Idempotents in Categories

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In this note we give a brief introduction to idempotent morphisms and idempotent completion of categories, focusing on categories which are preadditive. Idempotent morphisms and idempotent completion is also covered in [Bor94, Section 6.5] (where "idempotent completion" is called "Cauchy completion") but no special attention is paid to preadditive categories.

1 Definitions and basic results

Let $\mathcal{C}$ be a category, which for now we do not assume is preadditive.

Definition 1.1. An endomorphism $e : C \to C$ in $\mathcal{C}$ is an *idempotent* if $e^2 = e$.

Consider a pair of morphisms $s : R \leftrightarrows C : r$ such that $rs = \mathrm{id}_R$. Then $e = sr$ is an idempotent.

Definition 1.2. We call an idempotent $e : C \to C$ *split* if there exist morphisms $s : R \leftrightarrows C$ and $r : C \to R$ such that $rs = \mathrm{id}_R$. We call the category $\mathcal{C}$ *idempotent complete* if all idempotents split.

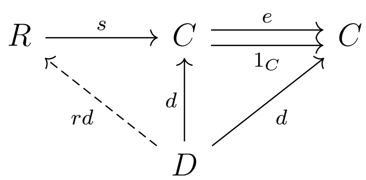
Lemma 1.3 (Proposition 6.5.4 [Bor94])**.** *Let $e : C \to C$ be an idempotent. The following are equivalent:*

(1) $e = sr$ is split, where $s : R \to C$ and $r : C \to R$.

(2) The equaliser $\mathrm{eq}(e, 1_C)$ exists and is equal to (R, s).

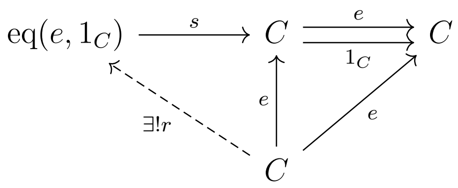
(3) The coequaliser $\mathrm{coeq}(1_C, e)$ exists and is equal to (R, r).

Proof. Suppose e is split, so we have morphisms $s : R \leftrightarrows C$ and $r : C \to R$ such that $rs = \mathrm{id}_R$. We now prove that $\mathrm{eq}(e, 1_C) = (R, s)$ by showing the universal property is satisfied. We have $es = s$, and given another morphism $d : D \to C$ where $ed = d$, we have



where all three triangles commute. Indeed, setting $n = rd$ we have $sn = srd = ed = d$. Moreover if $n' : D \to R$ is another morphism which satisfies $sn' = d$ we have $sn = sn'$ and so $rsn = rsn' = n = n'$. This shows $\mathrm{eq}(e, 1_C) = (R, s)$ and so (1) $\implies$ (2). This also shows (1) $\implies$ (3), since this is statement is equivalent to (1) $\implies$ (2) holding in $\mathcal{C}^{\mathrm{op}}$.

Supposing (2), there exists $r : C \to \mathrm{eq}(e, 1_C)$ such that



commutes. By applying the universal property to the morphism $s : \mathrm{eq}(e, 1_C) \to C$ and appealing to uniqueness we have $rs = 1$, which proves (2) $\implies$ (1). By making use of $\mathcal{C}^{\mathrm{op}}$ this also shows (3) $\implies$ (1). □

Lemma 1.4. *If $\mathcal{C}$ is a preadditive category then the following are equivalent:*

(1) $\mathcal{C}$ idempotent complete,

(2) All idempotents have a kernel,

(3) All idempotents have a cokernel.

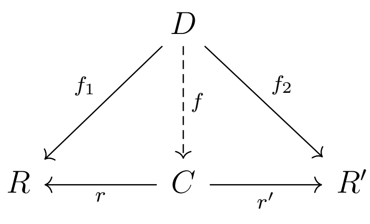
Proof. The equivalence (1) $\iff$ (2) can be proved using Lemma 1.3 by observing that if $e : C \to C$ is an idempotent then so is $1_C - e$, and that $\mathrm{eq}(1_C - e, 1_C) = \ker(e)$. The equivalence (1) $\iff$ (3) can be proved in the same way in the opposite category. □

As a corollary note that any abelian category is idempotent complete. For an additive category, the property of "being idempotent complete" can be viewed as a weakening of "being abelian".

Lemma 1.5. *Suppose $\mathcal{C}$ is preadditive. Let $e : C \to C$ be an idempotent such that the idempotents e and $1 - e$ both split: $e = sr$ and $1 - e = s'r'$ where $s : R \to C$, $r : C \to R$, $s' : R' \to C$ and $r' : C \to R'$. Then $C \cong R \oplus R'$.*

Proof. Since $rs = 1_R$ and $r's' = 1_{R'}$ we have that s and s' are monomorphisms, and r and r' are epimorphisms. Also note that $rs' = 0$ since $rs'r' = r(1 - e) = 0$ and r' is an epimorphism. Likewise $r's = 0$.

