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Lecture 7 – Representation theory of Lie algebras

We have associated to each matrix Lie group G its Lie algebra $\mathfrak{g}$ and proven that there is a functor $\mathrm{Lie} : \mathrm{rep}(G) \longrightarrow \mathrm{rep}(\mathfrak{g})$ sending a representation of G on a finite-dimensional complex vector space V to a representation of $\mathfrak{g}$ on the same space. We now turn to examples of such representations, beginning of course with $G = SO(3)$, $\mathfrak{g} = \mathfrak{so}(3)$ and $V = \mathcal{H}_k(S^2)$ for some $k \geq 0$.

There is a slight technicality: recall that $\delta : SO(3)^{op} \longrightarrow \mathrm{Aut}_{\mathbb{C}}(\mathcal{H}_k(S^2))$ is actually a representation of the group $SO(3)^{op}$, which is inconvenient if we wish to view our matrix Lie groups as subgroups of $GL(n, \mathbb{C})$. So we now identify $SO(3) \cong SO(3)^{op}$ as follows:

Lemma L7-1 For $G \subseteq GL(n, \mathbb{C})$ a matrix Lie group the function $i : G \longrightarrow G$ defined by $i(A) = A^{-1}$ is continuous.

Proof This follows from Cramer's rule (Lemma L1-2) as explained in the proof of Lemma L1-1. $\square$

Recall from Lecture 4 the representation δ of $SO(3)$ on spherical harmonics.

Lemma L7-2 The function $\delta : SO(3) \longrightarrow \mathrm{Aut}_{\mathbb{C}}(\mathcal{H}_k(S^2))$ is continuous for $k \geq 0$.

Proof Recall that our conventions, introduced in Lecture 4, is that a function like δ is continuous if for some (hence every) ordered $\mathbb{C}$-basis β of $\mathcal{H}_k(S^2)$ that the map $g \in SO(3) \longmapsto [\delta(g)]_{\beta}^{\beta} \in GL(2k+1, \mathbb{C})$ is continuous.
As explained in the proof of Lemma L4-2 it suffices to prove that

$$\rho : SO(3) \longrightarrow \mathrm{Aut}_{\mathbb{C}}(P_k) \qquad (1.1)$$
$$\rho(g) = \varphi_A$$

is continuous, where A is the matrix of g in the standard basis, and by L4 (2.3)

$$\varphi_A(P) = P\left(\sum_{j=1}^3 A_{1j} x_j,\ \sum_{j=1}^3 A_{2j} x_j,\ \sum_{j=1}^3 A_{3j} x_j \right)$$

If we choose the basis $\beta = \{x^\alpha\}_{|\alpha|=k}$ of degree k monomials $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$ for $\mathcal{P}_k$ then $[\rho(g)]_\beta^\beta$ has entries which are polynomial functions of the entries in the matrix A (see L4 p. 8) and hence $\rho : SO(3) \longrightarrow Aut_{\mathbb{C}}(\mathcal{P}_k)$ is continuous. $\square$

<u>Lemma L7-3</u> For $k \geq 0$ the function

$$\begin{aligned} SO(3) &\longrightarrow GL(2k+1, \mathbb{C}) \\ A &\longmapsto [\rho(A^{-1})]_{\mathcal{C}}^{\mathcal{C}} \end{aligned} \qquad (2.1)$$

is a morphism of matrix Lie groups, and thus determines a representation of $SO(3)$ on $\mathcal{H}_k(S^2)$ given by $A.f = f \circ A^{-1}$, for any ordered basis $\mathcal{C}$ of $\mathcal{H}_k(S^2)$.

<u>Proof</u> The function (2.1) is the composite

$$SO(3) \xrightarrow{(-)^{-1}} SO(3)^{op} \xrightarrow{\ \delta\ } Aut_{\mathbb{C}}(\mathcal{H}_k(S^2)) \xrightarrow{[-]_\beta^\beta} GL(2k+1, \mathbb{C})$$

of continuous group homomorphisms, and is thus a continuous group homomorphism (we are using Lemma L7-1, Lemma L7-2). $\square$

<u>Defⁿ</u> We write δ^- for the continuous group homomorphism $SO(3) \longrightarrow Aut_{\mathbb{C}}(\mathcal{H}_k(S^2))$ given by $\delta^- = \delta \circ (-)^{-1}$, that is, $\delta^-(R_\alpha^{\hat{n}})(f) = f \circ R_{-\alpha}^{\hat{n}}$, and we do not distinguish this from the representation (2.1) of $SO(3)$ as a matrix Lie group (although technically the latter involves the additional data of a choice of basis).

When we refer to $\mathcal{H}_k(S^2)$ as a representation of $SO(3)$ as a matrix Lie group, δ^- is what we mean. By Lemma L6-15 there is a corresponding representation of $\mathfrak{so}(3)$ on $\mathcal{H}_k(S^2)$. This is <u>almost</u> the content of Theorem L6-4, except for the difference between δ and δ^- (irritating).

<u>Lemma L7-4</u> Under the functor $\mathrm{Lie} : \mathrm{rep}(SO(3)) \longrightarrow \mathrm{rep}(\mathfrak{so}(3))$ the representation of $\mathfrak{so}(3)$ on $\mathcal{H}_k(S^2)$ corresponding the representation γ^- of $SO(3)$ is

$$\begin{aligned} \mathfrak{so}(3) &\xrightarrow{\quad D\gamma^- \quad} \mathrm{End}_{\mathbb{C}}(\mathcal{H}_k(S^2)) \\ \delta^{\hat{n}} &\longmapsto -\eta^{\hat{n}} \end{aligned} \tag{3.1}$$

Or written as an action, $\delta^{\hat{n}}.f = -\eta^{\hat{n}}(f)$ for $\hat{n} \in S^2$, $f \in \mathcal{H}_k(S^2)$.

<u>Proof</u> By definition

$$\delta^{\hat{n}}.f = \frac{d}{dt}\big(\exp(t\delta^{\hat{n}}).f\big)\big|_{t=0} = \frac{d}{dt}\big(\gamma(\exp(t\delta^{\hat{n}})^{-1})(f)\big)\big|_{t=0}$$

$$= \frac{d}{dt}\big(\gamma(\exp(-t\delta^{\hat{n}}))(f)\big)\big|_{t=0}$$

$$\overset{\text{Thm L6-4}}{=} \frac{d}{dt}\big(\exp(-t\eta^{\hat{n}})(f)\big)\big|_{t=0} = -\eta^{\hat{n}}(f). \square$$

We learn from Lemma L7-4 that (3.1) is a <u>homomorphism of Lie algebras</u> (previously we only knew it was $\mathbb{R}$-linear) which means in light of Lemma L6-11 that for $\hat{n}, \hat{m} \in S^2$

$$\begin{aligned} \eta^{\hat{n}\times\hat{m}} &= -D\gamma^-(\delta^{\hat{n}\times\hat{m}}) \\ &= -D\gamma^-([\delta^{\hat{n}}, \delta^{\hat{m}}]) \\ &= -[D\gamma^-(\delta^{\hat{n}}), D\gamma^{-1}(\delta^{\hat{m}})] \\ &= -[-\eta^{\hat{n}}, -\eta^{\hat{m}}] \\ &= -[\eta^{\hat{n}}, \eta^{\hat{m}}] \\ &= [\eta^{\hat{m}}, \eta^{\hat{n}}]. \end{aligned} \tag{3.2}$$

This could of course be checked directly, but don't look a gift horse in the mouth, as they say.

So we have finally made <u>$\mathcal{H}_k(S^2)$ into a representation of the real Lie algebra</u> $\mathfrak{so}(3)$. Now what?

Questions that remain to be answered

(1) What is the structure of the representation $\mathcal{C}$ of $SO(3)$ on $L^2(S^2, \mathbb{C})$? As we commented at the beginning of Lecture 4, using the Laplacian we know this representation decomposes into subrepresentations of $SO(3)$ on each $\mathcal{H}_k(S^2)$, but to this point we know nothing about these subrepresentations. Our motivation for proceeding to study infinitesimal generators of symmetries in Lecture 6 was the claim that these generators, which we now know form a real Lie algebra by Theorem L6-12, are the key to understanding the structure of these subrepresentations. We must now make good on this claim.

(2) Which representation of $SO(3)$ on $L^2(S^2, \mathbb{C})$ is chosen by Nature? Starting in Lectures 1, 2 and continuing in p. 4 L4, p. 1 L5 we have placed the idea of "observers of a quantum system with state space $\mathcal{H} = L^2(S^2, \mathbb{C})$ related by unitary transformations of $\mathcal{H}$ which constitute a representation of $SO(3)$" at the center of this subject. This state space might describe, for example, the angular part of the wavefunction of some particle. On p. 4 of L4 we emphasised that while Wigner's theorem tells us that for any such particle there is some (possibly projective!) representation of $SO(3)$ on $L^2(S^2, \mathbb{C})$, the theorem on its own cannot tell us which one Nature has "chosen" (i.e. that corresponds to actual measurements of that kind of particle). We now proceed to dig into this question, by revisiting the topic of angular momentum from L5.

