Lecture 6 – Lie Groups and Lie Algebras

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We have defined a group representation for $k \geq 0$

$$\delta : SO(3)^{op} \longrightarrow \mathrm{End}_{\mathbb{C}}(\mathcal{H}_k(S^2))$$
$$\delta(R^{\hat{n}}_\alpha)(f) = f \circ R^{\hat{n}}_\alpha = \exp(\alpha \mathcal{L}^{\hat{n}})(f) \qquad (1.0)$$

where $\mathcal{L}^{\hat{n}}$ is a differential operator, the infinitesimal generator of rotations around the axis $\hat{n} \in S^2$. If we let $\mathrm{Aut}_{\mathbb{C}}(\mathcal{H}_k(S^2))$ denote the automorphisms of $\mathcal{H}_k(S^2)$, that is, the invertible linear transformations, then δ is a group homomorphism $SO(3)^{op} \longrightarrow \mathrm{Aut}_{\mathbb{C}}(\mathcal{H}_k(S^2))$. We have spent much of our time understanding the codomain: spherical harmonics and their operators. In particular we have seen that every $\delta(R^{\hat{n}}_\alpha)$ is unitary and that the generator $\mathcal{L}^{\hat{n}}$ is skew self-adjoint (Lemma L5-5). We now turn our attention to the domain, where we will discover similar structure.

Lemma L6-1 As linear operators on $\mathbb{R}^3$, we have for $\alpha \in \mathbb{R}$

$$R^x_\alpha = \exp\left(\alpha \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\right) \qquad (1.1)$$

Proof Set $T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$. The exponential is the limit in $M_3(\mathbb{R})$ with respect to the Frobenius norm (Remark B1-1) of the sequence of partial sums

$$a_n = I_3 + \alpha T + \tfrac{1}{2}\alpha^2 T^2 + \cdots + \frac{1}{(2n)!}\alpha^{2n} T^{2n} + \frac{1}{(2n+1)!}\alpha^{2n+1} T^{2n+1}$$
$$= \sum_{i=0}^{n} \frac{1}{(2i)!}\alpha^{2i} T^{2i} + \sum_{i=0}^{n} \frac{1}{(2i+1)!}\alpha^{2i+1} T^{2i+1}. \qquad (1.2)$$

For all n we have $a_n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{b_n} \\ 0 & & \end{pmatrix}$ where b_n is (1.2) with T replaced by the 2×2 matrix $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Since limits in $M_3(\mathbb{R})$ with respect to the Frobenius norm are just limits in $\mathbb{R}^9$ thinking of matrices as vectors, we see that $\exp(\alpha T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\exp(\alpha S)} \\ 0 & & \end{pmatrix}$.

Since $S^2 = -I_2$ we have

$$b_n = \sum_{i=0}^{n} \frac{1}{(2i)!} \alpha^{2i} S^{2i} + \sum_{i=0}^{n} \frac{1}{(2i+1)!} \alpha^{2i+1} S^{2i+1}$$

$$= \sum_{i=0}^{n} \frac{1}{(2i)!} (-1)^i \alpha^{2i} I_2 + \sum_{i=0}^{n} \frac{1}{(2i+1)!} (-1)^i \alpha^{2i+1} S \qquad (2\text{-}1)$$

and since we recognise these as the partial sums of the Taylor series expansion of cos, sin

$$\lim_{n\to\infty} b_n = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}. \qquad (2\text{-}2)$$

as claimed. $\square$

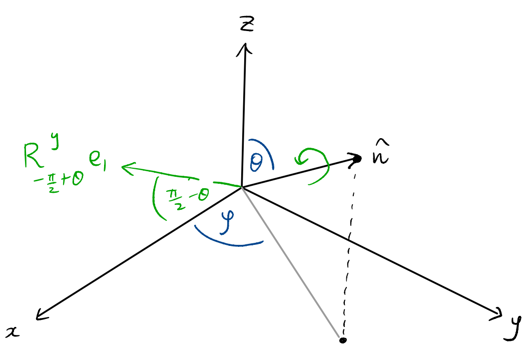
<u>Def^n</u> We define

$$\delta^x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \delta^y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \delta^z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

By the same argument $R^y_\alpha = \exp(\alpha\delta^y)$, $R^z_\alpha = \exp(\alpha\delta^z)$. Note that R^x_α is unitary (it preserves the dot product) and δ^x is skew symmetric $(\delta^x)^T = -\delta^x$, as it has to be by Ex L5-3, L5-4.

Let $\hat{n}$ have spherical angles θ, φ as per our usual notation (L3 p. 8), so that

$$R^{\hat{n}}_\alpha = R^z_\varphi \, R^y_{\theta-\frac{\pi}{2}} \, R^x_\alpha \, R^y_{\frac{\pi}{2}-\theta} \, R^z_{-\varphi}$$



Then using Ex B1-3 and Lemma L6-1 we compute

$$
\begin{aligned}
R^{\hat{n}}_{\alpha} &= R^{z}_{\varphi}\, R^{y}_{\theta-\frac{\pi}{2}} \exp(\alpha\, \delta^{x})\, R^{y}_{\frac{\pi}{2}-\theta}\, R^{z}_{-\varphi} \\
&= \exp\left(\alpha\, R^{z}_{\varphi}\, R^{y}_{\theta-\frac{\pi}{2}}\, \delta^{x}\, R^{y}_{\frac{\pi}{2}-\theta}\, R^{z}_{-\varphi} \right) \qquad (3.1) \\
&= \exp\left(\alpha\, (T^{\hat{n}})^{-1} \delta^{x}\, T^{\hat{n}} \right)
\end{aligned}
$$

where $T^{\hat{n}} = R^{y}_{\frac{\pi}{2}-\theta} R^{z}_{-\varphi}$ as in L4 p. 11. Explicitly

$$
\begin{aligned}
T^{\hat{n}} &= \begin{pmatrix} \cos(\frac{\pi}{2}-\theta) & 0 & \sin(\frac{\pi}{2}-\theta) \\ 0 & 1 & 0 \\ -\sin(\frac{\pi}{2}-\theta) & 0 & \cos(\frac{\pi}{2}-\theta) \end{pmatrix} \begin{pmatrix} \cos(-\varphi) & -\sin(-\varphi) & 0 \\ \sin(-\varphi) & \cos(-\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \sin(\theta) & 0 & \cos(\theta) \\ 0 & 1 & 0 \\ -\cos\theta & 0 & \sin\theta \end{pmatrix} \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (3.2) \\
&= \begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ -\sin\varphi & \cos\varphi & 0 \\ -\cos\theta\cos\varphi & -\cos\theta\sin\varphi & \sin\theta \end{pmatrix}
\end{aligned}
$$

This matrix is orthogonal (see Ex L5-4 (iii)) so $(T^{\hat{n}})^{-1} = (T^{\hat{n}})^{T}$. Hence

$$
(T^{\hat{n}})^{-1} \delta^{x}\, T^{\hat{n}} = \begin{pmatrix} \sin\theta\cos\varphi & -\sin\varphi & -\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & \cos\varphi & -\cos\theta\sin\varphi \\ \cos\theta & 0 & \sin\theta \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ -\sin\varphi & \cos\varphi & 0 \\ -\cos\theta\cos\varphi & -\cos\theta\sin\varphi & \sin\theta \end{pmatrix}
$$

(3.3)

$$= \begin{pmatrix} 0 & -\cos\theta\cos\varphi & \sin\varphi \\ 0 & -\cos\theta\sin\varphi & -\cos\varphi \\ 0 & \sin\theta & 0 \end{pmatrix} \begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ -\sin\varphi & \cos\varphi & 0 \\ -\cos\theta\cos\varphi & -\cos\theta\sin\varphi & \sin\theta \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\cos\theta\cos^2\varphi - \cos\theta\sin^2\varphi & \sin\theta\sin\varphi \\ \cos\theta\sin^2\varphi + \cos\theta\cos^2\varphi & 0 & -\sin\theta\cos\varphi \\ -\sin\theta\sin\varphi & \sin\theta\cos\varphi & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\cos\theta & \sin\theta\sin\varphi \\ \cos\theta & 0 & -\sin\theta\cos\varphi \\ -\sin\theta\sin\varphi & \sin\theta\cos\varphi & 0 \end{pmatrix} = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}$$

$$= n_1 \delta^x + n_2 \delta^y + n_3 \delta^z$$

where $\hat{n} = (n_1, n_2, n_3)$. We have proven

<u>Lemma L6-2</u> Given $\hat{n} = (n_1, n_2, n_3) \in S^2$ we have

$$R^{\hat{n}}_{\alpha} = \exp(\alpha[n_1 \delta^x + n_2 \delta^y + n_3 \delta^z])$$

where $\delta^{\hat{n}} = n_1 \delta^x + n_2 \delta^y + n_3 \delta^z$ is called the <u>infinitesimal generator of rotations</u> around $\hat{n}$.
The matrices $\delta^x, \delta^y, \delta^z$ are a basis for the subspace of skew symmetric matrices

$$so(3) = \{ X \in M_3(\mathbb{R}) \mid X^T = -X \} \qquad (4.1)$$

<u>Def^n</u> Where convenient we write $\delta^{x_1}, \delta^{x_2}, \delta^{x_3}$ for $\delta^x, \delta^y, \delta^z$.

<u>Remark</u> Note that if $n = (n_1, n_2, n_3) \in \mathbb{R}^3$ is nonzero and $\hat{n} = \frac{1}{\|n\|} n$ then

$$\exp\left(\alpha \sum_{i=1}^{3} n_i \delta^{x_i}\right) = \exp\left(\alpha \|n\| \sum_{i=1}^{3} \hat{n}_i \delta^{x_i}\right) = R^{\hat{n}}_{\alpha\|n\|} \qquad (5.1)$$

<u>Lemma L6-3</u> There is a surjective map

$$\exp(-) : \mathfrak{so}(3) \longrightarrow SO(3) \qquad (5.2)$$

<u>Proof</u> By Ex L3-6 every $g \in SO(3)$ is of the form $R^{\hat{n}}_{\alpha}$ for some $\hat{n} \in S^2$ and $\alpha \in [0, 2\pi)$. So the claim follows from Lemma L6-2. $\square$

We have now shown that every rotation $R^{\hat{n}}_{\alpha}$, a unitary operator on the <u>real</u> inner product space $\mathbb{R}^3$, is the exponential of a skew self-adjoint matrix $\alpha \delta^{\hat{n}}$. We may therefore rewrite (1.0) as

$$\begin{aligned} \mathcal{Z} : SO(3) &\longrightarrow \operatorname{Aut}_{\mathbb{C}}(\mathcal{H}_k(S^2)) \\ \mathcal{Z}\left(\exp(\alpha \delta^{\hat{n}})\right) &= \exp(\alpha \eta^{\hat{n}}) \end{aligned} \qquad (5.3)$$

This strongly suggests that the content of this representation lies in the <u>relationship between the infinitesimal generators</u> $\delta^{\hat{n}}, \eta^{\hat{n}}$ from which $\mathcal{Z}$ is obtained by exponentiation.

