}}$ which gives a self-adjoint operator on $\mathcal{H}_k(S^2)$ for $k \geq 0$. In physics this is called the <u>angular momentum operator</u> associated to $\hat{n}$ and for the special cases of the standard coordinate axes we write $(x = x_1, y = x_2, z = x_3)$

$$L_x = -i\left[x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2}\right] \qquad \delta(R^x_\alpha) = \exp(i\alpha L_x)$$

$$L_y = -i\left[x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_3}\right] \qquad \delta(R^y_\alpha) = \exp(i\alpha L_y)$$

$$L_z = -i\left[x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}\right] \qquad \delta(R^z_\alpha) = \exp(i\alpha L_z)$$

We will not explain what angular momentum is here, but see [F, Ch. 18] for the best presentation I know. You will find there the formula [F, (18.16)] for the angular momentum of a particle moving in the xy-plane to be $xp_y - yp_x$ where the normal momentum is $p = (p_x, p_y, p_z)$. The process of quantisation replaces e.g. p_x by $-i\hbar \frac{\partial}{\partial x}$ so that $xp_y - yp_x$ becomes the operator $x(-i\hbar \frac{\partial}{\partial y}) - y(-i\hbar\frac{\partial}{\partial x})$ $= i\hbar[y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}]$ which with $\hbar = 1$ is L_z above.

<u>Def^n</u> A linear operator $T : \mathcal{H} \to \mathcal{H}$ on an inner product space (over either $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) is called <u>skew self-adjoint</u> if

$$\langle x, Ty \rangle = -\langle Tx, y \rangle \qquad \forall x, y \in \mathcal{H}.$$

<u>Exercise L5-2</u> If $\mathbb{F} = \mathbb{C}$ prove that T is skew self-adjoint if and only if iT is self-adjoint.

Recall that an operator T on an inner product space $\mathcal{H}$ is unitary if $\langle x, y \rangle = \langle Tx, Ty \rangle$. Some authors reserve the word "unitary" for <u>complex</u> inner product spaces and call such a T <u>orthogonal</u> if $\mathbb{F} = \mathbb{R}$. We are not such an author.

<u>Exercise L5-3</u> Let $\mathcal{H}$ be a finite-dimensional inner product space and $T : \mathcal{H} \longrightarrow \mathcal{H}$ a linear operator. Prove that T is skew self-adjoint if and only if $e^{\alpha T}$ is unitary for all $\alpha \in \mathbb{R}$.

<u>Exercise L5-4</u> Let $\mathcal{H} = \mathbb{R}^n$ with the standard inner product. Prove that if $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear operator with matrix $A \in M_n(\mathbb{R})$ that

(i) T is self-adjoint iff. A is symmetric (i.e. $A^T = A$)

(ii) T is skew self-adjoint iff A is skew-symmetric (i.e. $A^T = -A$)

(iii) T is unitary iff. A is orthogonal (i.e. $A^T A = I_n$).

References

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