Structure of the representation

We consider $\mathfrak{so}(3)$ acting on $\mathcal{H}_k(S^2)$ for all $k \geq 0$, as in (3.1). To understand these representations we will make use of a trick we have exploited several times already: study instead the complex-vector space $\mathcal{P}_k = \mathcal{P}_k(3)$ of degree k homogeneous polynomials in $x = x_1$, $y = x_2$, $z = x_3$, which has a simple basis $\{x^\beta\}_{|\beta|=k}$ of monomials $x^\beta = x_1^{\beta_1} x_2^{\beta_2} x_3^{\beta_3}$ indexed by tuples $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{N}^3$ with $|\beta| = \sum_{i=1}^{3} \beta_i$ equal to k (here $0 \in \mathbb{N}$). The subspace $\mathcal{H}_k = \{ P \in \mathcal{P}_k \mid \Delta P = 0\}$ of harmonic polynomials is isomorphic to $\mathcal{H}_k(S^2)$ by Lemma L3-7. We start our investigations with $\mathcal{P}_k$ rather than $\mathcal{H}_k(S^2)$ because it is quite involved to construct a basis of the latter (although we will do so).

Def^n Let $\mathfrak{g}$ be a real Lie algebra and V a complex finite-dimensional representation of $\mathfrak{g}$. A $\mathbb{C}$-vector subspace $W \subseteq V$ is a <u>subrepresentation</u> if $X.v \in W$ for all $X \in \mathfrak{g}$, $v \in W$.

Recall that $\tau^{\hat{n}}$ is defined as a $\mathbb{C}$-linear operator on $\mathcal{P}_k$ on p. 4 L5.

Lemma L7-5 The complex vector space $\mathcal{P}_k$ is a representation of $\mathfrak{so}(3)$ with action $\delta^{\hat{n}}.P = -\tau^{\hat{n}}(P)$ for $P \in \mathcal{P}_k$. The subspace $\mathcal{H}_k$ is a subrepresentation and the map $\mathcal{H}_k \longrightarrow \mathcal{H}_k(S^2)$ sending P to $P|_{S^2}$ is an isomorphism of representations of $\mathfrak{so}(3)$.

Proof In the proof of Lemma L7-2 we have already checked $g \mapsto \rho(g^{-1})$ is a representation of the matrix Lie group $SO(3)$ on $\mathcal{P}_k$, and by the same calculation as in the proof of Lemma L7-4 the induced representation of $\mathfrak{so}(3)$ is the given one. It follows from Ex L4-7 that $\mathcal{H}_k \subseteq \mathcal{P}_k$ is a subrepresentation, and the final claim is clear. $\square$

Recall that $\mathcal{P}_k$ is a Hilbert space with pairing $\langle P, Q \rangle_k = [\partial(Q)\overline{P}]_{\text{const}}$ by Lemma L3-2, where for $Q = \sum_{|\alpha|=k} c_\alpha x^\alpha$, $\partial(Q) = \sum_{|\alpha|=k} c_\alpha \partial^{|\alpha|} / \partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$, and $\overline{P}$ denotes the polynomial with all coefficients conjugated.

<u>Lemma L7-6</u> The $\mathbb{C}$-linear operator $\mathcal{T}^{\hat{n}} : \mathcal{P}_k \longrightarrow \mathcal{P}_k$ is skew self-adjoint with respect to the pairing $\langle -, - \rangle_k$.

<u>Proof</u> A linear combination of skew self-adjoint operators (with real coefficients) is skew self-adjoint, so as $\mathcal{T}^{\hat{n}} = \sum_i n_i \mathcal{T}^{x_i}$ it suffices to show $\mathcal{T}^x, \mathcal{T}^y, \mathcal{T}^z$ are skew self-adjoint. But for $i \neq j$ we have for $P, Q \in \mathcal{P}_k$ by Lemma L3-3

$$\begin{aligned} \left\langle P, x_i \frac{\partial}{\partial x_j} Q \right\rangle_k &= \left\langle \frac{\partial}{\partial x_i} P, \frac{\partial}{\partial x_j} Q \right\rangle_{k-1} \\ &= \left\langle x_j \frac{\partial}{\partial x_i} P, Q \right\rangle_k \end{aligned} \tag{6.1}$$

From this it easily follows that $\mathcal{T}^{x_i}$ is skew self-adjoint for $1 \leq i \leq 3$. □

Compare this to Lemma L5-5. Since $i\mathcal{T}^{\hat{n}}$ is self-adjoint on $\mathcal{P}_k$ we have by the spectral theorem (Lemma L5-2) that all eigenvalues of $\mathcal{T}^{\hat{n}}$ are pure imaginary, and there is an orthogonal basis of $\mathcal{P}_k$ consisting of eigenvectors of $\mathcal{T}^{\hat{n}}$.

<u>Example L7-1</u> $\mathcal{P}_0 = \mathbb{C} \cdot 1$ is not interesting as all $\mathcal{T}^{\hat{n}}$ act as zero. On $\mathcal{P}_1$ with respect to the basis $\{x, y, z\}$ we have

$$[\mathcal{T}^x] = \left[y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = -\delta^x$$

$$[\mathcal{T}^y] = \left[z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right] = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = -\delta^y$$

$$[\mathcal{T}^z] = \left[x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right] = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -\delta^z$$

Now $\delta^{\hat{n}}$ acts on $\mathcal{P}_1$ via $-\mathcal{T}^{\hat{n}}$, so this says that $\delta^{\hat{n}}$ acts on $\mathcal{P}_1 \cong \mathbb{C}^3$ as itself!

That is, the elements of $\mathfrak{so}(3)$ just happen to themselves be matrices (for an abstract real Lie algebra we can make no such statement) and therefore can be viewed as linear operators on $\mathbb{R}^3$. In the standard terminology, P_0 is the trivial representation of $\mathfrak{so}(3)$ and P_1 is the defining representation. Note that $\mathcal{H}_0 = P_0$, $\mathcal{H}_1 = P_1$ so $\mathcal{H}_0(S^2)$ is the trivial representation and $\mathcal{H}_1(S^2)$ is the defining representation.

Def^n Let $\mathfrak{g}$ be a real Lie algebra. The trivial representation of $\mathfrak{g}$, denoted $\mathbb{1}$, is the vector space $\mathbb{C}$ with $X.v = 0$ for all $X \in \mathfrak{g}, v \in \mathbb{C}$.

Def^n Let $\mathfrak{g}$ be a real Lie algebra of $\mathfrak{gl}(n, \mathbb{C})$. The defining representation of $\mathfrak{g}$ is the vector space $\mathbb{C}^n$ with $X.v = Xv$ (matrix multiplication) for $X \in \mathfrak{g}, v \in \mathbb{C}^n$.

To be a bit pedantic, P_1 is only isomorphic to the defining representation (that is, it is isomorphic to the defining representation as an object of $\operatorname{rep}(\mathfrak{g})$), since as sets $P_1 \neq \mathbb{C}^3$.

Exercise L7-1 Given a matrix Lie group G construct the complex finite-dimensional G-representation V with $\operatorname{Lie}(V)$ the defining representation of $\mathfrak{g}$.

Example L7-2 Next we consider the representation P_2. The basis of monomials is

$$\{ x^2, xy, xz, y^2, yz, z^2 \} \tag{7.1}$$

so this is a six-dimensional $\mathbb{C}$-vector space. Writing down the matrices of η^x, η^y, η^z in this basis seems like a chore. The subspace $\mathcal{H}_2 \subseteq P_2$ is the set of linear combinations $ax^2 + bxy + cxz + dy^2 + eyz + fz^2$ in the kernel of $\partial = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, that is

$$0 = \partial\left(ax^2 + bxy + cxz + dy^2 + eyz + fz^2 \right) = 2a + 2d + 2f \tag{7.2}$$

So b, c, e are free parameters and $a+d+f=0$. This gives $\dim_{\mathbb{C}} \mathcal{H}_2 = 5 = 2\cdot 2+1$ which is consistent with Ex L4-2. By Theorem L3-4 we have a direct sum decomposition

$$P_2 = \mathcal{H}_2 \oplus \|x\|^2 P_0 \tag{8.1}$$

where in this case $\|x\|^2 P_0$ is the linear span of $\|x\|^2 = x^2+y^2+z^2$ (that is just what this notation means). Note that for $(i,j,k) \in \{(1,2,3),(2,3,1),(3,1,2)\}$ we have $\eta^{x_i} = x_j \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_j}$ and

$$\eta^{x_i}(x_i^2) = 0, \quad \eta^{x_i}(x_j^2) = -2x_j x_k, \quad \eta^{x_i}(x_k^2) = 2x_j x_k \tag{8.2}$$

Hence $\eta^{x_i}(\|x\|^2) = 0$ which shows $\|x\|^2 P_0$ is not just a $\mathbb{C}$-vector subspace it is a <u>subrepresentation</u> and moreover it is (isomorphic to) the trivial representation. Let us now turn our attention to $\mathcal{H}_2$.