<u>Theorem L6-4</u> Let $D\mathcal{Z}$ be the linear transformation $\mathfrak{so}(3) \longrightarrow \operatorname{End}_{\mathbb{C}}(\mathcal{H}_k(S^2))$ between real vector spaces defined by $D\mathcal{Z}(\delta^{x_i}) = \eta^{x_i}$. Then the diagram below commutes

$$\begin{tikzcd} SO(3) \arrow[r, "\mathcal{Z}"] & \operatorname{Aut}_{\mathbb{C}}(\mathcal{H}_k(S^2)) \\ \mathfrak{so}(3) \arrow[u, "\exp"] \arrow[r, "D\mathcal{Z}"'] & \operatorname{End}_{\mathbb{C}}(\mathcal{H}_k(S^2)) \arrow[u, "\exp"'] \end{tikzcd} \qquad (5.4)$$

Before proving the theorem we observe the relation between $\eta^{\hat{n}}$ and the η^x, η^y, η^z.

Lemma L6-5 Given $\hat{n} = (n_1, n_2, n_3) \in S^2$

$$\eta^{\hat{n}} = n_1 \eta^x + n_2 \eta^y + n_3 \eta^z$$

Proof By def^N (L5 p.④) we have $\eta^{\hat{n}} = t_2 \frac{\partial}{\partial t_3} - t_3 \frac{\partial}{\partial t_2}$ where $t_i = \sum_{j=1}^3 T^{\hat{n}}_{ij} x_j$. From this we deduce $x_i = \sum_{j=1}^3 (T^{\hat{n}})^{-1}_{ij} t_j$ and hence by the chain rule

$$\begin{aligned} \frac{\partial}{\partial t_i} &= \sum_{j=1}^3 \frac{\partial}{\partial x_j} \frac{\partial x_j}{\partial t_i} \\ &= \sum_{j=1}^3 (T^{\hat{n}})^{-1}_{ji} \frac{\partial}{\partial x_j} \end{aligned} \qquad (6.1)$$

Hence

$$\begin{aligned} t_2 \frac{\partial}{\partial t_3} - t_3 \frac{\partial}{\partial t_2} &= \Big(\sum_{j=1}^3 T^{\hat{n}}_{2j} x_j \Big)\Big(\sum_{j=1}^3 (T^{\hat{n}})^{-1}_{j3} \frac{\partial}{\partial x_j} \Big) \\ &\quad - \Big(\sum_{j=1}^3 T^{\hat{n}}_{3j} x_j \Big)\Big(\sum_{j=1}^3 (T^{\hat{n}})^{-1}_{j2} \frac{\partial}{\partial x_j} \Big) \\ &\overset{T^{\hat{n}} \text{ orthogonal}}{=} \sum_{j,k=1}^3 T^{\hat{n}}_{2j} T^{\hat{n}}_{3k} x_j \frac{\partial}{\partial x_k} \\ &\quad - \sum_{j,k=1}^3 T^{\hat{n}}_{2j} T^{\hat{n}}_{3k} x_k \frac{\partial}{\partial x_j} \\ &= \sum_{j,k=1}^3 T^{\hat{n}}_{2j} T^{\hat{n}}_{3k} \Big[x_j \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_j} \Big] \end{aligned} \qquad (6.2)$$

But the term in the bracket vanishes if $j = k$, and so only the pairs (j,k) in the set $\{(1,2),(1,3),(2,3),(2,1),(3,1),(3,2)\}$ contribute:

$$= \sum_{j<k} \left(T^{\hat{n}}_{2j} T^{\hat{n}}_{3k} - T^{\hat{n}}_{2k} T^{\hat{n}}_{3j} \right) \left[x_j \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_j} \right]$$

$$= \left(T^{\hat{n}}_{22} T^{\hat{n}}_{33} - T^{\hat{n}}_{23} T^{\hat{n}}_{32} \right) \eta^x \qquad (j,k) = (2,3)$$

$$- \left(T^{\hat{n}}_{21} T^{\hat{n}}_{33} - T^{\hat{n}}_{23} T^{\hat{n}}_{31} \right) \eta^y \qquad (j,k) = (1,3) \qquad (7.1)$$

note this!

$$+ \left(T^{\hat{n}}_{21} T^{\hat{n}}_{32} - T^{\hat{n}}_{22} T^{\hat{n}}_{31} \right) \eta^z \qquad (j,k) = (1,2)$$

We recognise the terms in round brackets as determinants of minors of $T^{\hat{n}}$, which we may compute by (3.2)

$$= \sin\theta \cos\varphi \, \eta^x + \sin\theta \sin\varphi \, \eta^y + \cos\theta \, \eta^z$$

$$= n_1 \eta^x + n_2 \eta^y + n_3 \eta^z. \quad \square$$

<u>Proof of Theorem L6-4</u> By Lemma L6-5 we have for $\hat{n} \in S^2$ that

$$\begin{aligned} D\mathfrak{z}(\delta^{\hat{n}}) &= D\mathfrak{z}\left(\textstyle\sum_{i=1}^3 n_i \delta^{x_i} \right) \\ &= \textstyle\sum_{i=1}^3 n_i \, D\mathfrak{z}(\delta^{x_i}) \\ &= \textstyle\sum_{i=1}^3 n_i \eta^{x_i} = \eta^{\hat{n}} \end{aligned} \qquad (7.2)$$

The diagram (5.4) commutes on $O \in so(3)$ since, using that $\mathfrak{z}$ is a group homomorphism

$$\mathfrak{z}(\exp(0)) = \mathfrak{z}(I_3) = 1_{\mathcal{H}_k(S^2)} = \exp(0) = \exp(D\mathfrak{z}(0)).$$

If $Y \in so(3)$ is nonzero, it can be written as $Y = \alpha \delta^{\hat{n}}$ for some $\hat{n} \in S^2$ and $\alpha \in \mathbb{R}$. Then we have

$$\mathfrak{z}(\exp(Y)) \overset{\text{Lemma L6-2}}{=} \mathfrak{z}(R^{\hat{n}}_{\alpha}) \overset{\text{Thm L4-5}}{=} \exp(\alpha \eta^{\hat{n}}) \overset{(7.2)}{=} \exp(\alpha \, D\mathfrak{z}(\delta^{\hat{n}}))$$

$$= \exp(D\mathfrak{z}(\alpha\delta^{\hat{n}})) = \exp(D\mathfrak{z}(Y)). \quad \square$$

From Theorem L6-4 we learn that the representation $\mathcal{Z}$ of $SO(3)$ can be viewed as the exponential of a linear map $D\mathcal{Z} : \mathfrak{so}(3) \longrightarrow \mathrm{End}_{\mathbb{C}}(\mathcal{H}_k(S^2))$. What kind of mathematical object is this, and to what degree can we infer useful information about $\mathcal{Z}$ from $D\mathcal{Z}$?
To answer these questions we develop some general theory. We note that Theorem L6-4 is about one-parameter families of invertible operators $\{\exp(tX)\}_{t\in\mathbb{R}}$ and so our first goal to characterise such families abstractly.

One-parameter subgroups

Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$ and let $\mathfrak{gl}(n,\mathbb{F})$ denote the $\mathbb{F}$-vector space of $n\times n$ matrices over $\mathbb{F}$ (previously denoted $M_n(\mathbb{F})$). Let $GL(n,\mathbb{F})$ denote the group of invertible $n\times n$ matrices. By Theorem B1-11 applied to the Banach space $V=\mathbb{F}^n$ (with say the $\|-\|_2$ norm) we have the exponential map

$$\exp : \mathfrak{gl}(n,\mathbb{F}) \longrightarrow GL(n,\mathbb{F}) \tag{8.1}$$

Lemma L6-6 Given $X \in \mathfrak{gl}(n,\mathbb{F})$ the function

$$\begin{aligned} \mathbb{R} &\longrightarrow \mathfrak{gl}(n,\mathbb{F}) \\ t &\longmapsto \exp(tX) \end{aligned} \tag{8.2}$$

is smooth and

$$\frac{d}{dt}\exp(tX) = X\exp(tX) = \exp(tX)X. \tag{8.3}$$

Proof We are asserting that a function $\mathbb{R}\longrightarrow \mathbb{F}^{n^2}$ is differentiable, i.e. that all its component real-valued functions are differentiable (with $\mathbb{C}^{n^2} = \mathbb{R}^{2n^2}$).
This and (8.3) were both proven in the proof of Theorem L4-5 using Picard's theorem.
From (8.3) we infer by induction that all higher derivatives exist as well. □

Remark In particular for $X \in \mathfrak{gl}(n,\mathbb{F})$ we have by Theorem B1-11 (i)

$$\frac{d}{dt}\exp(tX)\Big|_{t=0} = [X\exp(tX)]\Big|_{t=0} = X \qquad (9.1)$$

By induction it is easy to see that $\frac{d^n}{dt^n}\exp(tX) = X^n\exp(tX)$

Def^n We say a smooth function $X(t) : \mathbb{R} \longrightarrow \mathfrak{gl}(n,\mathbb{F})$ vanishes to order k at $t=a$ if $\frac{d^i}{dt^i}X(t)\big|_{t=a}$ is the zero matrix for $0 \leq i \leq k$.

Def^n The commutator of $X, Y \in \mathfrak{gl}(n,\mathbb{F})$ is $[X,Y] = XY - YX$. We say X, Y commute if $XY = YX$ i.e. $[X,Y] = 0$.

If X, Y commute then $\exp(tX)\exp(tY) = \exp(t(X+Y))$ by Theorem B1-11 (ii). In general this is false, and while the full formula for $\exp(tX)\exp(tY)$ is quite complicated (the Baker–Campbell–Hausdorff formula) the low order terms in t are easy to compute:

Lemma L6-7 For $X, Y \in \mathfrak{gl}(n,\mathbb{F})$

$$\exp(tX)\exp(tY) = \exp\left(t(X+Y) + \tfrac{t^2}{2}[X,Y]\right) + R_2(t) \qquad (9.2)$$

where $R_2(t)$ is a smooth function of t vanishing to order 2 at $t=0$.

Proof We have

$$\exp(tX)\exp(tY) = \left(I + tX + \tfrac{t^2}{2}X^2 + A(t)\right)\cdot\left(I + tY + \tfrac{t^2}{2}Y^2 + B(t)\right)$$

for some matrix-valued functions $A(t), B(t)$ smooth and vanishing to order 2 at $t=0$ (since e.g. $A(t) = \exp(tX) - I - tX - \frac{t^2}{2}X^2$ and so $A(t)$ is smooth and $d^2/dt^2\, A(t) = X^2\exp(tX) - X^2$ which vanishes at $t=0$. Similarly for $B(t)$).