Suppose we have morphisms $f_1 : D \to R$ and $f_2 = D \to R'$. Then we have



where $f = sf_1 + s'f_2$. Clearly both triangles in this diagram commute. Suppose $g : D \to C$ also makes both triangles in the diagram above commute. Then $r'g - r'sf_1 = f_2 = r's'f_2$ and, since r' is an epimorphism this gives $g = f$, so f is unique and hence $C = R \times R'$. A similar argument shows that C is also the coproduct of R and R', which proves the lemma. □

2 Idempotent completion

Definition 2.1. The *idempotent completion* of $\mathcal{C}$ is an idempotent complete category $\mathcal{C}^\omega$ together with a full and faithful functor $\mathcal{C} \to \mathcal{C}^\omega$ such that, given a functor $F : \mathcal{C} \to \mathcal{D}$ where $\mathcal{D}$ is idempotent complete, there exists functor $F^\omega : \mathcal{C}^\omega \to \mathcal{D}$ such that

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{C}^\omega \\ & \overset{F}{\searrow} & \Big\downarrow {\scriptstyle F^\omega} \\ & & \mathcal{D} \end{array}$$

commutes, and moreover F^ω is unique up to isomorphism of functors.

Using the standard argument for objects defined via universal properties one can show that if $\mathcal{C}^\omega$ exists it is unique up to equivalence of categories. If $\mathcal{C}^\omega$ exists we can without loss of generality consider $\mathcal{C}$ to be a full subcategory of $\mathcal{C}^\omega$. In [Bor94, Proposition 6.5.9] it is proved that $\mathcal{C}^\omega$ exists when $\mathcal{C}$ is small.

Our goal now is to prove that when $\mathcal{C}$ is a subcategory of an preadditive, idempotent complete category $\mathcal{A}$, that $\mathcal{C}^\omega$ exists and is the full subcategory of $\mathcal{A}$ of direct summands of objects of $\mathcal{C}$. Let A and B be objects of the same category. We say B is a *retract* of A if there exist morphisms

$$B \xrightarrow{\ s\ } A \xrightarrow{\ r\ } B$$

such that $rs = 1_B$.

Lemma 2.2. *Let $\mathcal{C} \to \mathcal{D}$ be a fully faithful functor. Suppose $\mathcal{D}$ is idempotent complete and that every object of $\mathcal{D}$ is a retract of an object of $\mathcal{C}$. Then $\mathcal{D}$ is the idempotent completion of $\mathcal{C}$.*

Proof. Without loss of generality suppose $\mathcal{C}$ is a full subcategory of $\mathcal{D}$. Let $F : \mathcal{C} \to \mathcal{E}$ be a functor to an idempotent complete category $\mathcal{E}$. We aim to construct a functor $\widetilde{F} : \mathcal{D} \to \mathcal{E}$ which fills in the diagram in Definition 2.1.

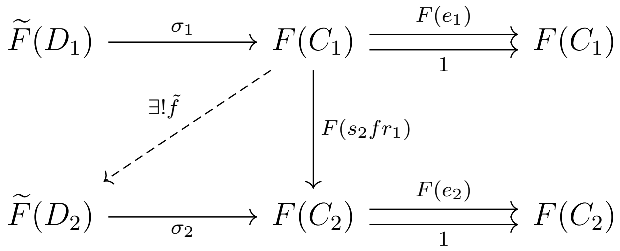
Let D be an object of $\mathcal{D}$ and C an object of $\mathcal{C}$ such that D is a retract of C, so we have

$$D \xrightarrow{\ s\ } C \xrightarrow{\ r\ } D$$

where $rs = 1_D$. In order to ensure $\widetilde{F}$ is equal to F when restricted to objects of $\mathcal{C}$, if D happens to be an object of $\mathcal{C}$ then choose $C = D$ and $s = r = 1_D$. Consider the morphism $e = sr : C \to C$, which is an idempotent in $\mathcal{C}$. Since $\mathcal{E}$ is idempotent complete $F(e)$ splits and, using Lemma 1.3, we can define $\widetilde{F}(D) = \mathrm{eq}(F(e), 1_{F(C)})$. In the case that D is an object of $\mathcal{C}$ choose the equaliser to be $(F(C), 1)$. We denote the associated equaliser morphism in $\mathcal{E}$ by $\sigma : \widetilde{F}(D) \to F(C)$. Also note that by the same argument as in Lemma 1.3 we have a morphism $\rho : F(C) \to \widetilde{F}(D)$ in $\mathcal{E}$ such that $F(e) = \sigma\rho$ and $\rho\sigma = 1$.