A general element of $\mathcal{H}_2$ can be written as

$$\begin{aligned} & ax^2 + bxy + cxz + (-a-f)y^2 + eyz + fz^2 \\ = \; & a(x^2-y^2) + bxy + cxz + eyz + f(z^2-y^2) \end{aligned} \tag{8.3}$$

so our basis of $\mathcal{H}_2$ is $\{x^2-y^2, xy, xz, yz, z^2-y^2\}$. We can of course compute all the η^{x_i} in this basis, but it isn't clear what this will accomplish. Recall that since the η^{x_i} are skew self-adjoint we can find an orthogonal basis of eigenvectors for <u>any one of the η^{x_i}</u> (but not all at once). This will at least diagonalise one of the operators, and then we can worry about the other two. It is arbitrary which operator we choose to diagonalise, but the convention is to choose η^z. We compute (recall $\eta^z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$)

$$\begin{aligned} &\eta^z(x^2-y^2) = -4xy, \quad \eta^z(xy) = x^2-y^2, \quad \eta^z(xz) = -yz \\ &\eta^z(yz) = xz \qquad\qquad\quad \eta^z(z^2-y^2) = -2xy \end{aligned} \tag{8.4}$$

We observe that $\mathrm{span}_{\mathbb{C}}\{x^2-y^2, xy, z^2-y^2\}$ and $\mathrm{span}_{\mathbb{C}}\{xz, yz\}$ are η^z-invariant subspaces, on which η^z can be depicted as

$$\begin{tikzcd} \boxed{z^2-y^2} \arrow[r, "-2"] & \boxed{xy} \arrow[r, shift left, "1"] & \boxed{x^2-y^2} \arrow[l, shift left, "-4"] \end{tikzcd} \tag{9.1}$$

$$\begin{tikzcd} \boxed{xz} \arrow[r, shift left, "-1"] & \boxed{yz} \arrow[l, shift left, "1"] \end{tikzcd}$$

A vector $cxz + yz$ is an eigenvector with eigenvalue λ of η^z if and only if

$$\begin{aligned} -cyz + xz = \lambda cxz + \lambda yz &\iff \lambda c = 1,\ \lambda = -c \\ &\iff c^2 = -1,\ \lambda = -c \\ &\iff (c,\lambda) \in \{(i,-i), (-i,i)\} \end{aligned}$$

This produces eigenvectors $ixz + yz$, $-ixz + yz$ with respective eigenvalues $-i, i$, and these vectors span $\mathrm{span}_{\mathbb{C}}\{xz, yz\}$. They are orthogonal by Lemma L5-2(b).

A vector $a(x^2-y^2) + bxy + f(z^2-y^2)$ is an eigenvector with eigenvalue λ of η^z iff.

$$-4axy + b(x^2-y^2) - 2fxy = \lambda a(x^2-y^2) + \lambda bxy + \lambda f(z^2-y^2)$$

$$\iff \lambda a = b,\ \lambda b = -2(f+2a),\ \lambda f = 0$$

This gives an eigenvector $-\frac{1}{2}(x^2-y^2) + (z^2-y^2)$ with eigenvalue $\lambda = 0$, and if $\lambda \neq 0$ then

$\lambda \neq 0$
$f = 0$
$(\therefore a, b \neq 0)$

$$\left\{ \begin{aligned} &\iff \lambda a = b,\ \lambda b = -4a \\ &\iff \lambda^2 a = -4a,\ b = \lambda a \\ &\iff \lambda^2 = -4,\ \tfrac{b}{a} = \lambda \end{aligned} \right.$$

This produces eigenvectors $(x^2-y^2) + 2ixy$, $(x^2-y^2) - 2ixy$ of eigenvalues $2i, -2i$ respectively.

We have thus constructed an orthogonal basis of eigenvectors of η^z for $\mathcal{H}_2$:

$$
\begin{aligned}
|2\rangle &:= (x^2-y^2)+2ixy & \eta^z|2\rangle &= 2i|2\rangle \\
|1\rangle &:= -ixz+yz & \eta^z|1\rangle &= i|1\rangle \\
|0\rangle &:= -\tfrac{1}{2}(x^2-y^2)+(z^2-y^2) & \eta^z|0\rangle &= 0 \\
|-1\rangle &:= ixz+yz & \eta^z|-1\rangle &= -i|-1\rangle \\
|-2\rangle &:= (x^2-y^2)-2ixy & \eta^z|-2\rangle &:= -2i|-2\rangle.
\end{aligned}
\tag{10.1}
$$

The notation reflects that the z-component of the angular momentum $L_z = -i\eta^z$ has eigenvalue $m \in \{-2,-1,0,1,2\}$ on $|m\rangle$. That is, $L_z|m\rangle = m|m\rangle$. In particular we have computed that the spectrum of η^z on $\mathcal{H}_2$ is $\{im \mid m\in\mathbb{Z} \text{ and } -2\leq m\leq 2\}$. Note $\overline{|m\rangle} = |-m\rangle$.

Now let us return to the operators η^x, η^y and try to understand how they behave with respect to the above basis: recall $\eta^x = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$, $\eta^y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$

$$
\begin{aligned}
\eta^x|2\rangle &= 2zy-2ixz = 2|1\rangle \\
\eta^x|1\rangle &= -ixy+y^2-z^2 = -|0\rangle - \tfrac{1}{2}|2\rangle \\
\eta^x|0\rangle &= 2yz-yz+2yz = 3yz = \tfrac{3}{2}|1\rangle + \tfrac{3}{2}|-1\rangle \\
\eta^x|-1\rangle &= iyx+y^2-z^2 = -|0\rangle - \tfrac{1}{2}|-2\rangle \\
\eta^x|-2\rangle &= 2yz+2ixz = 2|-1\rangle
\end{aligned}
\tag{10.2}
$$

Well that isn't so ugly. We see that with some coefficients η^x maps $|m\rangle$ to $|m\pm1\rangle$:

$$
\begin{tikzcd}
|-2\rangle \arrow[r, bend left, "2"] & |-1\rangle \arrow[l, bend left, "-\frac{1}{2}"] \arrow[r, bend right, "-1"'] & |0\rangle \arrow[l, bend right, "\frac{3}{2}"'] \arrow[r, bend left, "\frac{3}{2}"] & |1\rangle \arrow[l, bend left, "-1"] \arrow[r, bend right, "-\frac{1}{2}"'] & |2\rangle \arrow[l, bend right, "2"']
\end{tikzcd}
\tag{10.3}
$$

Let's continue and do the same for η^y:

$$
\begin{aligned}
\eta^y|2\rangle &= 2zx + 2iyz = 2i|1\rangle \\
\eta^y|1\rangle &= -iz^2 + ix^2 - xy = i(x^2-z^2) - xy = -i|0\rangle + \tfrac{1}{2}i|2\rangle \\
\eta^y|0\rangle &= -zx - xz = -3xz = -\tfrac{3}{2}i|1\rangle + \tfrac{3}{2}i|-1\rangle \qquad (11.1)\\
\eta^y|-1\rangle &= iz^2 - ix^2 - xy = i(z^2-x^2) - xy = i|0\rangle - \tfrac{1}{2}i|-2\rangle \\
\eta^y|-2\rangle &= 2zx - 2iyz = -2i|-1\rangle
\end{aligned}
$$

Note that for any complex polynomial P, $\overline{\eta^{x_i}(P)} = \eta^{x_i}(\overline{P})$ so the final two equations in (11.1) can be deduced from the first two via $\eta^y|-m\rangle = \eta^y(\overline{|m\rangle}) = \overline{\eta^y|m\rangle}$. The diagram is

$$
\begin{tikzcd}
|-2\rangle \arrow[r, bend left, "-2i"] & |-1\rangle \arrow[l, bend left, "-\frac{1}{2}i"] \arrow[r, bend right, "i"'] & |0\rangle \arrow[l, bend right, "\frac{3}{2}i"'] \arrow[r, bend left, "-\frac{3}{2}i"] & |1\rangle \arrow[l, bend left, "-i"] \arrow[r, bend right, "\frac{1}{2}i"'] & |2\rangle \arrow[l, bend right, "2i"']
\end{tikzcd} \qquad (11.2)
$$

In presenting $\mathcal{H}_2$ as a representation of $\mathfrak{so}(3)$ we get to choose (a) a basis of $\mathcal{H}_2$ but also (b) <u>a basis of $\mathfrak{so}(3)$</u>. So far we have made a choice to prioritise η^z (and thus δ^z) in order to fix our choice of basis $\{|-2\rangle, |-1\rangle, |0\rangle, |1\rangle, |2\rangle\}$ of $\mathcal{H}_2$, but if we wish we can adopt some linear combinations $\alpha\delta^x + \beta\delta^y$, $\gamma\delta^x + \mu\delta^y$ in place of δ^x, δ^y to augment δ^z in forming a basis of $\mathfrak{so}(3)$ (so we need $\begin{pmatrix}\alpha & \gamma\\ \beta & \mu\end{pmatrix}$ to be invertible). This will be useful if the matrices of $\alpha\eta^x + \beta\eta^y$, $\gamma\eta^x + \mu\eta^y$ are particularly simple in the basis $\{|m\rangle\}_{-2\le m\le 2}$.