Expanding gives

$$
\begin{aligned}
\exp(tX)\exp(tY) = \; & I + tY + \tfrac{t^2}{2}Y^2 + \underline{B(t)} \\
& + tX + t^2XY + \underline{\tfrac{1}{2}t^3XY^2} + \underline{tXB(t)} \\
& + \tfrac{t^2}{2}X^2 + \underline{\tfrac{1}{2}t^3X^2Y} + \underline{\tfrac{1}{4}t^4X^2Y^2} + \underline{\tfrac{1}{2}t^2X^2B(t)} \\
& + \underline{A(t)} + \underline{t.A(t)Y} + \underline{\tfrac{1}{2}t^2A(t)Y^2} + \underline{A(t)B(t)}
\end{aligned}
$$

$$= I + t(X+Y) + \tfrac{t^2}{2}(Y^2 + 2XY + X^2) + P(t)$$

where $P(t)$ is smooth and vanishes to order 2 at $t=0$. On the other hand

$$\exp\left(t(X+Y) + \tfrac{t^2}{2}[X,Y]\right) = I + t(X+Y) + \tfrac{t^2}{2}[X,Y] + \tfrac{t^2}{2}(X+Y)^2 + Q(t)$$

$$= I + t(X+Y) + \tfrac{t^2}{2}\left(X^2 + XY + YX + Y^2 + XY - YX\right) + Q(t)$$

$$= I + t(X+Y) + \tfrac{t^2}{2}\left(X^2 + 2XY + Y^2\right) + Q(t)$$

where $Q(t)$ is smooth and vanishes to order 2 at $t=0$. Setting $R_2(t) = Q(t) - P(t)$ we are done. $\square$

<u>Def^n</u> A <u>one-parameter subgroup</u> of $GL(n,\mathbb{F})$ is a continuous function $f:\mathbb{R}\to GL(n,\mathbb{F})$ such that

$$f(s+t) = f(s)f(t) \qquad \forall s,t\in\mathbb{R}$$

<u>Exercise L6-1</u> Prove that the image of a one-parameter subgroup $f:\mathbb{R}\to GL(n,\mathbb{F})$ is in fact an <u>abelian</u> subgroup of $GL(n,\mathbb{F})$, and that $f(0) = I$.

<u>Lemma L6-8</u> Let $f:\mathbb{R}\to GL(n,\mathbb{F})$ be a one-parameter subgroup. Then f is differentiable.

<u>Proof</u> Since f is continuous it is (entry-wise) integrable, so for $a>0$ we have $\int_0^a f(t)\,dt \in \mathfrak{gl}(n,\mathbb{F})$. We claim that if a is sufficiently small this matrix is invertible. The function $\det : \mathfrak{gl}(n,\mathbb{F})\to\mathbb{F}$ is continuous so $GL(n,\mathbb{F})\subseteq \mathfrak{gl}(n,\mathbb{F})$ is an open subset with respect to the Frobenius norm $\|-\|$. Since $I\in GL(n,\mathbb{F})$ we can find $\epsilon>0$ such that $\|X-I\|<\epsilon$ implies X is invertible. Now since f is continuous and $f(0)=I$ there exists $\delta>0$ with $\|f(t)-I\|<\epsilon/n$ whenever $|t|<\delta$. Thus $|f(t)_{ij}-I_{ij}|<\epsilon/n$ for all $1\le i,j\le n$, and so

$$\int_0^a |f(t)_{ij}-I_{ij}|\,dt < \epsilon a/n$$

Hence if $a<\delta$

$$\left\|\frac{1}{a}\int_0^a f(t)dt - I\right\| = \left\|\frac{1}{a}\int_0^a (f(t)-I)\,dt\right\|$$

$$= \frac{1}{a}\left\{\sum_{i,j}\left|\int_0^a (f(t)_{ij}-I_{ij})\,dt\right|^2\right\}^{1/2} \qquad (11.1)$$

$$< \frac{1}{a}\left\{n^2\cdot\frac{\epsilon^2a^2}{n^2}\right\}^{1/2} = \epsilon$$

So $\frac{1}{a}\int_0^a f(t)dt$ is invertible and hence so is $\int_0^a f(t)dt$. But then for $s\in\mathbb{R}$

$$\int_0^a f(t+s)\,dt = f(s)\int_0^a f(t)\,dt \qquad (11.2)$$

$$\int_0^a f(t+s)\,dt = \int_s^{s+a} f(t)\,dt \qquad (11.3)$$

The second integral may be written as (since $a>0$) $\int_s^p f(t)dt + \int_p^{s+a} f(t)dt$ for some fixed p with $s<p<s+a$ and this shows $\int_0^a f(t+s)dt$ is differentiable in s (by the Fundamental Theorem of calculus, keeping in mind that to prove differentiability at s we need only values in $(s-\delta, s+\delta)$ so a fixed p may be found).

But by (11.2) this shows $g(s) = f(s)\int_0^a f(t)\,dt$ is a differentiable function of s, and hence so too is

$$f(s) = g(s)\left[\int_0^a f(t)\,dt\right]^{-1}$$

as claimed. $\square$

Remark By Lemma L6-6 for any $X \in \mathfrak{gl}(n,\mathbb{F})$ we have a one-parameter subgroup $f(t) = \exp(tX)$. Given a one-parameter subgroup $f : \mathbb{R} \longrightarrow GL(n,\mathbb{F})$ is by Lemma L6-8 a smooth function and hence we may define $X := \frac{d}{dt}f(t)\big|_{t=0} \in \mathfrak{gl}(n,\mathbb{F})$. We now claim these are mutually inverse maps.

Lemma L6-9 There is a bijection

$$\mathfrak{gl}(n,\mathbb{F}) \xrightarrow{\quad \Psi \quad} \{ f : \mathbb{R} \longrightarrow GL(n,\mathbb{F}) \mid f \text{ is a one-parameter subgroup} \}$$

where Ψ and its inverse Φ are given by

$$\Psi(X)(t) = \exp(tX)$$
$$\Phi(f) = \frac{d}{dt}f(t)\Big|_{t=0}.$$

Proof Given $X \in \mathfrak{gl}(n,\mathbb{F})$ let $f(t) = \exp(tX)$. This is continuous by Lemma L6-6, takes values in $GL(n,\mathbb{F})$ by Theorem B1-11 (iv). By Theorem B1-11 (ii) we have

$$\begin{aligned} f(t+s) &= \exp(tX + sX) \\ &= \exp(tX)\exp(sX) = f(t)f(s) \end{aligned}$$

so f is a one-parameter subgroup. By Lemma L6-8 the function Φ is well-defined, and it remains to show that $\Phi\Psi = 1$ and $\Psi\Phi = 1$.

By Lemma L6-6 (or more precisely its proof) we have that $t \mapsto \exp(tX)$ is the <u>unique</u> solution of the differential equation

$$\frac{d}{dt} Y(t) = X\, Y(t). \tag{13.1}$$

Hence

$$\begin{aligned} \Phi\Psi(X) &= \frac{d}{dt}\big(\Psi(X)\big)\Big|_{t=0} \\ &= \frac{d}{dt}\big(\exp(tX)\big)\Big|_{t=0} \\ &= \big(X\exp(tX)\big)\Big|_{t=0} \\ &= X. \end{aligned} \tag{13.2}$$

To show $\Psi\Phi(f) = f$ it suffices to show that $\frac{d}{dt} f(t) = X f(t)$ where $X = \Phi(f)$. But

$$\begin{aligned} \frac{d}{dt} f(t)\Big|_{t=a} &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \qquad \text{(recall this all happens entry-wise)} \\ &= \lim_{h\to 0} \frac{f(h)f(a) - f(a)}{h} \\ &= \lim_{h\to 0} \frac{f(h) - 1}{h}\, f(a) \\ &= \left[\lim_{h\to 0} \frac{f(h) - f(0)}{h}\right] f(a) \\ &= \frac{d}{dt} f(t)\Big|_{t=0} f(a) \\ &= X f(a) \end{aligned} \tag{13.3}$$

That is, $f(t)$ is a solution of (13.1). Hence by uniqueness $\Psi\Phi(f) = f$. $\square$

Def^n If $f : \mathbb{R} \longrightarrow GL(n, \mathbb{F})$ is a one-parameter subgroup and X the unique matrix such that $f(t) = \exp(tX)$ for all $t \in \mathbb{R}$ we call X the infinitesimal generator of the one-parameter subgroup.

Example L6-1 For $\hat{n} \in S^2$ we have $\delta^{\hat{n}} \in so(3) \subseteq gl(3, \mathbb{R})$ and the corresponding one-parameter subgroup is by Lemma L6-2

$$f(t) = \exp(t\delta^{\hat{n}}) = R_t^{\hat{n}} \tag{14.1}$$

This function $f : \mathbb{R} \longrightarrow GL(3, \mathbb{R})$ is continuous (a continuous map from $\mathbb{R}$ is called a path see [MHS, L12]) and induces a continuous map $f : \mathbb{R} \longrightarrow SO(3)$. The union over all $\hat{n}$ of these paths is all of $SO(3)$ by Ex L3-6.

Example L6-2 For $\hat{n} \in S^2$ and $k \geqslant 0$ we have $\eta^{\hat{n}} \in gl(2k+1, \mathbb{C})$, by L5 p. 4 and Ex L4-2 (or at least the matrix of $\eta^{\hat{n}}$ is in $gl(2k+1, \mathbb{C})$. As nothing we will say depends on the basis, we may assume one has been chosen, and conflate $\eta^{\hat{n}}$ with this matrix). The corresponding one-parameter family is by Theorem L4-5

$$f(t) = \exp(t\eta^{\hat{n}}) = \mathcal{Z}(R_t^{\hat{n}}). \tag{14.2}$$

Remark L6-3 If $f : \mathbb{R} \longrightarrow GL(n, \mathbb{F})$ is a one-parameter subgroup then by Lemma L6-8 f is differentiable (when we say a function f into $GL(n, \mathbb{F})$ is continuous, differentiable, C^1, smooth etc. we mean the function $\iota \circ f$ has this property where $\iota : GL(n, \mathbb{F}) \to gl(n, \mathbb{F})$ is the inclusion, and we identify $gl(n, \mathbb{F})$ with $\mathbb{F}^{n^2}$). But Lemma L6-9 improves this: since $\frac{d}{dt} f(t) = Xf(t)$ we see that f is in fact smooth (Lemma L6-6).

