Lecture 8 — Complex Lie algebras and spin

In Lecture 7 we achieved a complete understanding of $L^2(S^2, \mathbb{C})$ as an $SO(3)$-representation, which was the goal of the subject. However there are a few loose ends:

(1) The characterisation of $\mathcal{H}_k(S^2)$ as an $\mathfrak{so}(3)$-representation was in terms of operators $L_z, L_\pm$ which are <u>complex</u> linear combinations of the actions of $\delta^x, \delta^y, \delta^z \in \mathfrak{so}(3)$, so the structure cannot be neatly described in terms of elements of $\mathfrak{so}(3)$ and their action on $\mathcal{H}_k(S^2)$ (see p. 12 of L7).

(2) The functor $T: \mathrm{rep}(SO(3)) \longrightarrow \mathrm{rep}(\mathfrak{so}(3))$ is fully faithful, but we do not know if this is an <u>equivalence</u> of categories, that is, we do not know if <u>every</u> representation V is induced by an $SO(3)$ representation on V (see p. 27 L7). By the Theorem cited on p. 35 of L6 we know T is an equivalence for any G which is connected and simply connected. While $G = SO(3)$ is connected it is <u>not</u> simply connected, as we will recall below, so the Theorem tells us nothing.

(3) Going back to Wigner's theorem, if we take $L^2(S^2, \mathbb{C})$ as (one component of) the Hilbert space of a particle (say an electron in an atom at the origin), <u>which</u> representation of $SO(3)$ on $L^2(S^2, \mathbb{C})$ relates observers in $\mathbb{R}^3$ of the particle? Is it the representation $\mathcal{E}^-$ we have spent so much time studying? Or is it a more exotic example, and perhaps even only a representation up to phases (a projective representation)?

It is a profound and beautiful fact that all three of these loose ends are part of a single thread, which leads us to discover a new fundamental symmetry of nature and its associated observable, whose quantum number is called <u>spin</u>.

- Observers agree about total angular momentum

    - Fourier decomp (ref MHS) and "degrees of freedom"

    - case of $\mathbb{R}$ (Ex)

- Poincaré group

<u>Next lecture</u>

- Complete reducibility
- irreducibles as primes
- Schur's lemma
- $SO(3)_{\mathbb{C}} = su(2)$

- construct irrep of $SO(3)$ outside.
- Δ_{S^2} commutes with everything

$$\begin{tikzcd}
\mathrm{rep}(SU(2)) \arrow[r, "\cong"] & \mathrm{rep}_{\mathbb{C}}(su(2)) \arrow[d, dash, "\cong"] \\
\mathrm{rep}(SO(3)) \arrow[u, dashed] \arrow[r, "\cancel{\text{full}}\ \text{full \& faithful}"'] & \mathrm{rep}(so(3))
\end{tikzcd}$$

$SU(2) \longrightarrow SO(3)$