Inspecting (10.3), (11.2) we see that there are <u>sign asymmetries</u> in the latter (e.g. in the lines emanating from $|0\rangle$) but not the former, which invites us to achieve the following cancellations:

$$
\begin{aligned}
(\eta^y - i\eta^x)|2\rangle &= 2i|1\rangle - 2i|1\rangle = 0 \qquad (11.3)\\
(\eta^y - i\eta^x)|1\rangle &= -i|0\rangle + \tfrac{1}{2}i|2\rangle + i|0\rangle + \tfrac{i}{2}|2\rangle = i|2\rangle \\
(\eta^y - i\eta^x)|0\rangle &= -\tfrac{3}{2}i|1\rangle + \tfrac{3}{2}i|-1\rangle - \tfrac{3i}{2}|1\rangle - \tfrac{3i}{2}|-1\rangle = -3i|1\rangle \\
(\eta^y - i\eta^x)|-1\rangle &= i|0\rangle - \tfrac{1}{2}i|-2\rangle + i|0\rangle + \tfrac{i}{2}|-2\rangle = 2i|0\rangle \\
(\eta^y - i\eta^x)|-2\rangle &= -2i|-1\rangle - 2i|-1\rangle = -4i|-1\rangle
\end{aligned}
$$

Taking conjugates yields

$$\begin{aligned}
(\eta^y + i\eta^x)|-2\rangle &= 0 \\
(\eta^y + i\eta^x)|-1\rangle &= -i|-2\rangle \\
(\eta^y + i\eta^x)|0\rangle &= 3i|-1\rangle \\
(\eta^y + i\eta^x)|1\rangle &= -2i|0\rangle \\
(\eta^y + i\eta^x)|2\rangle &= 4i|1\rangle
\end{aligned} \tag{12.1}$$

If we set $L_+ := \eta^y - i\eta^x$, $L_- = -\eta^y - i\eta^x$ then

[signs are chosen to make L_+, L_- adjoint to one another, see below]

$$\begin{tikzcd}
 & & L_+ & & \\
{|-2\rangle} \arrow[r, bend left, "-4i"] & {|-1\rangle} \arrow[l, bend left, "i"] \arrow[r, bend left, "2i"] & {|0\rangle} \arrow[l, bend left, "-3i"] \arrow[r, bend left, "-3i"] & {|1\rangle} \arrow[l, bend left, "2i"] \arrow[r, bend left, "i"] & {|2\rangle} \arrow[l, bend left, "-4i"] \\
 & & L_- & &
\end{tikzcd} \tag{12.2}$$

In the space of $\mathbb{C}$-linear operators on $\mathcal{H}_2$ of course $\mathrm{span}_{\mathbb{C}}\{\eta^x, \eta^y\} = \mathrm{span}_{\mathbb{C}}\{L_+, L_-\}$ and clearly the structure of $\mathcal{H}_2$ is more visible in the latter basis than the former. We call L_+ a <u>raising operator</u> (since it increases the eigenvalue of L_z) and L_- a <u>lowering operator</u>. Together L_+, L_- are known as <u>ladder operators</u> due to the "ladder of states" in (12.2). Since

$$\eta^x = \frac{i}{2}(L_+ + L_-), \quad \eta^y = \frac{1}{2}(L_+ - L_-) \tag{12.3}$$

the presentation of the action of $so(3)$ on $\mathcal{H}_2$ via the eigenbasis $\{|m\rangle\}_{-2\le m\le 2}$ of η^z and the ladder operators L_+, L_- <u>completely characterises the structure of this representation</u>. There are some fine points to return to (e.g. we might like to choose a normalisation which gets rid of the odd 2, 3, 4 in (12.2), and it is strange that <u>nothing in $so(3)$ acts as L_+ on $\mathcal{H}_2$</u> since we can't lift the complex linear combination, i.e. $-\delta^y + i\delta^x \notin so(3)$), but for now let us give an application of this characterisation of $\mathcal{H}_2$ in terms of ladder operators.

Def^n Let $\mathfrak{g}$ be a real Lie algebra and V a complex finite-dimensional representation. We say V is irreducible if it is nonzero and has no subrepresentations other than 0 and V. If $V \neq 0$ is not irreducible it is called reducible

Of course the trivial representation is irreducible, and we have seen that the representation P_2 is reducible since it contains $\mathcal{H}_2$ as a proper subrepresentation.

Exercise L7-2 Prove directly from Example L7-1 that P_1 is an irreducible representation of $so(3)$.

Lemma L7-7 The representation $\mathcal{H}_2$ of $so(3)$ is irreducible.

Proof Suppose $V \subseteq \mathcal{H}_2$ is a $\mathbb{C}$-vector subspace which is closed under the action of $so(3)$. Equivalently, it is closed under L_+, L_-, L_z. If $V \neq 0$ we can choose a nonzero vector $v \in V$, and write $v = \sum_{m=-2}^{2} a_m |m\rangle$ for some $a_m \in \mathbb{C}$. Set $M = \inf\{m \mid a_m \neq 0\}$. Then
$v = a_M |M\rangle + a_{M+1} |M+1\rangle + \cdots + a_2 |2\rangle$ and so

$$(L_+)^{2-M} v = \mu a_M |2\rangle \qquad (13.1)$$

for some nonzero $\mu \in \mathbb{C}$. Hence $|2\rangle \in V$. But then $(L_-)^n |2\rangle \in V$ for all $n \geq 0$, from which we deduce $|m\rangle \in V$ for all $-2 \leq m \leq 2$, and hence $V = \mathcal{H}_2$. $\square$

We have now discovered three essentially distinct representations of $so(3)$: $\mathcal{H}_0 = P_0 = \mathbb{1}$, the trivial representation (dimension is 1), $\mathcal{H}_1 = P_1 = \mathbb{C}^3$ the defining representation, and now we know $\mathcal{H}_2$ cannot be written as a direct sum of these since it is irreducible, so this is a genuinely new representation (unlike P_2 which is just $P_2 \cong \mathcal{H}_2 \oplus \mathcal{H}_1$) of dimension 5. This leads to the following questions:

Question 1 Can every representation of $so(3)$ be decomposed as a direct sum of irreducible representations?

Question 2 Are the $\mathcal{H}_k$ all irreducible? Are there any other irreducible representations?

Ladder operators

Our study of $\mathcal{H}_2$ was very "hands on" in the sense that we found an explicit basis of eigenvectors for η^z and observed how η^x, η^y behave on this basis, leading us to introduce the ladder operators. Let us now give a more abstract explanation for why the ladder operators behave the way they do, which focuses on their commutation relations.

Def^n For any $k \geq 0$ we define the raising operator $L_+ = \eta^y - i\eta^x$ and lowering operator $L_- = -\eta^y - i\eta^x$ on $\mathcal{H}_k$. Collectively these are known as ladder operators.

Remark L7-1 Recall from p. 10 that $L_{\hat{n}} = -i\eta^{\hat{n}}$ is the self-adjoint angular momentum operator in physics. It is easily checked that $L_+ = L_x + iL_y$, $L_- = L_x - iL_y$, which are the usual expressions for the raising and lowering operators you will find in any quantum mechanics text. As far as I know it was Dirac who first introduced ladder operators, in his famous textbook "Principles of Quantum mechanics" in 1930.

Recall from (3.2) that since $D\delta^-$ is a representation of $so(3)$ we have (as operators on any $\mathcal{H}_k$)

$$[\eta^{\hat{m}}, \eta^{\hat{n}}] = -\eta^{\hat{m}\times\hat{n}} \qquad \begin{bmatrix} i\times j = k,\ j\times k = i,\ k\times i = j \\ a\times b = -b\times a \end{bmatrix} \qquad (14.1)$$

and in particular $[\eta^x, \eta^y] = -\eta^z$, $[\eta^y, \eta^z] = -\eta^x$, $[\eta^z, \eta^x] = -\eta^y$. Hence $[L_x, L_y] = -[\eta^x, \eta^y] = \eta^z = iL_z$, and similarly $[L_y, L_z] = iL_x$, $[L_z, L_x] = iL_y$. Writing $L_1 = L_x$, $L_2 = L_y$, $L_3 = L_z$ it is typical in physics to write

$$[L_i, L_j] = i\,\epsilon_{ijk} L_k \qquad (14.2)$$

where ϵ_{ijk} is the Levi-Cevita symbol which is $+1$ if (i,j,k) is an even permutation of $(1,2,3)$ and -1 if it is an odd permutation.

We compute that

$$
\begin{aligned}
[L_z, L_\pm] &= [L_z, L_x \pm i L_y] \\
&= [L_z, L_x] \pm i [L_z, L_y] \\
&= i L_y \mp i^2 L_x \\
&= i L_y \pm L_x \\
&= \pm L_\pm
\end{aligned}
\tag{15.1}
$$

$$
\begin{aligned}
[L_+, L_-] &= [L_x + i L_y, L_x - i L_y] \\
&= -i[L_x, L_y] + i[L_y, L_x] \\
&= L_z + L_z = 2L_z
\end{aligned}
\tag{15.2}
$$

<u>Lemma L7-8</u> If $v \in \mathcal{H}_k$ is an eigenvector of L_z with eigenvalue $\lambda \in \mathbb{R}$ then if $L_\pm v$ is nonzero it is an eigenvector of L_z with eigenvalue $\lambda \pm 1$.

<u>Proof</u> We have by (15.1)

$$
\begin{aligned}
L_z L_\pm v &= ([L_z, L_\pm] + L_\pm L_z) v \\
&= \pm L_\pm v + \lambda L_\pm v \\
&= (\lambda \pm 1) L_\pm v. \quad \square
\end{aligned}
$$

This reproduces in part the diagram (12.2) for $\mathcal{H}_2$. But to <u>precisely</u> capture what is going on in that diagram we would need to show the spectrum of L_+ is $\{-2,-1,0,1,2\}$. It would follow that each eigenspace is one-dimensional, since we know a priori that $\dim_{\mathbb{C}} \mathcal{H}_2 = 5$. Of course we would prefer to extend the argument and prove that the spectrum of L_+ on $\mathcal{H}_k$ is $\{-k, -k+1, \dots, 0, \dots, k-1, k\}$.

Exercise L7-3 Prove that L_+ is adjoint to L_- as operators on the Hilbert space $\mathcal{H}_k$ equipped with the pairing $\langle -,- \rangle_k$.