Remark L6-4 Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a smooth function. By Taylor's theorem (using the Lagrange form of the remainder) for any $a \in \mathbb{R}$ we have for $k \geq 1$ an integer

$$f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + R_k(a,x) \qquad (15.1)$$

as functions on $\mathbb{R}$, where the <u>remainder</u> $R_k(a,x)$ is equal to $\frac{f^{(k+1)}(b)}{(k+1)!}(x-a)^{k+1}$ for some b between x and a (depending on x). In L5 p.3 we learned a different way to think about this: if we evaluate the Taylor series expansion (15.1) at $x+a$ we have

$$f(x+a) = \sum_{i=0}^{k} \frac{x^i}{i!} f^{(k)}(a) + R_k(a, x+a) \qquad (15.2)$$

If we now swap the role of x, a

$$f(x+a) = \sum_{i=0}^{k} \frac{a^i}{i!} f^{(i)}(x) + r_k(a,x) \qquad (15.3)$$

where $r_k(a,x) = \frac{f^{(k+1)}(b)}{(k+1)!} a^{k+1}$ for some b between a and $x+a$. That is,
We consider the $\mathbb{R}$-vector space $C^\infty(\mathbb{R})$ of all smooth functions, which is an $\mathbb{R}$-algebra, on which we have a linear operator $\frac{\partial}{\partial x}$. Then (15.3) says

$$f(x+a) = \Big[\sum_{i=0}^{k} \frac{a^i}{i!} \frac{\partial^i}{\partial x^i} \Big](f) + r_k(a,x). \qquad (15.4)$$

Thinking now of an $n \times n$ matrix of smooth functions $X : \mathbb{R} \longrightarrow \mathfrak{gl}(n, \mathbb{R})$ (so we have $X(t) = (f_{ij}(t))_{1 \le i,j \le n}$ where the $f_{ij} : \mathbb{R} \longrightarrow \mathbb{R}$ are smooth) if this vanishes to order k at $t = 0$ (as defined on p. 9) then by (15.1)

$$X(t) = \frac{1}{(k+1)!} \underbrace{\Big(f_{ij}^{(k+1)}(b_{ij}) \Big)_{ij}}_{n \times n \text{ matrix}} t^{k+1} \qquad (15.5)$$

for some b_{ij} between t and 0 (see Rudin "Principles of mathematical analysis")

Matrix Lie groups and Lie algebras (finally?)

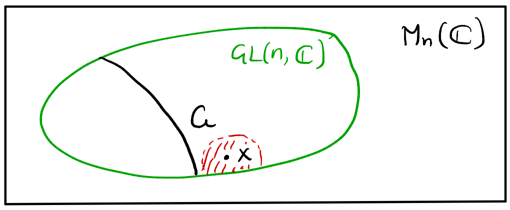
Recall that $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$, and the <u>general linear group</u> $GL(n,\mathbb{F})$ is the group of invertible $n\times n$ matrices over $\mathbb{F}$. The set of all such matrices is denoted $M_n(\mathbb{F})$. This vector space is a normed space with the Frobenius norm $\|-\|_F$, and the associated metric is such that a sequence $(X_m)_{m=1}^{\infty}$ of matrices converges to X if and only if for all i,j the entries $(X_m)_{ij} \longrightarrow X_{ij}$ as $m\to\infty$. The subset $GL(n,\mathbb{F}) \subseteq M_n(\mathbb{F})$ becomes a metric space with the induced metric. Since $\det: M_n(\mathbb{C}) \longrightarrow \mathbb{C}$ is continuous and $GL(n,\mathbb{F}) = \det^{-1}(\mathbb{C}\setminus\{0\})$ this is an <u>open</u> subset of $M_n(\mathbb{F})$ (so if A is invertible there exists $\varepsilon > 0$ with $\{B \mid \|B-A\| < \varepsilon\} \subseteq GL(n,\mathbb{F})$).

<u>Def^n</u> A <u>matrix Lie group</u> is a subgroup G of $GL(n,\mathbb{C})$ which is <u>closed</u> in $GL(n,\mathbb{C})$.

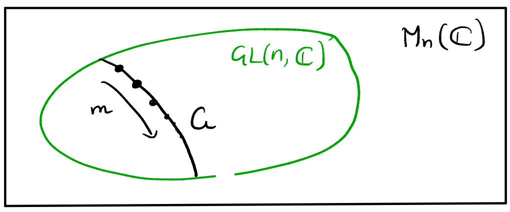
That is, a matrix Lie group is a <u>closed subgroup</u> of $GL(n,\mathbb{C})$.

Here are some conditions equivalent to $G \subseteq GL(n,\mathbb{C})$ being closed in the subspace topology on $GL(n,\mathbb{C})$:

- if $X \in GL(n,\mathbb{C}) \setminus G$ there exists $\varepsilon > 0$ such that for all $B \in GL(n,\mathbb{C})$ with $\|B - X\| < \varepsilon$ we have $B \notin G$.



- if $(A_m)_{m=1}^{\infty}$ is a sequence in G converging to $A \in GL(n,\mathbb{C})$ then $A \in G$.



<u>Remark L6-5</u> It is possible for sequences in G to converge to <u>non-invertible</u> matrices (think of the sequence $(\frac{1}{m} I_n)_{m=1}^{\infty}$ with $G = GL(n,\mathbb{C})$) and on such sequences there is no constraint imposed by G being a matrix subgroup.

<u>Example L6-3</u> $GL(n,\mathbb{C})$ is a matrix Lie group, and so is $GL(n,\mathbb{R})$ since $\mathbb{R}$ is closed in $\mathbb{C}$.

Remark L6-6 Any matrix Lie group G is a metric space (and thus topological space) with the metric induced from $GL(n, \mathbb{C})$.

Exercise L6-2 (i) Prove that any subgroup H of a matrix Lie group G which is closed in the subspace topology on G is also closed in $GL(n,\mathbb{C})$, and is thus itself a matrix Lie group.

(ii) If G, H are matrix Lie groups so is $G \cap H$.

Example L6-4 The following subgroups of $GL(n, \mathbb{C})$ are closed and thus matrix Lie groups:

(1) $SL(n,\mathbb{F}) = \{X \in GL(n,\mathbb{F}) \mid \det(X) = 1\}$ called the special linear group

This is closed since $\det$ is continuous and $SL(n,\mathbb{F}) = \det^{-1}(\{1\})$.

(2) $O(n) = \{X \in M_n(\mathbb{R}) \mid X^\top X = I_n\}$ the orthogonal group. $(-)^\top$ meaning transpose

It is easy to see this is a subgroup of $GL(n,\mathbb{R})$ (since $1 = \det(X^\top X) = \det(X)^2$ an orthogonal matrix is invertible) and if $A_m \in O(n)$ for $m \geq 1$ converges to a matrix A then since the transpose and multiplication are continuous

$$\begin{aligned} A^\top A &= \left(\lim_{m\to\infty} A_m\right)^\top \left(\lim_{m\to\infty} A_m\right) \\ &= \left(\lim_{m\to\infty} (A_m)^\top\right)\left(\lim_{m\to\infty} A_m\right) \qquad (17.1) \\ &= \lim_{m\to\infty} (A_m)^\top A_m = 1 \end{aligned}$$

so $A \in O(n)$

(3) $SO(n) = O(n) \cap SL(n,\mathbb{R})$ is the special orthogonal group

(4) $U(n) = \{ X \in M_n(\mathbb{C}) \mid X^* X = I_n \}$ where X^* is the conjugate transpose.
By the same argument as in (17.1) this is a matrix Lie group, the <u>unitary group</u>

(5) $SU(n) = U(n) \cap SL(n, \mathbb{C})$ the <u>special unitary group</u>.

(6) $\{ I_n \} \subseteq GL(n, \mathbb{C})$ is the <u>trivial Lie group</u>.

We will see more examples but these will do for now. We note that for us the embedding of G into $GL(n, \mathbb{C})$ is part of the data of a matrix Lie group.

<u>Def^n</u> A <u>one-parameter subgroup</u> of a matrix Lie group G is a continuous function $f : \mathbb{R} \longrightarrow G$ such that $f(s+t) = f(s) f(t)$ for all $s, t \in \mathbb{R}$.

If G is a matrix Lie group and $\iota : G \longrightarrow GL(n, \mathbb{C})$ is the inclusion then we have an injective map

$$\begin{tikzcd}
\{ f : \mathbb{R} \longrightarrow G \mid f \text{ is a one-parameter subgroup} \} \arrow[d, "\iota \circ (-)"] \\
\{ f : \mathbb{R} \longrightarrow GL(n, \mathbb{C}) \mid f \text{ is a one-parameter subgroup} \}
\end{tikzcd} \tag{18.1}$$

whose image is precisely the set of $f : \mathbb{R} \longrightarrow GL(n, \mathbb{C})$ with $f(t) \in G$ for all $t \in \mathbb{R}$. But we know by Lemma L6-9 that the exponential map establishes a bijection between $\mathfrak{gl}(n, \mathbb{C})$ (the space of infinitesimal generators) and the set of one-parameter subgroups of G. The diagram above leads us to wonder <u>which infinitesimal symmetries exponentiate to symmetries in G?</u>

$$\begin{tikzcd}
? \arrow[r, dashed, leftrightarrow] \arrow[d] & \{ f : \mathbb{R} \longrightarrow G \mid f \text{ is a one-parameter subgroup} \} \arrow[d, "\iota \circ (-)"] \\
\mathfrak{gl}(n, \mathbb{C}) \arrow[r, "\cong", "\exp"'] & \{ f : \mathbb{R} \longrightarrow GL(n, \mathbb{C}) \mid f \text{ is a one-parameter subgroup} \}
\end{tikzcd} \tag{18.2}$$

Def^{n} Let G be a matrix Lie group. The Lie algebra $\mathrm{Lie}(G)$ of G is

$$\mathrm{Lie}(G) = \{ X \in \mathfrak{gl}(n,\mathbb{C}) \mid \exp(tX) \in G \text{ for all } t \in \mathbb{R} \} \tag{19.1}$$

Typically Lie algebras are denoted with lowercase "fraktur" letters $\mathfrak{g}, \mathfrak{h}, \ldots$

For the moment this is just a set and it remains unclear what additional structure on this set is induced by "taking the logarithm" of the group structure of G.

Remark L6-7 (i) If $G \subseteq H$ are matrix Lie groups with Lie algebras $\mathfrak{g}, \mathfrak{h}$ resp. then $\mathfrak{g} \subseteq \mathfrak{h}$.
(ii) If G, H are matrix Lie groups then $\mathrm{Lie}(G \cap H) = \mathrm{Lie}(G) \cap \mathrm{Lie}(H)$.

Example L6-5 (i) Clearly $\mathrm{Lie}(GL(n,\mathbb{C})) = \mathfrak{gl}(n,\mathbb{C})$, $\mathrm{Lie}(\{I_n\}) = \{0\}$.

(ii) $\mathrm{Lie}(GL(n,\mathbb{R})) = \mathfrak{gl}(n,\mathbb{R})$. By B1 p. 21 we have $\mathfrak{gl}(n,\mathbb{R}) \subseteq \mathrm{Lie}(GL(n,\mathbb{R}))$. If X is a complex matrix with $\exp(tX)$ real for all $t \in \mathbb{R}$ then by Lemma L6-9, $X = \frac{d}{dt}\exp(tX)\big|_{t=0}$ must be real.