Exercise L7-4 Prove that the function $\mathcal{P}_k \longrightarrow \mathcal{P}_k$ sending $P = \sum_{|\alpha|=k} c_\alpha x^\alpha$ to its conjugate $\overline{P} = \sum_{|\alpha|=k} \overline{c_\alpha} x^\alpha$ is an $\mathbb{R}$-linear transformation and that

$$
\begin{aligned}
&(i) \quad \eta^{\hat{n}}(\overline{P}) = \overline{\eta^{\hat{n}}(P)} \qquad \text{for } \hat{n} \in S^2 \\
&(ii) \quad L_{\hat{n}}(\overline{P}) = -\overline{L_{\hat{n}}(P)} \\
&(iii) \quad L_{\pm}(\overline{P}) = -\overline{L_{\mp}(P)}
\end{aligned}
$$

Let $\mathfrak{s}(L_z)$ denote the spectrum of L_z acting on $\mathcal{H}_k$, that is, its set of eigenvalues. We make the following observations about this set:

- $\mathfrak{s}(L_z) \subseteq \mathbb{R}$ since L_z is self-adjoint (Lemma L5-2 (a))
- $\mathfrak{s}(L_z)$ contains at most $2k+1$ elements (Lemma L5-2 (b)) (16.1)
- if $\lambda \in \mathfrak{s}(L_z)$ then $-\lambda \in \mathfrak{s}(L_z)$ (Ex L7-4 (ii))

Since $\mathfrak{s}(L_z)$ is a finite set of real numbers it contains a largest element λ^{max}. Let v be an eigenvector for this maximal eigenvalue. If $L_+ v$ were nonzero it would by Lemma L7-8 be an eigenvector of eigenvalue $\lambda^{max}+1$, which is a contradiction. Hence $L_+ v = 0$. It is natural to wonder about $L_- v$ (especially given our strategy in the proof of Lemma L7-7). Note that by (15.2) we have

$$[L_+, L_-]v = 2L_z v = 2\lambda^{max} v \tag{16.2}$$

but also

$$[L_+, L_-]v = L_+ L_- v - L_- L_+ v = L_+ L_- v \tag{16.3}$$

Hence $2\lambda^{max} v = L_+ L_- v$. Now it is possible that $\lambda^{max} = 0$ (e.g. for $k=0$ this holds) but if $\lambda^{max}=0$ then $L_z \equiv 0$ on $\mathcal{H}_k$. We'll worry about this in a moment. For now observe that if $\lambda^{max} > 0$ then $v = \frac{1}{2\lambda^{max}} L_+ L_- v$ and so in particular $L_- v$ is nonzero and thus by Lemma L7-8 it is an eigenvector of eigenvalue $\lambda^{max} - 1$. We can keep applying L_- to construct eigenvectors $(L_-)^n v = (\lambda^{max} - n) v$, at least while these vectors remain nonzero. If we set

$$V_\lambda = \{ v \in \mathcal{H}_k \mid L_z v = \lambda v \} \tag{17.1}$$

This shows that at least <u>a part of $\mathcal{H}_k$</u> looks like (12.2)

$$\begin{tikzcd} \cdots \arrow[r, shift left] & V_{\lambda^{max}-2} \arrow[l, shift left] \arrow[r, shift left] & V_{\lambda^{max}-1} \arrow[l, shift left] \arrow[r, shift left, "L_+"] & V_{\lambda^{max}} \arrow[l, shift left, "L_-"] \end{tikzcd} \tag{17.2}$$

But how do we know that $\{ \lambda^{max} - n \mid n \in \mathbb{N} \}$ exhausts the spectrum? By the spectral theorem we know $\mathcal{H}_k \cong \bigoplus_\lambda V_\lambda$ but this on its own is not enough. We need one more ingredient to complete our analysis of the structure of $\mathcal{H}_k$ as an $\mathfrak{so}(3)$ representation.

<u>Return of the Laplacian</u>

Let us study this operator $L_+ L_-$ more closely

$$\begin{aligned} L_+ L_- &= (L_x + iL_y)(L_x - iL_y) \\ &= L_x^2 - iL_xL_y + iL_yL_x + L_y^2 \\ &= L_x^2 + L_y^2 + i[L_y, L_x] \\ &= L_x^2 + L_y^2 + i(-iL_z) \\ &= L_x^2 + L_y^2 + L_z \end{aligned} \tag{17.3}$$

That sure seems like it might be related somehow to the Laplacian ...

<u>Exercise L7-5</u> Let x_i, $\frac{\partial}{\partial x_i}$ respectively denote the $\mathbb{C}$-linear operators on P_k of multiplication by, and the derivative with respect to, the variable x_i. Prove that

1 if $i=j$ and zero otherwise

$$\left[\frac{\partial}{\partial x_j}, x_i\right] = \delta_{ij} \qquad 1 \le i,j \le 3 \qquad (18.1)$$

<u>Lemma L7-9</u> The $\mathbb{C}$-linear operator $L_x^2 + L_y^2 + L_z^2$ on $\mathcal{H}_k$ is equal to $-\Delta_{S^2}$, the Laplacian on S^2, as it acts by multiplication with $k(k+1)$.

<u>Proof</u> We compute that for $(i,j,k) \in \{(1,2,3),(2,3,1),(3,1,2)\}$ as operators on P_k

$$(\eta^{x_i})^2 = \left[x_j \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_j}\right]\left[x_j \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_j}\right]$$

$$= x_j^2 \frac{\partial^2}{\partial x_k^2} - x_j \frac{\partial}{\partial x_k} x_k \frac{\partial}{\partial x_j} - x_k \frac{\partial}{\partial x_j} x_j \frac{\partial}{\partial x_k} + x_k^2 \frac{\partial^2}{\partial x_j^2} \qquad (18.2)$$

$$= x_j^2 \frac{\partial^2}{\partial x_k^2} + x_k^2 \frac{\partial^2}{\partial x_j^2} - x_j \frac{\partial}{\partial x_k} x_k \frac{\partial}{\partial x_j} - x_k \frac{\partial}{\partial x_j} x_j \frac{\partial}{\partial x_k}$$

$$= x_j^2 \frac{\partial^2}{\partial x_k^2} + x_k^2 \frac{\partial^2}{\partial x_j^2} - x_j\left(1 + x_k \frac{\partial}{\partial x_k}\right)\frac{\partial}{\partial x_j} - x_k\left(1 + x_j \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_k}$$

$$= x_j^2 \frac{\partial^2}{\partial x_k^2} + x_k^2 \frac{\partial^2}{\partial x_j^2} - x_j \frac{\partial}{\partial x_j} - x_k \frac{\partial}{\partial x_k} - 2 x_j x_k \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}$$

Hence, writing $\Delta = \sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2}$, $\|x\|^2 = \sum_{i=1}^3 x_i^2$

$$\sum_{i=1}^3 (\eta^{x_i})^2 = x_1^2\left(\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right) + x_2^2\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_3^2}\right) + x_3^2\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)$$

$$- 2\sum_{i=1}^3 x_i \frac{\partial}{\partial x_i} - \sum_{i \ne j} x_i x_j \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} \qquad (18.3)$$

$$= \|x\|^2 \Delta - \sum_{i=1}^3 x_i^2 \frac{\partial^2}{\partial x_i^2} - 2\sum_{i=1}^3 x_i \frac{\partial}{\partial x_i} - \sum_{i \ne j} x_i x_j \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}$$

Let f be a smooth function on $\mathbb{R}^3$. Then using the usual spherical coordinates of L3

$$r\frac{\partial f}{\partial r} = \sum_{i=1}^{3} r\frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial r} = \sum_{i=1}^{3} x_i \frac{\partial f}{\partial x_i} \qquad (19.1)$$

If we agree to write $r\frac{\partial}{\partial r}$ for $\sum_{i=1}^{3} x_i \frac{\partial}{\partial x_i}$ on P_k, then for $f \in P_k$

$$\begin{aligned} r\frac{\partial}{\partial r} r\frac{\partial}{\partial r}(f) &= \sum_{i=1}^{3} r\frac{\partial}{\partial r}\left(x_i \frac{\partial f}{\partial x_i}\right) \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3} x_j \frac{\partial}{\partial x_j}\left(x_i \frac{\partial f}{\partial x_i}\right) \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3} x_j\left[\delta_{ij}\frac{\partial f}{\partial x_i} + x_i \frac{\partial^2 f}{\partial x_i \partial x_j}\right] \qquad (19.2) \\ &= \sum_{i=1}^{3} x_i\frac{\partial f}{\partial x_i} + \sum_{i,j=1}^{3} x_i x_j \frac{\partial^2 f}{\partial x_i \partial x_j} \\ &= \sum_{i=1}^{3} x_i^2\frac{\partial^2 f}{\partial x_i^2} + \sum_{i=1}^{3} x_i \frac{\partial f}{\partial x_i} + \sum_{i\neq j} x_i x_j \frac{\partial^2 f}{\partial x_i \partial x_j} \end{aligned}$$

and so as operators on P_k

$$\begin{aligned} \sum_{i=1}^{3}(\eta^{x_i})^2 &= \|x\|^2\Delta - r\frac{\partial}{\partial r} r\frac{\partial}{\partial r} - r\frac{\partial}{\partial r} \qquad (19.3) \\ &= \|x\|^2\Delta - r\frac{\partial}{\partial r}\left(1 + r\frac{\partial}{\partial r}\right) \end{aligned}$$

Hence, since $L_i = -i\eta^{x_i}$, we have

$$\sum_{i=1}^{3} L_i^2 = -\sum_{i=1}^{3}(\eta^{x_i})^2 = -\|x\|^2\Delta + r\frac{\partial}{\partial r}\left(1 + r\frac{\partial}{\partial r}\right). \qquad (19.4)$$

Restricting to $\mathcal{H}_k$ we have $\sum_{i=1}^{3} L_i^2 = r\frac{\partial}{\partial r}\left(1 + r\frac{\partial}{\partial r}\right)$ as claimed, and it is easy to see that $r\frac{\partial}{\partial r}$ acts by multiplication with k on $\mathcal{H}_k$, since

$$\begin{aligned} r\frac{\partial}{\partial r}(x^\alpha) &= \sum_{i=1}^{3} x_i \frac{\partial}{\partial x_i}\left(x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}\right) \\ &= x_1\left(\alpha_1 x_1^{\alpha_1-1}x_2^{\alpha_2}x_3^{\alpha_3} + \alpha_2 x_1^{\alpha_1}x_2^{\alpha_2-1}x_3^{\alpha_3} \right. \\ &\qquad \left. + \alpha_3 x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3-1}\right) \\ &= (\alpha_1+\alpha_2+\alpha_3)x^\alpha = |\alpha|\, x^\alpha . \quad \square \end{aligned}$$

<u>Remark L7-2</u> Recall from Ex L3-13 that for a smooth function on $\mathbb{R}^3$

$$r^2 \Delta_{\mathbb{R}^3} f = \frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \Delta_{S^2} f. \qquad (20.1)$$

But $\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) = 2r\frac{\partial f}{\partial r} + r^2\frac{\partial^2 f}{\partial r^2} = \left[2r\frac{\partial}{\partial r} + r^2\frac{\partial}{\partial r}\frac{\partial}{\partial r}\right] f$

while

$$\begin{aligned} r\frac{\partial}{\partial r} + r\frac{\partial}{\partial r}r\frac{\partial}{\partial r} &= r\frac{\partial}{\partial r} + r\left(1 + r\frac{\partial}{\partial r}\right)\frac{\partial}{\partial r} \\ &= r\frac{\partial}{\partial r} + r\frac{\partial}{\partial r} + r^2\frac{\partial^2}{\partial r^2} \\ &= 2r\frac{\partial}{\partial r} + r^2\frac{\partial^2}{\partial r^2} \end{aligned}$$

Hence $\Delta_{S^2} f = r^2\Delta_{\mathbb{R}^3} f - r\frac{\partial}{\partial r}\left(1 + r\frac{\partial}{\partial r}\right) f = -\sum_{i=1}^{3} L_i^2(f).$

Returning now to the representation of $SO(3)$ on $\mathcal{H}_k$ and specifically (17.3) we have as operators on $\mathcal{H}_k$

$$\begin{aligned} L_+L_- &= L_x^2 + L_y^2 + L_z \\ &= L_x^2 + L_y^2 + L_z^2 - L_z^2 + L_z \qquad (20.5) \\ &= k(k+1) + L_z(1 - L_z) \end{aligned}$$

But of course this acts as a <u>constant</u> on each eigenspace of L_z.

Theorem L7-10 For each $k \geq 0$ the $so(3)$ representation $\mathcal{H}_k$ is irreducible. The spectrum of L_z acting on $\mathcal{H}_k$ is $\{-k, -k+1, \ldots, 0, \ldots, k-1, k\}$, and if we denote by $|m\rangle$ some chosen eigenvector with eigenvalue m for $-k \leq m \leq k$ then the structure of $\mathcal{H}_k$ is given by:

(i) $\{|m\rangle\}_{m=-k}^{k}$ is an orthogonal basis for $\mathcal{H}_k$

(ii) $L_+|k\rangle = 0, \quad L_-|-k\rangle = 0$

(iii) for $-k \leq m < k \quad L_+|m\rangle = \mu|m+1\rangle$ for some $\mu \neq 0$.

(iv) for $-k < m \leq k \quad L_-|m\rangle = \mu|m-1\rangle$ for some $\mu \neq 0$.

Proof Since $\mathcal{H}_k$ is a nonzero finite-dimensional complex vector space L_z has some eigenvalue, and they are all real as discussed above. Let λ^{max} be the largest eigenvalue (possibly $\lambda^{max} = 0$), and v an eigenvector. Now by (16.2), (16.3)

$$2\lambda^{max} v = L_+ L_- v \qquad (21.1)$$

but by (20.5)

$$\begin{aligned} L_+ L_- v &= \left[k(k+1) + L_z(1-L_z) \right] v \\ &= \left[k(k+1) + \lambda^{max}(1-\lambda^{max}) \right] v \end{aligned} \qquad (21.2)$$

Hence

$$\left(2\lambda^{max} - k(k+1) - \lambda^{max}(1-\lambda^{max})\right) v = 0. \qquad (21.3)$$

Now if $\lambda^{max} = 0$ then L_z acts as zero on $\mathcal{H}_k$ so by (15.1) also $L_+ = L_- = 0$, but by (20.5) this means $k(k+1)v = 0$ in $\mathcal{H}_k$ for all $v \in \mathcal{H}_k$, and since $\dim_{\mathbb{C}} \mathcal{H}_k = 2k+1$ so $\mathcal{H}_k \neq 0$ this implies $k=0$. That is, $\lambda^{max} = 0$ iff. $k = 0$, in which case all the claims are vacuously true. So we now assume $\lambda^{max} > 0$.

If $\lambda^{max} > 0$ then by (21.3)

$$\begin{aligned} 2\lambda^{max} - k(k+1) - \lambda^{max}(1-\lambda^{max}) &= 0 \\ \Longleftrightarrow \quad k(k+1) &= \lambda^{max}(\lambda^{max}+1) \end{aligned} \tag{22.1}$$

The function $t \mapsto t(t+1)$ has positive derivative for $t > 0$ and is therefore monotonic, so we deduce that $\lambda^{max} = k$. Let $|k\rangle$ denote some eigenvector for L_z with eigenvalue k.

Of course the strategy now is to repeat the proof of Lemma L7-7, namely we act on $|k\rangle$ with lowering operators and argue that this is everything. Suppose that for some $-k < m \le k$ we have shown that $m \in \mathfrak{z}(L_z)$, that is, m is an eigenvalue of L_z. Let v be an eigenvector for this eigenvalue. If $L_-v \neq 0$ then it is an eigenvector with eigenvalue $m-1$ (Lemma L7-8). But we need somehow to argue $L_-v \neq 0$. But by (20.5)

$$\begin{aligned} L_+L_-v &= \big(k(k+1) + L_z(1-L_z) \big) v \\ &= \big(k(k+1) + m(1-m) \big) v. \end{aligned}$$

If $L_-v = 0$ then since $v \neq 0$, $k(k+1) = m(m-1)$. The function $f(t) = t(t-1) - k(k+1)$ vanishes for $t = -k$ and $t = k+1$ and is nonzero otherwise, so since $-k < m \le k$ we have $f(m) \neq 0$ and so $L_-v \neq 0$ is an eigenvector with eigenvalue $m-1$ as desired. By induction we have shown that $\{-k, -k+1, \dots, 0, \dots, k-1, k\} \subseteq \mathfrak{z}(L_z)$ and we may let $\{|m\rangle\}_{m=-k}^{k}$ denote eigenvectors $L_z|m\rangle = m|m\rangle$. Since eigenvectors for distinct eigenvalues are linearly independent (Lemma L5-2), if $V = \operatorname{span}_{\mathbb{C}}\{|m\rangle\}_{m=-k}^{k}$ then $\dim_{\mathbb{C}} V = 2k+1$. Since $\mathcal{H}_k$ has the same dimension (Ex L4-2), we must have $\mathcal{H}_k = V$ and $\mathfrak{z}(L_z) = \{-k, \dots, k\}$, proving (i).

By Lemma L7-8 if $L_+|k\rangle$ were nonzero we would have $k+1 \in \mathfrak{z}(L_z)$. This is false so $L_+|k\rangle = 0$ and similarly $L_-|-k\rangle = 0$, proving (ii).

For $-k \le m < k$ we know by Lemma L7-8 that $L_+|m\rangle$ is an eigenvector with eigenvalue $m+1$, and thus is orthogonal to $|n\rangle$ for $n \neq m+1$. Since the $\{|n\rangle\}_{n=-k}^{k}$ form an orthogonal basis it follows that $L_+|m\rangle \propto |m+1\rangle$, proving (iii). The proof of (iv) is similar.