(iii) By Ex B1-5 for $X \in \mathfrak{gl}(n,\mathbb{C})$

$$\det(\exp(tX)) = \exp(t\,\mathrm{tr}(X)) \tag{19.2}$$

Hence $\det(\exp(tX)) = 1$ for all $t \in \mathbb{R}$ if and only if $\mathrm{tr}(X) = 0$. This proves that $\mathfrak{sl}(n,\mathbb{R}) := \mathrm{Lie}(SL(n,\mathbb{R}))$ and $\mathfrak{sl}(n,\mathbb{C}) := \mathrm{Lie}(SL(n,\mathbb{C}))$ are given for $\mathbb{F} \in \{\mathbb{R},\mathbb{C}\}$ by

$$\mathfrak{sl}(n,\mathbb{F}) = \{ X \in \mathfrak{gl}(n,\mathbb{F}) \mid \mathrm{tr}(X) = 0 \}. \tag{19.3}$$

(iv) With $\mathfrak{o}(n) = \mathrm{Lie}(O(n))$ we have by Ex L5-3, L5-4

$$\mathfrak{o}(n) = \{ X \in \mathfrak{gl}(n,\mathbb{R}) \mid X^T = -X \} \qquad (20.1)$$

That is, $\mathfrak{o}(n)$ is the set of skew-symmetric real matrices.

(v) With $\mathfrak{so}(n) := \mathrm{Lie}(SO(n))$ we have by Remark L6-7 (ii)

$$\begin{aligned} \mathfrak{so}(n) &= \mathrm{Lie}(O(n) \cap SL(n,\mathbb{R})) \\ &= \mathrm{Lie}(O(n)) \cap \mathrm{Lie}(SL(n,\mathbb{R})) \qquad (20.2) \\ &= \mathfrak{o}(n) \cap \mathfrak{sl}(n,\mathbb{R}) \\ &= \{ X \in \mathfrak{gl}(n,\mathbb{R}) \mid X^T = -X \text{ and } \mathrm{tr}(X) = 0 \} \\ &= \{ X \in \mathfrak{gl}(n,\mathbb{R}) \mid X^T = -X \} \end{aligned}$$

since an anti-symmetric matrix necessarily has zero diagonal entries and thus zero trace. So $SO(n)$, $O(n)$ <u>have the same Lie algebra</u>, typically denoted $\mathfrak{so}(n)$ (i.e. we will not write $\mathfrak{o}(n)$ ever again). This has to do with the global topological structure of these groups, which is "missed" by passing to infinitesimals at $t=0$. We'll return to this later. Note that if $n=3$ then (20.2) agrees with (4.1).

(vi) With $\mathfrak{u}(n) := \mathrm{Lie}(U(n))$ we have by Lemma L5-4

$$\begin{aligned} X \in \mathfrak{u}(n) &\Longleftrightarrow \exp(tX) \text{ is unitary for all } \alpha \in \mathbb{R} \qquad (20.3) \\ &\overset{\text{L5-4}}{\Longleftrightarrow} -iX \text{ is self-adjoint as an operator} \\ &\Longleftrightarrow (-iX)^* = -iX \text{ as a matrix } (* \text{ being conjugate transpose}) \\ &\Longleftrightarrow iX^* = -iX \text{ as a matrix} \\ &\Longleftrightarrow -X = X^* \end{aligned}$$

Thus $\mathfrak{u}(n) = \{ X \in \mathfrak{gl}(n,\mathbb{C}) \mid X^* = -X \}$ is the set of anti-self-adjoint matrices.

(vii) we have

$$
\begin{aligned}
su(n) &:= \mathrm{Lie}(SU(n)) \\
&= \mathrm{Lie}(U(n)) \cap \mathrm{Lie}(SL(n,\mathbb{C})) \\
&= u(n) \cap sl(n,\mathbb{C}) \\
&= \{ X \in gl(n,\mathbb{C}) \mid X^* = -X,\ tr(X) = 0 \}
\end{aligned} \tag{21.1}
$$

Note that the condition of vanishing trace is no longer vacuous, as e.g. $(i) \in u(1) \setminus su(1)$.

Regarding the structure of Lie algebras, the most obvious feature in Example L6-5 is that all the Lie algebra examples are <u>real vector spaces</u> (some are complex vector spaces).

<u>Lemma L6-10</u> Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}$ is a real vector subspace of $gl(n,\mathbb{C})$, that is

(a) if $X, Y \in \mathfrak{g}$ then $X + Y \in \mathfrak{g}$.

(b) if $X \in \mathfrak{g}$ and $\lambda \in \mathbb{R}$ then $\lambda X \in \mathfrak{g}$.

<u>Proof</u> (b) is immediate from the definition. For (a) we need the Lie product formula (Theorem B1-16) according to which for $X, Y \in gl(n,\mathbb{C})$ and $t \in \mathbb{R}$

$$\exp(t(X+Y)) = \lim_{m\to\infty} \left(\exp\left(\frac{tX}{m}\right) \exp\left(\frac{tY}{m}\right) \right)^m \tag{21.2}$$

This limit is with respect to the metric associated the operator norm on $\mathcal{B}(\mathbb{C}^n) = gl(n,\mathbb{C})$, but since all norms on $\mathcal{B}(\mathbb{C}^n)$ are Lipschitz-equivalent (Lemma B1-1), (21.2) holds with respect to <u>any</u> norm you like on matrices, including the Frobenius norm (so that the RHS converges entry-wise to the LHS). But if $X, Y \in \mathfrak{g}$ then $\exp(\frac{t}{m}X), \exp(\frac{t}{m}Y) \in G$ for all $t \in \mathbb{R}$ and integers m, and since G is a subgroup $[\exp(\frac{t}{m}X)\exp(\frac{t}{m}Y)]^m \in G$. Since $\exp(t(X+Y))$ is invertible and G is closed, the limit in (21.2) also belongs to G. $\square$

<u>Def^n</u> A matrix Lie group is called <u>complex</u> if $\mathfrak{g} = \mathrm{Lie}(G)$ is a complex subspace of $\mathfrak{gl}(n,\mathbb{C})$, equivalently $iX \in \mathfrak{g}$ whenever $X \in \mathfrak{g}$.

<u>Example L6-6</u> (i) If $G \subseteq GL(n,\mathbb{R})$ then $\mathfrak{g} \subseteq \mathfrak{gl}(n,\mathbb{R})$ so G is complex iff. it is the trivial Lie group.

(ii) $SL(n,\mathbb{C})$ is complex.

(iii) If $X^* = -X$ then $(iX)^* = -iX^* = iX$ so if $X \in \mathfrak{so}(n)$ and $iX \in \mathfrak{so}(n)$ then $X = 0$, so $U(n), SU(n)$ are <u>not</u> complex for $n \geqslant 1$, <u>despite the Lie algebras having complex entries</u>.

Beyond being real subspaces of $\mathfrak{gl}(n,\mathbb{C})$ there is one additional piece of structure on a Lie algebra that remains for us to discover. At least for matrix Lie groups that do not contain "nontrivial loops" (we will see what this means later) it is a remarkable fact that this one additional piece of structure, the <u>Lie bracket</u>, allows us to capture everything about the representation theory of the Lie group using just the infinitesimal information in the Lie algebra. Taylor series win!

The origin of the Lie bracket

Recall that in our quest to understand the natural representation of $SO(3)$ on $L^2(S^2, \mathbb{C})$ the fact that $SO(3)$ is not abelian was the first obstacle (see p. 1 L4) or, put differently, if $SO(3)$ were abelian we could simultaneously diagonalise all the operators $\delta(g)$ and so the structure of the representation would be quite trivial. Since $SO(3)$ is not abelian, the representation δ at least stands a chance of being interesting. Let us now return to this comment and make a careful study of some pairs $g, h \in SO(3)$ with $gh \neq hg$ or what is the same $ghg^{-1}h^{-1} \neq e$, where e is the identity. Using the notation of p. 2 set

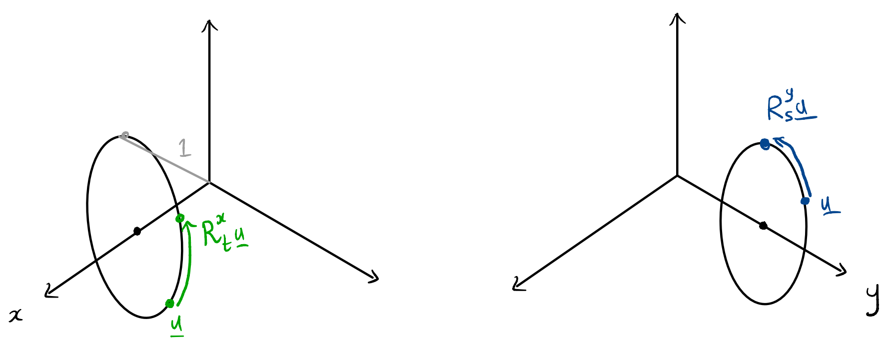
$$
\begin{aligned}
g &= \exp(t\delta^x) = R^x_t \\
h &= \exp(s\delta^y) = R^y_s
\end{aligned}
\tag{23.1}
$$

We wish to compare the two ways around the following diagram

hg gh

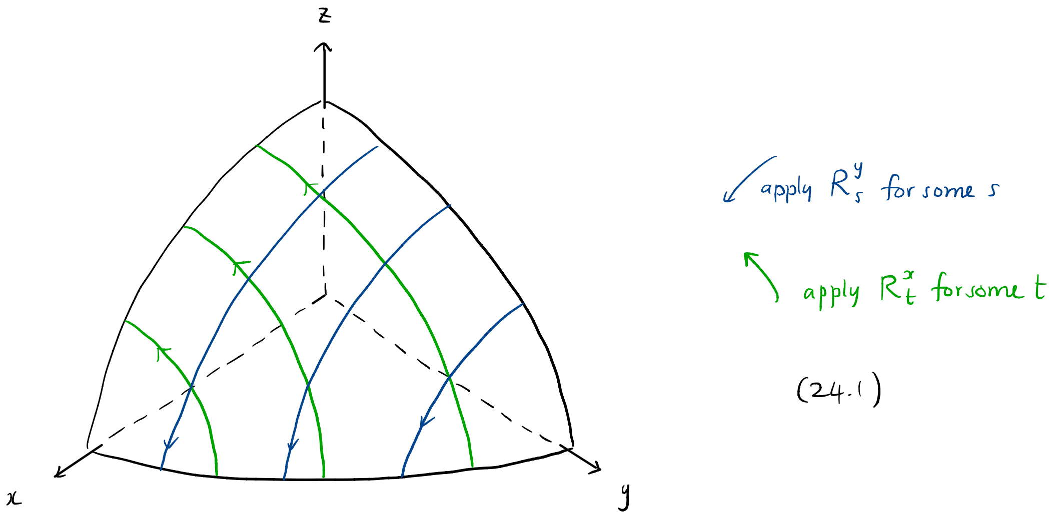
$$
\begin{tikzcd}
\mathbb{R}^3 \arrow[r, "R^x_t"] \arrow[d, "R^y_s"'] & \mathbb{R}^3 \arrow[d, "R^y_s"] \\
\mathbb{R}^3 \arrow[r, "R^x_t"'] & \mathbb{R}^3
\end{tikzcd}
\tag{23.2}
$$

We can visualise R^x_t, R^y_s by picking $\underline{u} \in S^2$ and noting that $a \longmapsto R^x_a(\underline{u})$ for $a \in [0, t]$ is a continuous path joining $\underline{u}$ to $R^x_t \underline{u}$, and similarly for R^y_s (we assume s, t small in the pictures)



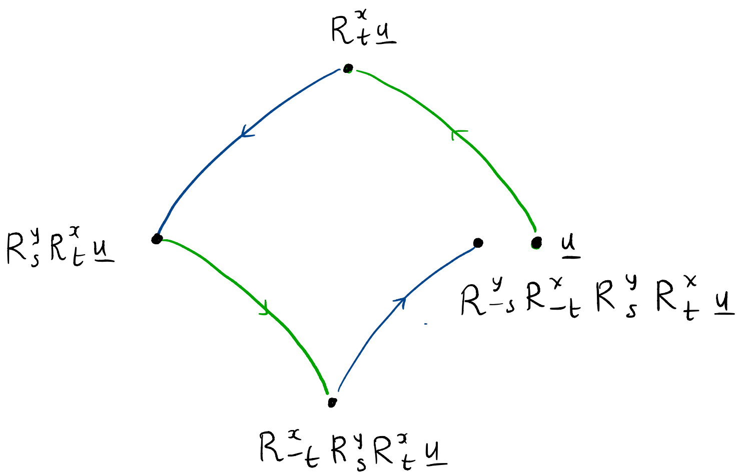
(23.3)

These "flowlines" or orbits of R^x_t, R^y_s obviously intersect in a kind of patchwork, which we visualise below on the intersection of S^2 with the region $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$



(24.1)

The diagram (23.2) invites us to consider the following kind of path on the sphere



(24.2)

Does this path "close" to a loop, at least for "infinitesimal" rectangles on the sphere?
Let us first do a rough calculation using (23.1) and keeping only second order terms in s, t:

(24.3)

$$
\begin{aligned}
R^x_t \underline{u} &= \underline{u} + t\,\delta^x \underline{u} + O(t^2) \\
R^y_s R^x_t \underline{u} &= \underline{u} + t\,\delta^x \underline{u} + O(t^2) + s\,\delta^y \underline{u} + st\,\delta^y \delta^x \underline{u} + O(s^2) \\
R^x_{-t} R^y_s R^x_t \underline{u} &= \underline{u} + t\,\delta^x \underline{u} + s\,\delta^y \underline{u} + st\,\delta^y\delta^x \underline{u} + O(t^2) + O(s^2) \\
&\quad - \left[t\delta^x \underline{u} + t^2 (\delta^x)^2 \underline{u} + ts\,\delta^x \delta^y \underline{u} + st^2 \delta^x \delta^y \delta^x \underline{u} \right]
\end{aligned}
$$

$$= \underline{u} + \cancel{t\delta^x \underline{u}} + s\delta^y \underline{u} + st\,\delta^y\delta^x \underline{u} + O(t^2) + O(s^2)$$
$$\quad - \cancel{t\delta^x \underline{u}} \qquad\qquad - st\,\delta^x\delta^y$$

$$= \underline{u} + s\delta^y \underline{u} + st\left(\delta^y\delta^x - \delta^x\delta^y\right)\underline{u} + O(t^2) + O(s^2)$$

(25.1)

$$R^y_{-s}R^x_{-t}R^y_s R^x_t\,\underline{u} = \underline{u} + \cancel{s\delta^y \underline{u}} + st\left(\delta^y\delta^x - \delta^x\delta^y\right)\underline{u} + O(t^2)O(s^2)$$
$$- \left[\cancel{s\delta^y \underline{u}} + s^2(\delta^y)^2\underline{u} + s^2t\left(\delta^y\delta^y\delta^x - \delta^y\delta^x\delta^y\right)\underline{u}\right]$$
$$= \underline{u} + st\left(\delta^y\delta^x - \delta^x\delta^y\right)\underline{u} + O(t^2) + O(s^2)$$

The "error" term $st\left(\delta^y\delta^x - \delta^x\delta^y\right)\underline{u} + O(t^2) + O(s^2)$ certainly goes to zero as $s,t \to 0$ but this was obvious anyway since $R^y_{-s}R^x_{-t}R^y_s R^x_t$ is continuous in s,t. What we want to know is if "flowing" $\underline{u}$ around an <u>infinitesimal</u> rectangle returns us back to $\underline{u}$ which means we need to scale by the area st, i.e. compute

$$\frac{\partial^2}{\partial s\partial t}\left[R^y_{-s}R^x_{-t}R^y_s R^x_t\,\underline{u}\right]\Big|_{t=s=0} = \left(\delta^y\delta^x - \delta^x\delta^y\right)\underline{u} \qquad (25.2)$$
$$= \left[\delta^y, \delta^x\right]\underline{u}$$

where $[-,-]$ is the matrix commutator. But $[\delta^y,\delta^x] = -[\delta^x,\delta^y]$ and

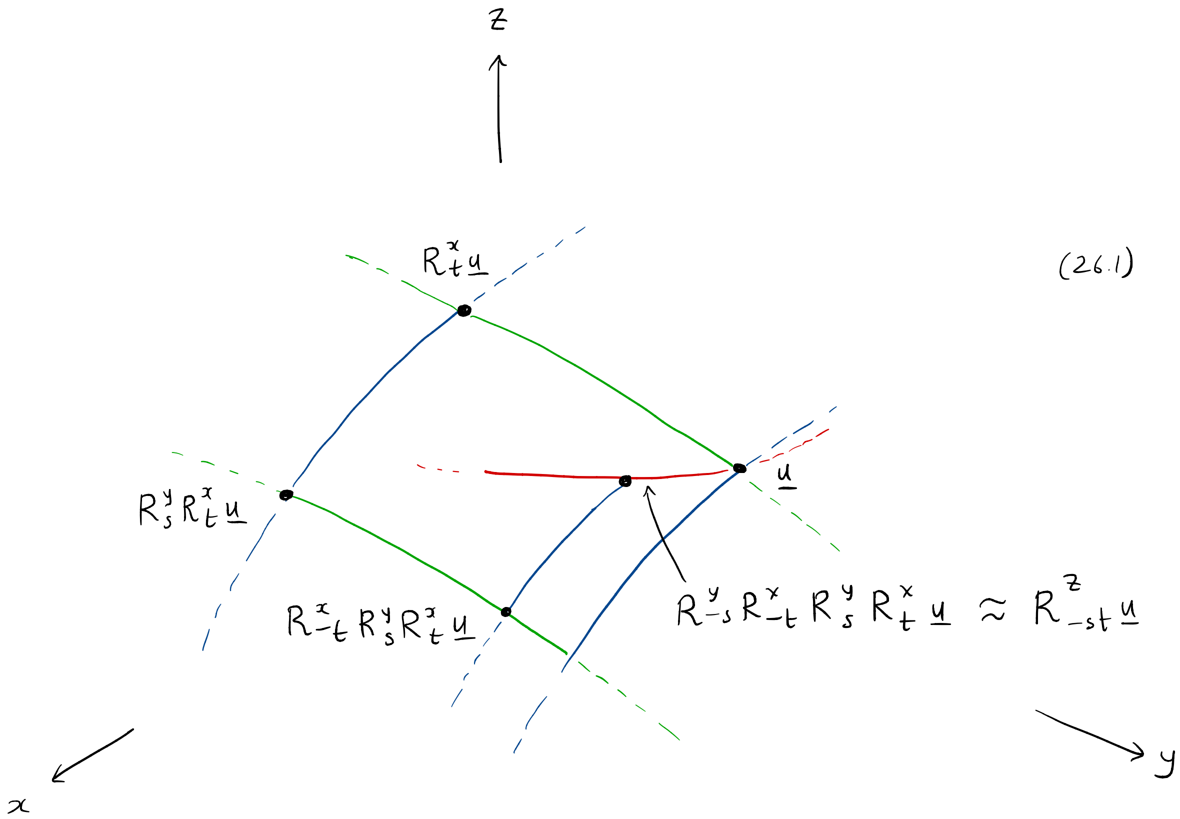
$$[\delta^x,\delta^y] = \begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix}\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix} - \begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix}$$

(25.3)

$$= \begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix} - \begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix} = \delta^z$$

So $R^y_{-s}R^x_{-t}R^y_s R^x_t\,\underline{u} \approx \underline{u} - st\,\delta^z\underline{u}$ for s,t small. The error is a <u>rotation around the z-axis</u>!

Or said differently, transporting $\underline{u}$ around the "rectangle" is approximately $R^z_{-st}\underline{u}$, rotating $\underline{u}$ by an angle of $-st$ around the z-axis:



(26.1)

<u>The upshot is</u>: the diagram (23.2) fails to commute ($gh \neq hg$ or $ghg^{-1}h^{-1} \neq e$) and the failure is measured by the commutator $[\delta^y, \delta^x]$.

<u>Lemma L6-11</u> We have $[\delta^x, \delta^y] = \delta^z$, $[\delta^y, \delta^z] = \delta^x$, $[\delta^z, \delta^x] = \delta^y$, and in general

$$[\delta^{\hat{n}}, \delta^{\hat{m}}] = \delta^{\hat{n} \times \hat{m}} \tag{26.2}$$

where $\hat{n}, \hat{m} \in S^2$ and $(-)\times(-)$ denotes the cross product.

<u>Proof</u> We check the first claims about $\delta^x, \delta^y, \delta^z$ by direct calculation. For (26.2) note that the cross product is bilinear so $\hat{n} \times \hat{m} = \sum_{i,j=1}^3 n_i m_j (e_i \times e_j)$. On the other hand the commutator is also bilinear in each variable so $[\delta^{\hat{n}}, \delta^{\hat{m}}] = \sum_{i,j=1}^3 n_i m_j [\delta^{x_i}, \delta^{x_j}]$ which is $\sum_{i,j=1}^3 n_i m_j \, \delta^{\hat{e}_i \times \hat{e}_j}$ by the first set of claims. $\square$

Generalising (26.1) then we have the following approximation of the error

$$R^{\hat{m}}_{-s} R^{\hat{n}}_{-t} R^{\hat{m}}_{s} R^{\hat{n}}_{t} \underline{u} \approx R^{\hat{n} \times \hat{m}}_{-st} \underline{u} \qquad (27.1)$$

between the two ways around the (not necessarily commutative!) diagram

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "R^{\hat{n}}_t"] \arrow[d, "R^{\hat{m}}_s"'] & \mathbb{R}^3 \arrow[d, "R^{\hat{m}}_s"] \\ \mathbb{R}^3 \arrow[r, "R^{\hat{n}}_t"'] & \mathbb{R}^3 \end{tikzcd} \qquad (27.2)$$

The same thing occurs in the general case:

<u>Exercise L6-3</u> Have a nutritious snack.