It only remains to prove that $\mathcal{H}_k$ is irreducible. The proof is the same as Lemma L7-7. Suppose $V \subseteq \mathcal{H}_k$ is nonzero and closed under $L_z, L_\pm$. If $0 \neq v \in V$ is $v = \sum_{m=-k}^{k} a_m |m\rangle$ and $M = \inf\{ m \mid a_m \neq 0\}$ then $(L_+)^{k-m} v = \mu |k\rangle$ for some $\mu \neq 0$ and hence $|k\rangle \in V$. Applying powers of L_- we deduce $|m\rangle \in V$ for all $-k \le m \le k$ hence $V = \mathcal{H}_k$. $\square$

<u>Remark L7-3</u> Recall $[L_z, L_\pm] = \pm L_\pm$, $[L_+, L_-] = 2L_z$. Suppose for $-k \le m < k$ that $L_+|m\rangle = \mu^+_{m+1}|m+1\rangle$ and for $-k < m \le k$ $L_-|m\rangle = \mu^-_{m-1}|m-1\rangle$. Then for $-k < m < k$

$$\begin{aligned} L_+L_-|m\rangle &= L_+\left(\mu^-_{m-1}|m-1\rangle\right) \\ &= \mu^-_{m-1}\mu^+_m|m\rangle \\ L_-L_+|m\rangle &= L_-\left(\mu^+_{m+1}|m+1\rangle\right) \\ &= \mu^-_m \mu^+_{m+1}|m\rangle \end{aligned} \tag{23.1}$$

Hence from this and $L_-L_+|-k\rangle = 2k|-k\rangle$, $L_+L_-|k\rangle = 2k|k\rangle$,

$$\begin{aligned} \mu^-_{m-1}\mu^+_m - \mu^-_m\mu^+_{m+1} &= 2m \\ \mu^-_{-k}\mu^+_{-k+1} &= 2k \\ \mu^-_{k-1}\mu^+_k &= 2k \end{aligned} \tag{23.2}$$

We might as well assume the $|m\rangle$ have been chosen to be <u>orthonormal</u> (up to now we have not made any use of this).

$$\begin{tikzcd} {|-k\rangle} \arrow[r, bend left, "\mu^+_{-k+1}"] & {|-k+1\rangle} \arrow[l, bend left, "\mu^-_{-k}"] & \cdots & {|m-1\rangle} \arrow[r, bend left, "\mu^+_m"] & {|m\rangle} \arrow[l, bend left, "\mu^-_{m-1}"] \arrow[r, bend left, "\mu^+_{m+1}"] & {|m+1\rangle} \arrow[l, bend left, "\mu^-_m"] & \cdots & {|k-1\rangle} \arrow[r, bend left, "\mu^+_k"] & {|k\rangle} \arrow[l, bend left, "\mu^-_{k-1}"] \end{tikzcd} \tag{24.1}$$

$$\circlearrowleft = 2k \qquad\qquad \circlearrowright = \circlearrowleft + 2m \qquad\qquad \circlearrowright = 2k$$

Note that if we scale the action of L_+ by $\tau \neq 0$ and L_- by $\frac{1}{\tau}$ then the commutation relations $[L_z, L_\pm] = \pm L_\pm$, $[L_+, L_-] = 2L_z$ are unchanged, so we have no chance of determining the $\mu^\pm_m$ purely from the fact that $\mathcal{H}_k$ is a representation of $so(3)$. However, we know beyond this that L_+ is adjoint to L_- (Ex L7-3), using which we obtain for $m<k$

$$\begin{aligned} \overline{\mu^+_{m+1}}\,\mu^+_{m+1} &= \langle L_+|m\rangle, L_+|m\rangle\rangle \\ &= \langle L_-L_+|m\rangle, |m\rangle\rangle \\ &= \langle \mu^+_{m+1}\mu^-_m|m\rangle, |m\rangle\rangle \\ &= \overline{\mu^+_{m+1}}\;\overline{\mu^-_m} \end{aligned} \tag{24.2}$$

Hence

$$\mu^-_m = \overline{\mu^+_{m+1}} \tag{24.3}$$

From (24.3) for $m=k-1$ and (23.2) we obtain

$$2k = \mu^-_{k-1}\mu^+_k = \overline{\mu^+_k}\,\mu^+_k = |\mu^+_k|^2$$

Hence $|\mu^+_k| = \sqrt{2k}$. Suppose $\mu^+_k = \sqrt{2k}\,e^{i\theta_k}$. Then by (23.2), $\mu^-_{k-1} = \sqrt{2k}\,e^{-i\theta_k}$. Then by (23.2)

$$\mu^-_{k-2}\mu^+_{k-1} = \mu^-_{k-1}\mu^+_k + 2(k-1) = 2(k+k-1) = 4k-2$$

Hence by (24.3), $|\mu^+_{k-1}|^2 = 4k-2$ and so $|\mu^+_{k-1}| = \sqrt{4k-2}$. In general, we can prove by induction that for $m \leq k$

$$
\begin{aligned}
|\mu_m^+|^2 &= 2\sum_{j=m}^{k} j \\
&= 2\left[\sum_{j=1}^{k-m+1}(j+m-1)\right] \\
&= 2\left[(m-1)(k-m+1) + \sum_{j=1}^{k-m+1} j\right] \qquad (25.1)\\
&= 2(m-1)(k-m+1) + (k-m+2)(k-m+1) \\
&= (2m-2+k-m+2)(k-m+1) = (k+m)(k-m+1)
\end{aligned}
$$

If we write $\mu_m^+ = \sqrt{(k+m)(k-m+1)}\, e^{i\theta_m}$ then $\mu_{m-1}^- = \overline{\mu_m^+}$. This determines all the coefficients μ_m^+, μ_m^- in (24.1) <u>up to these phase factors θ_m</u>. But these phases are simply the remaining degrees of freedom we have in choosing our basis $\{|m\rangle\}_{m=-k}^{k}$ once we insist each basis vector has norm 1, in the sense that if we now change to the basis $|m\rangle' = e^{i\rho_m}|m\rangle$ then for $m<k$

$$
\begin{aligned}
L_+|m\rangle' &= e^{i\rho_m}L_+|m\rangle \\
&= e^{i\rho_m}\mu_{m+1}^+|m+1\rangle \qquad (25.2)\\
&= e^{i\rho_m}\sqrt{(k+m)(k-m+1)}\, e^{i\theta_{m+1}}|m+1\rangle \\
&= e^{i(\theta_{m+1}-\rho_{m+1}+\rho_m)}\sqrt{(k+m)(k-m+1)}\, |m+1\rangle'
\end{aligned}
$$

We can solve the equations $\rho_m = \rho_{m+1} - \theta_{m+1}$ recursively for decreasing m, starting with $\rho_k = 0$, so $\rho_{k-1} = -\theta_k$, $\rho_{k-2} = \rho_{k-1} - \theta_{k-1} = -\theta_k - \theta_{k-1}$, and so on, such that in our new orthonormal basis all the phase factors have been eliminated. Thus, in conclusion, we may choose our orthonormal basis of eigenvectors $\{|m\rangle\}_{m=-k}^{k}$ such that

$$
\begin{aligned}
L_+|m\rangle &= \sqrt{(k-m)(k+m+1)}\;|m+1\rangle \qquad -k\le m<k \\
L_-|m\rangle &= \sqrt{(k+m)(k-m+1)}\;|m-1\rangle \qquad -k< m\le k
\end{aligned}
\qquad (25.3)
$$

Summary

Theorem L7-10 combined with Remark L7-3 constitute a complete analysis of $\mathcal{H}_k$ as an $so(3)$ representation, and thus also $\mathcal{H}_k(S^2)$ as an $SO(3)$ representation. Since

$$L^2(S^2, \mathbb{C}) = \widehat{\bigoplus_{k \geq 0}} \mathcal{H}_k(S^2) \qquad (26.1)$$

in the sense explained in Theorem L3-12 (the right hand side being a completion of Hilbert spaces, or what amounts to the same thing, every $f \in L^2(S^2, \mathbb{C})$ can be uniquely written as a convergent series $f = \sum_{k \geq 0} f_k$, $f_k \in \mathcal{H}_k(S^2)$) we have now completed the analysis of $L^2(S^2, \mathbb{C})$ as an $SO(3)$ representation begun in Lecture 3. While this is "just" a single example, historically the entire representation theory of Lie algebras (and much else) grew out of this example. In the following we recapitulate what we have found, highlighting those aspects which generalise to other Lie groups:

- The structure of $L^2(S^2, \mathbb{C})$ was organised by four self-adjoint operators: the spherical Laplacian Δ_{S^2} and the three components L_x, L_y, L_z of the angular momentum, or equivalently (but less symmetrically) L_z together with the ladder operators $L_\pm = L_x \pm i L_y$. In fact by Lemma L7-9 we need only the angular momentum operators since

$$-\Delta_{S^2} = L_x^2 + L_y^2 + L_z^2 \qquad (26.2)$$

  and these are given by $L_x = -i\gamma^x$, $L_y = -i\gamma^y$, $L_z = -i\gamma^z$ where $\delta^{\hat{n}} \in so(3)$ acts on $L^2(S^2, \mathbb{C})$ by the operator $-\gamma^{\hat{n}}$.

- The spherical harmonics $\mathcal{H}_k(S^2)$ of degree k are the $-k(k+1)$ eigenspace of Δ_{S^2}, so the decomposition (26.1) is the eigenspace decomposition of $L^2(S^2, \mathbb{C})$ determined by the Laplacian.

- For each $k \geq 0$ we can further decompose $\mathcal{H}_k(S^2)$ into eigenspaces of L_z. We can by Theorem L7-10 and Remark L7-3 choose an orthonormal basis $\{|k,m\rangle\}_{m=-k}^{k}$ of $\mathcal{H}_k(S^2)$ such that $L_+|k,k\rangle = L_-|k,-k\rangle = 0$ and

$$
\begin{aligned}
\Delta_{S^2}|k,m\rangle &= -k(k+1)|k,m\rangle & -k \leq m \leq k \\
L_z|k,m\rangle &= m|k,m\rangle & -k \leq m \leq k \\
L_+|k,m\rangle &= \sqrt{(k-m)(k+m+1)}\,|k,m+1\rangle & -k \leq m < k \\
L_-|k,m\rangle &= \sqrt{(k+m)(k-m+1)}\,|k,m-1\rangle & -k < m \leq k
\end{aligned}
\tag{27.1}
$$

- The representation $\mathcal{H}_k(S^2)$ of $so(3)$ is irreducible (Theorem L7-10).