<u>Exercise L6-4</u> Let $X, Y \in \mathfrak{gl}(n, \mathbb{C})$. Then

(a) $$[Y, X] = \frac{\partial^2}{\partial s \partial t} \left(\exp(-sY) \exp(-tX) \exp(sY) \exp(tX) \right) \Big|_{s=t=0}.$$

(b) $$\exp(-tY)\exp(-tX)\exp(tY)\exp(tX) = \exp\left(t^2 [Y, X] + O(t^3) \right)$$

Note that (b) can be read as saying that $[Y, X]$ provides the "correction factor" to commutativity

$$\exp(tY)\exp(tX) = \exp(tX)\exp(tY)\exp(t^2[X,Y] + O(t^3))$$

This <u>infinitesimal measure of the failure to commute</u> of two one-parameter subgroups $\{\exp(tX)\}_{t \in \mathbb{R}}$ and $\{\exp(sY)\}_{s \in \mathbb{R}}$ of a matrix Lie group G equips the Lie algebra $\mathfrak{g}$ of G with additional structure beyond that of a real vector space:

<u>Def</u> A <u>real Lie algebra</u> is an $\mathbb{R}$-vector space $\mathfrak{g}$ together with a function $\mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ denoted $(X,Y) \longmapsto [X,Y]$ (called the "bracket") with the following properties

(1) $[\lambda X + \mu Y, Z] = \lambda[X,Z] + \mu[Y,Z]$ $\quad \forall X,Y,Z \in \mathfrak{g}, \ \forall \lambda, \mu \in \mathbb{R}$ (Bilinearity)

(2) $[X,Y] = -[Y,X]$ $\quad \forall X,Y \in \mathfrak{g}$ (Skew symmetry)

(3) $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$ $\quad \forall X,Y,Z \in \mathfrak{g}$ (Jacobi identity)

If $\mathfrak{g}$ is a real Lie algebra and $\mathfrak{h} \subseteq \mathfrak{g}$ is an $\mathbb{R}$-vector subspace we call $\mathfrak{h}$ a <u>real Lie subalgebra</u> if whenever $X, Y \in \mathfrak{h}$ then also $[X,Y] \in \mathfrak{h}$. Then this bracket makes $\mathfrak{h}$ a real Lie algebra. We will often drop "real" and simply speak of Lie algebras.

<u>Remark L6-8</u> (i) (1) & (2) imply $[Z,-] : \mathfrak{g} \longrightarrow \mathfrak{g}$ is also linear, so $[-,-]$ is <u>bilinear</u>.

(ii) (2) implies $[X,X] = 0$ for all $X \in \mathfrak{g}$.

(iii) $\mathfrak{gl}(n,\mathbb{C})$ is a real Lie algebra for $n \geqslant 1$, with $[X,Y] = XY - YX$. If $\mathfrak{g} \subseteq \mathfrak{gl}(n,\mathbb{C})$ is any $\mathbb{R}$-vector subspace with the property that $X, Y \in \mathfrak{g}$ implies $[X,Y] \in \mathfrak{g}$ then $\mathfrak{g}$ is also a real Lie algebra with the same bracket.

(iv) $\mathrm{End}_{\mathbb{F}}(V)$ is a real Lie algebra with $[X,Y] = XY - YX$ for any $\mathbb{F}$-vector space V. Sometimes we write $\mathfrak{gl}(V,\mathbb{F})$ for $\mathrm{End}_{\mathbb{F}}(V)$ viewed as a Lie algebra.

<u>Theorem L6-12</u> Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$. Then

(i) If $A \in G$ and $X \in \mathfrak{g}$ then $AXA^{-1} \in \mathfrak{g}$.

(ii) If $X, Y \in \mathfrak{g}$ then $XY - YX \in \mathfrak{g}$.

Thus $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(n,\mathbb{C})$.

<u>Proof</u> (i) We have $\exp(tAXA^{-1}) = A\exp(tX)A^{-1} \in G$ by Ex B1-3, for $t \in \mathbb{R}, A \in G, X \in \mathfrak{g}$.

(ii) By (i) we have $\exp(tX)Y\exp(-tX) \in \mathfrak{g}$ for any $X, Y \in \mathfrak{g}$, and by the Leibniz rule (Ex L4-4) $\frac{d}{dt}\left(\exp(tX)Y\exp(-tX)\right)\Big|_{t=0} = XY - YX$ that is

$$\lim_{h \to 0} \frac{\exp(hX)Y\exp(-hX) - Y}{h} = XY - YX. \qquad (28.1)$$

The LHS is a limit in $\mathfrak{gl}(n,\mathbb{C})$ of a sequence of matrices in $\mathfrak{g}$, which is by Lemma L6-10 a real subspace and hence closed (since $\mathbb{R}^k \subseteq \mathbb{R}^n$ for $k \le n$ is cut out by linear equations, and is thus closed since those functions are continuous). Hence the RHS also belongs to $\mathfrak{g}$. $\square$

The Lie functor

We have now associated to any matrix Lie group G its real Lie algebra $\mathrm{Lie}(G)$. Next we prove that this construction is functorial (see Background 2 for a primer on categories and functors). In particular this means we can "take the logarithm" of any continuous representation of a matrix Lie group, giving the generalisation of Theorem L6-4 which motivated our study of one-parameter subgroups in the first place.

Def^n Let G, H be matrix Lie groups. A homomorphism of matrix Lie groups $\Phi : G \longrightarrow H$ is a continuous function which is also a group homomorphism, that is, $\Phi(e) = e$ and $\Phi(gg') = \Phi(g)\Phi(g')$ for all $g, g' \in G$.

Def^n The category LieGrp has matrix Lie groups as objects and homomorphisms of matrix Lie groups as morphisms.

Def^n Given real Lie algebras $\mathfrak{g}, \mathfrak{h}$ a homomorphism $\varphi : \mathfrak{g} \longrightarrow \mathfrak{h}$ is an $\mathbb{R}$-linear map satisfying $[\varphi X, \varphi Y] = \varphi([X,Y])$ for all $X, Y \in \mathfrak{g}$.

Def^n The category LieAlg $\mathbb{R}$ has real Lie algebras as objects and homomorphisms of real Lie algebras as morphisms.

As we will prove next lecture, $\sigma : SO(3) \longrightarrow GL(\mathcal{H}_k(S^2))$ is a morphism of matrix Lie groups and D3 of Theorem L6-4 is a morphism of Lie algebras $\mathfrak{so}(3) \longrightarrow \mathfrak{gl}(\mathcal{H}_k(S^2))$ and the theorem establishes that the latter is the image under a functor $\mathrm{Lie} :$ LieGrp $\longrightarrow$ LieAlg $\mathbb{R}$ of the former.

<u>Theorem L6-13</u> Let G, H be matrix Lie groups with respective Lie algebras $\mathfrak{g}, \mathfrak{h}$. If $\Phi : G \longrightarrow H$ is a homomorphism then there exists a unique $\mathbb{R}$-linear map $\varphi : \mathfrak{g} \longrightarrow \mathfrak{h}$ making the diagram

$$\begin{tikzcd} G \arrow[r, "\Phi"] & H \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "\varphi"'] & \mathfrak{h} \arrow[u, "\exp"'] \end{tikzcd} \tag{30.1}$$

commute. Moreover

(1) $\varphi(AXA^{-1}) = \Phi(A)\varphi(X)\Phi(A)^{-1}$ for all $A \in G$, $X \in \mathfrak{g}$

(2) $\varphi([X,Y]) = [\varphi X, \varphi Y]$ for all $X, Y \in \mathfrak{g}$.

(3) $\varphi(X) = \frac{d}{dt}\Phi(\exp(tX))\big|_{t=0}$ for all $X \in \mathfrak{g}$.

<u>Proof</u> Since Φ is a continuous group homomorphism $t \longmapsto \Phi(\exp(tX))$ is a one-parameter subgroup of H, for any $X \in \mathfrak{g}$. By Lemma L6-9 there is a unique $Y \in \mathfrak{h}$ with

$$\Phi(\exp(tX)) = \exp(tY) \qquad \forall t \in \mathbb{R} \tag{30.2}$$

and moreover $Y = \frac{d}{dt}\Phi(\exp(tX))\big|_{t=0}$. We set $\varphi(X) = Y$ and check it has the required properties. Firstly, to show φ is $\mathbb{R}$-linear: if $s \in \mathbb{R}$ then it is clear from (30.2) that $\varphi(sX) = s\varphi(X)$. Given $X, Y \in \mathfrak{g}$

$$\begin{aligned} \exp(t\varphi(X+Y)) &= \exp(\varphi(t(X+Y))) \\ &= \Phi(\exp(t(X+Y))) \end{aligned}$$

By the Lie product formula (Theorem B1-16) and the fact that Φ is a continuous homomorphism we can write this as

$$= \Phi\left(\lim_{m\to\infty} \left[\exp\left(\tfrac{tX}{m}\right) \exp\left(\tfrac{tY}{m}\right) \right]^m \right)$$

$$= \lim_{m\to\infty} \Phi\left(\left[\exp\left(\tfrac{tX}{m}\right) \exp\left(\tfrac{tY}{m}\right) \right]^m \right)$$

$$= \lim_{m\to\infty} \left[\Phi\left(\exp\left(\tfrac{tX}{m}\right)\right) \Phi\left(\exp\left(\tfrac{tY}{m}\right)\right) \right]^m \qquad (31.1)$$

$$= \lim_{m\to\infty} \left[\exp\left(\frac{t\varphi(X)}{m}\right) \exp\left(\frac{t\varphi(Y)}{m}\right) \right]^m$$

$$= \exp\left(t\left(\varphi(X) + \varphi(Y) \right) \right)$$

By Lemma L6-9 we must therefore have $\varphi(X+Y) = \varphi(X) + \varphi(Y)$ since these two matrices determine the same one-parameter subgroup. Hence φ is $\mathbb{R}$-linear.

To prove (1) note that

$$\exp(t\varphi(AXA^{-1})) = \Phi(\exp(tAXA^{-1}))$$

Ex B1-3

$$= \Phi(A \exp(tX) A^{-1})$$

$$= \Phi(A)\, \Phi(\exp(tX))\, \Phi(A^{-1}) \qquad (31.2)$$

$$= \Phi(A) \exp(t\varphi(X))\, \Phi(A^{-1})$$

Hence $\varphi(AXA^{-1}) = \frac{d}{dt} \exp(t\varphi(AXA^{-1}))\big|_{t=0} = \Phi(A)\, \varphi(X)\, \Phi(A)^{-1}$. Finally for (2) note that by (28.1) for $X, Y \in \mathfrak{g}$

$$[X,Y] = \frac{d}{dt} \exp(tX)\, Y \exp(-tX)\Big|_{t=0} \qquad (31.3)$$

$$= \lim_{h\to 0} \frac{\exp(hX)\, Y \exp(-hX) - Y}{h}$$

Now since φ is a linear transformation between finite-dimensional vector spaces it is continuous, so

$$
\begin{aligned}
\varphi([X,Y]) &= \varphi\Big(\lim_{t \to 0} \frac{\exp(tX)\, Y \exp(-tX) - Y}{h} \Big) \\
&= \lim_{t \to 0} \frac{1}{t}\, \varphi\big(\exp(tX)\, Y \exp(-tX) - Y \big) \\
&= \lim_{t \to 0} \frac{1}{t}\Big(\varphi\big(\exp(tX)\, Y \exp(-tX)\big) - \varphi Y \Big) \qquad (32.1) \\
&\overset{\text{by (1)}}{=} \lim_{t \to 0} \frac{1}{t}\Big(\Phi\big(\exp(tX)\big)\, \varphi Y\, \Phi\big(\exp(tX)^{-1}\big) - \varphi Y \Big) \\
&= \lim_{t \to 0} \frac{1}{t}\Big(\exp(t\varphi X)\, \varphi Y\, \exp(-t\varphi X) - \varphi Y \Big) \\
&= \frac{d}{dt}\Big(\exp(t\varphi X)\, \varphi Y \exp(-t\varphi X) \Big)\Big|_{t=0} \\
&\overset{(28.1)}{=} [\varphi X, \varphi Y]. \quad \square
\end{aligned}
$$

<u>Corollary L6-14</u> There is a functor $\mathrm{Lie} : \underline{\mathrm{LieGrp}} \longrightarrow \underline{\mathrm{LieAlg}\mathbb{R}}$ sending a matrix Lie group to its Lie algebra and a homomorphism of Lie groups $\Phi : G \longrightarrow H$ to the homomorphism of Lie algebras $\mathrm{Lie}(\Phi) : \mathrm{Lie}(G) \longrightarrow \mathrm{Lie}(H)$ defined by

$$
\mathrm{Lie}(\Phi)(X) = \frac{d}{dt}\, \Phi(\exp(tX))\Big|_{t=0} \qquad (32.2)
$$

<u>Proof</u> The fact that $\mathrm{Lie}(\Phi)$ is well-defined and a homomorphism of Lie algebras follows from Theorem L6-14. Clearly if $\Phi = 1_G$ then $\mathrm{Lie}(\Phi) = 1_{\mathrm{Lie}(G)}$. Suppose $\Phi : G \longrightarrow H$, $\Psi : H \longrightarrow K$ are homomorphisms of matrix Lie groups. Then $\mathrm{Lie}(\Phi)$, $\mathrm{Lie}(\Psi)$ are by Theorem L6-14 unique making the respective squares in the following diagram commute:

$$\begin{tikzcd} G \arrow[r, "\Phi"] & H \arrow[r, "\Psi"] & K \\ \mathrm{Lie}(G) \arrow[u, "\exp"] \arrow[r, "\mathrm{Lie}(\Phi)"'] & \mathrm{Lie}(H) \arrow[u, "\exp"] \arrow[r, "\mathrm{Lie}(\Psi)"'] & \mathrm{Lie}(K) \arrow[u, "\exp"] \end{tikzcd} \qquad (33.1)$$

But then outer diagram commutes, $\exp \circ (\mathrm{Lie}(\Psi) \circ \mathrm{Lie}(\Phi)) = (\Psi\Phi) \circ \exp$ so by uniqueness we have $\mathrm{Lie}(\Psi) \circ \mathrm{Lie}(\Phi) = \mathrm{Lie}(\Psi \circ \Phi)$ as claimed. $\square$

<u>Def^n</u> A <u>finite-dimensional complex representation</u> of a matrix Lie group G is a finite-dimensional complex vector space V together with a function $G \times V \longrightarrow V$ written $(g,v) \longmapsto g.v$ satisfying the following axioms

(R1) $g.(h.v) = (gh).v$ for all $g, h \in G$, $v \in V$

(R2) $e.v = v$ for all $v \in V$

(R3) the function $\alpha_g : V \longrightarrow V$, $\alpha_g(v) = g.v$ is $\mathbb{C}$-linear for all $g \in G$

(R4) for any basis β of V the function $G \longrightarrow GL(n, \mathbb{C})$ sending g to $[\alpha_g]^\beta_\beta$ is continuous, where $n = \dim(V)$ ⌜we allow $n = 0$, $GL(0, \mathbb{C}) = \{e\}$⌟

Thus for any basis β the map $g \longmapsto [\alpha_g]^\beta_\beta$ is a morphism of matrix Lie groups $G \to GL(n, \mathbb{C})$.

A homomorphism $\varphi : V \longrightarrow W$ of representations is a $\mathbb{C}$-linear map which satisfies

$$\varphi(g.v) = g.\varphi(v) \quad \text{for all } g \in G, v \in V.$$

The category of (complex, finite-dimensional) representations of G is denoted $\mathrm{rep}(G)$.

<u>Def^n</u> A <u>finite-dimensional complex representation</u> of a real Lie algebra $\mathfrak{g}$ is a finite-dimensional complex vector space V together with a function $\mathfrak{g} \times V \longrightarrow V$ written $(X, v) \longmapsto X.v$ satisfying the following axioms:

(S1) $[X,Y].v = X.(Y.v) - Y.(X.v)$ for all $X, Y \in \mathfrak{g}$, $v \in V$

(S2) $(\lambda X + \mu Y).v = \lambda(X.v) + \mu(Y.v)$ for all $X, Y \in \mathfrak{g}$, $\lambda, \mu \in \mathbb{R}$, $v \in V$.

(S3) the function $\alpha_X : V \longrightarrow V$, $\alpha_X(v) = X.v$ is $\mathbb{C}$-linear for all $X \in \mathfrak{g}$.

Thus for any basis β the map $X \longmapsto [\alpha_X]^\beta_\beta$ is a morphism of Lie algebras $\mathfrak{g} \longrightarrow \mathfrak{gl}(n, \mathbb{C})$.
A homomorphism $\varphi : V \longrightarrow W$ of representations is a $\mathbb{C}$-linear map satisfying

$n = \dim V$

$$\varphi(X.v) = X.\varphi(v) \qquad \text{for all } X \in \mathfrak{g},\ v \in V.$$

The category of (complex, finite-dimensional) representations of $\mathfrak{g}$ is denoted $\mathrm{rep}(\mathfrak{g})$.

<u>Lemma L6-15</u> Given a matrix Lie group G with Lie algebra $\mathfrak{g}$ there is a functor

$$T : \mathrm{rep}(G) \longrightarrow \mathrm{rep}(\mathfrak{g}) \qquad (34.1)$$

sending a representation V of G to a representation of $\mathfrak{g}$ on the same vector space, where for $X \in \mathfrak{g}$ we define

$$X.v = \frac{d}{dt}\big(\exp(tX).v\big)\Big|_{t=0} \qquad (34.2)$$

<u>Proof</u> Choose a basis β of V and encode the representation of G on V by a morphism $\Phi : G \longrightarrow GL(n, \mathbb{C})$, $\Phi(g) = [\alpha_g]^\beta_\beta$ of Lie groups. This induces by Corollary L6-14 a morphism $\phi = \mathrm{Lie}(\Phi) : \mathfrak{g} \longrightarrow \mathfrak{gl}(n, \mathbb{C})$ of Lie algebras, given by $\phi(X) = \frac{d}{dt}\Phi(\exp(tX))|_{t=0}$.
Now $\Phi(\exp(tX)) = [\alpha_{\exp(tX)}]^\beta_\beta$ where $\alpha_g(v) = g.v$. By definition the representation of $\mathfrak{g}$ determined by ϕ is $[X.v]_\beta = \phi(X)[v]_\beta$. Hence

Hence

$$
\begin{aligned}
[X.v]_\beta &= \frac{d}{dt}\Phi(\exp(tX))\Big|_{t=0}[v]_\beta \\
&= \frac{d}{dt}\Big([\alpha_{\exp(tX)}]^\beta_\beta [v]_\beta \Big)\Big|_{t=0} \\
&= \frac{d}{dt}\Big([\alpha_{\exp(tX)}(v)]_\beta \Big)\Big|_{t=0} \\
&= \frac{d}{dt}\Big([\exp(tX).v]_\beta \Big)\Big|_{t=0}
\end{aligned}
\tag{35.1}
$$

as claimed. This shows that T is well-defined on objects. Given a morphism $\varphi : V \longrightarrow W$ of representations of G we claim $T(\varphi) = \varphi$ is <u>also</u> a morphism of representations of $\mathfrak{g}$ (i.e. T is the identity on morphisms). To see this note that φ is continuous and $\mathbb{R}$-linear so

$$
\begin{aligned}
\varphi(X.v) &= \varphi\Big(\lim_{t\to 0} \frac{\exp(tX).v - v}{t} \Big) \\
&= \lim_{t\to 0} \frac{\varphi(\exp(tX).v) - \varphi(v)}{t} \\
&= \lim_{t\to 0} \frac{\exp(tX).\varphi(v) - \varphi(v)}{t} \\
&= X.\varphi(v)
\end{aligned}
\tag{35.2}
$$

as claimed. $\square$

Unfortunately we will not have time to prove the following fundamental result, which justifies the study of Lie algebras and their representations (see [H, Theorem 3.7]). But you should know it.

<u>Theorem</u> If G is connected and simply-connected then $T : \mathrm{rep}(G) \longrightarrow \mathrm{rep}(\mathfrak{g})$ is an equivalence of categories.

Exercise LG-5 Suppose that G is a matrix Lie group every element of which can be written as a product of the form $\exp(X_1) \cdots \exp(X_n)$ for some $X_1, \dots, X_n \in \mathfrak{g} = \mathrm{Lie}(G)$ (this is true if G is connected, i.e. every $g, h \in G$ are connected by a continuous path in G). Prove that the functor T is full, that is, for any pair of representations V, W of G if a linear map $\varphi : V \to W$ is a morphism of $\mathfrak{g}$-representations it is also a morphism of G representations.

References

[H] B. Hall "Lie Groups, Lie Algebras, and Representations" Springer GTM.