- The vectors $\{ |k,m\rangle \mid k \geq 0, -k \leq m \leq k\}$ are an orthonormal basis of $L^2(S^2;\mathbb{C})$ (see the proof of Theorem L3-12) so every $\Psi \in L^2(S^2,\mathbb{C})$ has a <u>unique</u> representation as a convergent series

$$
\Psi = \sum_{k=0}^{\infty} \sum_{m=-k}^{k} a_{k,m}\,|k,m\rangle \tag{27.2}
$$

where $a_{k,m} = \langle |k,m\rangle, \Psi\rangle$ is computed by an integral over S^2 as defined in L3 p.②
(it is admittedly awkward to half-adopt the physics notation : physicists would write $|\Psi\rangle$ and $\langle k,m|\Psi\rangle$ rather than the abomination $\langle |k,m\rangle, \Psi\rangle$. But we'll live). These are the <u>Fourier coefficients</u> of Ψ (cf. [MHS, Example L21-3]).

- Since $SO(3)$ satisfies the hypothesis of Ex L6-5 the functor

$$
T : \operatorname{rep}(SO(3)) \longrightarrow \operatorname{rep}(so(3)) \tag{27.3}
$$

of Lemma L6-15 is fully faithful: that is, for every pair of $SO(3)$-representations V, W, a $\mathbb{C}$-linear map $\varphi: V \longrightarrow W$ is a morphism of $SO(3)$-representations iff it is a morphism of $so(3)$-representations.

Remark L7-4 We know that $P \longmapsto P|_{S^2}$ gives an isomorphism of $so(3)$ representations $\mathcal{H}_k \longrightarrow \mathcal{H}_k(S^2)$ but we have not attempted to compare the inner product $\langle -,- \rangle_k$ on $\mathcal{H}_k$ to the integral pairing on $\mathcal{H}_k(S^2)$ of Lecture 3, because we do not need to. We know $L_z, L_\pm$ are self-adjoint operators on $\mathcal{H}_k(S^2)$ with respect to the integral pairing (Lemma L5-5) so everything in Remark L7-3 applies just as well to $\mathcal{H}_k(S^2)$ with this pairing as it does to $\mathcal{H}_k$ with $\langle -,- \rangle_k$.

Exercise L7-6 (i) Prove that the Laplacian $\Delta : \mathcal{P}_k \longrightarrow \mathcal{P}_{k-2}$ is a morphism of $so(3)$ representations (here $\Delta = \sum_{i=1}^{3} \partial^2/\partial x_i^2$).

(ii) Prove that multiplication by $\|x\|^2 = \sum_{i=1}^{3} x_i^2$ is a morphism of $so(3)$ representations $\|x\|^2 : \mathcal{P}_{k-2} \longrightarrow \mathcal{P}_k$.

(iii) We showed in the proof of Theorem L3-4 that the sequence of $so(3)$-representations (i is the inclusion)

$$0 \longrightarrow \mathcal{H}_k \underset{f}{\overset{i}{\rightleftarrows}} \mathcal{P}_k \underset{g}{\overset{\Delta}{\rightleftarrows}} \mathcal{P}_{k-2} \longrightarrow 0$$

is <u>exact</u>: that is, i is injective, Δ is surjective and $\mathcal{H}_k = \operatorname{Ker}(\Delta)$. Prove that this sequence is <u>split exact</u>, meaning that there are morphisms f, g of $so(3)$ representations such that $f \circ i = 1_{\mathcal{H}_k}$, $\Delta \circ g = 1_{\mathcal{P}_{k-2}}$ and $i \circ f + g \circ \Delta = 1_{\mathcal{P}_k}$.

Remark L7-5 The operator $L_x^2 + L_y^2 + L_z^2$ is the <u>Casimir invariant</u> of $so(3)$. For a general Lie algebra (under some conditions) the eigenspace decomposition with respect to the Casimir invariant plays a similar role to the eigenspace decomposition with respect to the Laplacian of $L^2(S^2, \mathbb{C})$.

Exercise L7-7 Using [MHS, Lemma L21-11] give a description of $L^2(S^2, \mathbb{R})$ as a representation of $SO(3)$ paralleling the above summary.

What is Real?

The spaces S^1 and S^2 do not appear at first to be complicated objects. However, it is by no means trivial to truly understand the spaces of continuous real-valued functions $Cts(S^1, \mathbb{R})$, $Cts(S^2, \mathbb{R})$. For reasons that we are familiar with ($e^{i\theta}$ being easier to deal with than $\sin(\theta)$, $\cos(\theta)$ separately) even if we ultimately care about real-valued functions it is convenient to work with complex-valued functions. It turns out that the fundamental degrees of freedom in such a function are nonlocal and wave-like, in the sense that the most natural "directions" in which to perturb a function Ψ correspond to the eigenbasis $\{e^{in\theta}\}_{n\in\mathbb{Z}}$ of the Laplacian $\Delta_{S^1} = \partial^2/\partial\theta^2$ in the case of S^1 and the eigenbasis $\{|k,m\rangle\}_{k\geq 0, -k\leq m\leq k}$ of Δ_{S^2} in the case of S^2. That is, the natural variations in a function are variations of its Fourier components (see [MHS, L21 p. 18] for some discussion of this for S^1).

Thus $L^2(S^1, \mathbb{C})$, $L^2(S^2, \mathbb{C})$ are the natural structures in which to study functions on S^1, S^2 (as introduced in [MHS, L18] these Hilbert spaces possess exactly enough structure to talk about convergent series such as Fourier decompositions).

In L1 and L2 we made the argument, following standard physics "lore", that real things have symmetries. In less vague terms, and accepting the basic postulates of quantum mechanics, the Hilbert space of states of any physical system must be a representation (either unitary or antiunitary, and possibly projective) of whatever Lie group governs the set of equivalent observers. In relativistic quantum mechanics this is the Poincaré group, but we have focused on the subgroup $SO(3)$, and thus observers at rest relative to one another. In this case the Hilbert space $\mathcal{H} = L^2(S^2, \mathbb{C})$ is a reasonable model of (part of) the state of many interesting systems, such as electrons in small atoms (see p. 4 of L4).

We argued on p. 4 L4 that if one observer sees the system in an eigenstate of Δ_{S^2}, say $\Psi \in \mathcal{H}_k(S^2)$, so that it has total angular momentum quantum number $k(k+1)$ (the eigenvalue of the self-adjoint operator $L_x^2 + L_y^2 + L_z^2$) then every equivalent observer also sees a state in $\mathcal{H}_k(S^2)$.

Observers do <u>not</u> necessarily agree on the quantum number m, i.e. the eigenbasis of L_z, because their L_z's may be different! Let $\hat{n} \in S^2$ and take $\hat{n}$ as the x-axis of a second observer O' (we call the first observer O) with coordinate system t_1, t_2, t_3. Then if O' chooses an eigenbasis for $\mathcal{H}_k(S^2)$ consisting of eigenvectors of $L_{t_3} = -i\left[t_1 \frac{\partial}{\partial t_2} - t_2 \frac{\partial}{\partial t_1}\right]$ then these will not in general be an eigenbasis for L_z. However one can show

$$L_x^2 + L_y^2 + L_z^2 = L_{t_1}^2 + L_{t_2}^2 + L_{t_3}^2 \qquad (30.1)$$

so that both observers <u>do agree</u> on the operator of total angular momentum, and hence its eigenvectors and eigenvalues (this being the aforementioned agreement on the decomposition of $L^2(S^2, \mathbb{C})$ into the $\mathcal{H}_k(S^2)$'s). In this sense the Casimir invariant (30.1) is <u>real</u>. The general definition of a "real quantity" for a system with Hilbert space $\mathcal{H}$ acted on via unitary transformations by a Lie group G with Lie algebra $\mathfrak{g}$ is an element of the <u>center of the universal enveloping algebra</u> $U(\mathfrak{g})$. These are polynomial expressions in the elements of $\mathfrak{g}$ (such as (30.1)) which commute with all the actions of elements of $\mathfrak{g}$ on any representation. For $\mathfrak{so}(3)$ this center is spanned by (30.1).

<u>Exercise L7-8</u> Let G be a matrix Lie group which is connected (see Ex L6-5). You may assume that this implies every element of G can be written in the form $\exp(X_1) \cdots \exp(X_n)$ for some $X_1, \dots, X_n \in \mathfrak{g}$. Let $\mathcal{H}$ be a finite-dimensional complex representation of G, where $\mathcal{H}$ is a Hilbert space and each $g \in G$ acts by a unitary transformation on $\mathcal{H}$. Let O, O' be two observers related by a group element $g_{O',O}$ so that if O observes a state $\psi \in \mathcal{H}$ then O' observes $g_{O',O} . \psi \in \mathcal{H}$.

Let $\{X_i\}_{i \in I}$ be an $\mathbb{R}$-basis for $\mathfrak{g}$ and C a polynomial in the X_i, which we may view as a $\mathbb{C}$-linear operator on $\mathcal{H}$ by interpreting it as a sum of products of the operators $X_i . (-)$ using the action of $\mathfrak{g}$ (i.e. $3X_1X_2 + X_2^2$ sends $\psi \in \mathcal{H}$ to $3X_1.(X_2.\psi) + X_2.(X_2.\psi)$). Suppose $[C, X_i] = 0$ for all $i \in I$ as operators on $\mathcal{H}$. Prove that $g_{O',O}^{-1} C g_{O',O} = C$ hence, <u>both observers agree on the eigenvectors and eigenvalues of C</u>, and these are hence "real".

Exercise L7-9 Prove that for $k \geqslant 0$ there is an isomorphism of $\mathfrak{so}(3)$-representations

$$\mathcal{P}_k \cong \bigoplus_{i=0}^{a} \mathcal{H}_{k-2i}$$

where $k = 2a+b$, $a, b \in \mathbb{N}$ and $b < 2$.