Let $f : D_1 \to D_2$ be a morphism in $\mathcal{D}$. For $i = 1, 2$ let C_i be an object of $\mathcal{C}$ such that D_i is a retract of C_i. Let $s_i : D_i \to C_i$ and $r_i : C_i \to D_i$ be the morphisms in $\mathcal{D}$ in this retract and $e_i = r_i s_i$. Let $\rho_i : F(C_i) \to \widetilde{F}(D_i)$ and $\sigma_i : \widetilde{F}(D_i) \to F(C_i)$ be the morphisms in $\mathcal{E}$ which split $F(e_i)$. Note that the composition $s_2 f r_1 : C_1 \to C_2$ is a morphism in $\mathcal{C}$

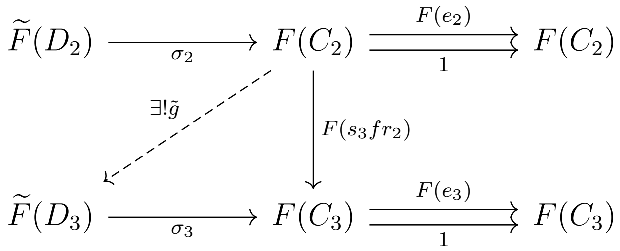
and so in $\mathcal{E}$ we have the diagram



where $\tilde{f}$ exists by the universal property of the equaliser. We define $\widetilde{F}(f) = \tilde{f}\sigma_1$.

To see that this defines a functor, first consider the case when $D_1 = D_2$ and $f = 1_{D_1}$. Then $\tilde{f}$ on the diagram above is a morphism such that $s_1\tilde{f} = e_1$, so by uniqueness $\tilde{f} = \rho_1$. Therefore $\widetilde{F}(1_{C_1}) = \rho_1\sigma_1 = 1_{\widetilde{F}(D_1)}$ as required.

Consider another morphism $g : D_2 \to D_3$ in $\mathcal{D}$. We have the following diagram in $\mathcal{E}$:



Let $\tilde{h} : F(C_1) \to \widetilde{F}(D_3)$ be the unique morphism in $\mathcal{E}$ satisfying $\sigma_3\tilde{h} = F(s_3 g f r_1)$, so by definition we have $\widetilde{F}(gf) = \tilde{h}\sigma_1$. Note that

$$\sigma_3(\tilde{g}\sigma_2\tilde{f}) = F(s_3 g r_2)F(s_2 f r_1) = F(s_3 g f r_1)$$

and so $\tilde{h} = \tilde{g}\sigma_3\tilde{f}$ by uniqueness. Therefore $\widetilde{F}(gf) = \widetilde{F}(g)\widetilde{F}(f)$ as required and hence we have shown that $\widetilde{F}$ defines a functor and by construction $\widetilde{F}|_{\mathcal{C}} = F$.

For uniqueness, suppose we have two functors $\widetilde{F}_1, \widetilde{F}_2 : \mathcal{D} \to \mathcal{E}$ such that $\widetilde{F}_1|_{\mathcal{C}} = \widetilde{F}_2|_{\mathcal{C}} = F$. Let D be an object in $\mathcal{D}$ and

$$D \xrightarrow{s} C \xrightarrow{r} D$$

be a retract with C an object of $\mathcal{C}$. Then for $i = 1, 2$ we have the retract

$$\widetilde{F}_i(D) \xrightarrow{\widetilde{F}_i(s)} F(C) \xrightarrow{\widetilde{F}_i(r)} \widetilde{F}_i(D)$$

in $\mathcal{E}$. Noting that $\widetilde{F}_i(s)\widetilde{F}_i(r) = F(e)$, we have $(\widetilde{F}_i(D), \widetilde{F}_i(s))$ is the equaliser $\mathrm{eq}(F(e), 1_{F(C)})$ by Lemma 1.3. Therefore the functors are naturally isomorphic. □

Corollary 2.3. *Let $\mathcal{C}$ be a subcategory of an preadditive, idempotent complete category $\mathcal{A}$. Then $\mathcal{C}^\omega$ is the full subcategory of $\mathcal{A}$ consisting of objects which are direct summands of objects of $\mathcal{C}$.*

Proof. Let $\mathcal{D}$ be this subcategory. Clearly $\mathcal{C}$ is a subcategory of $\mathcal{D}$, and every object of $\mathcal{D}$ is a retract of some object of $\mathcal{C}$.

We now show that $\mathcal{D}$ is idempotent complete. If $e : C \to C$ is an idempotent in $\mathcal{D}$ then it splits in $\mathcal{A}$ as $e = sr$ where $s : R \to C$ and $r : C \to R$. By Lemma 1.5 R is a direct summand of C. Since C is a direct summand of an object of $\mathcal{C}$ we have that R is a direct summand of the same object and hence R is in $\mathcal{D}$. Therefore the morphisms $s : R \to C$ and $r : C \to R$ are in $\mathcal{D}$ and e splits in $\mathcal{D}$. □

Corollary 2.4. *The idempotent completion of the category of free modules over a commutative ring R is the category of projective modules over R.*

Proof. It is well-known that a module is projective if and only if it is a direct summand of a free module. □

3 Results used to define $\mathcal{LG}$

Let R be a commutative ring, $f \in R$. We quote the following result without proof.

Theorem 3.1 (Proposition 1.6.8 [Nee14])**.** *Any triangulated category which admits all countable coproducts is idempotent complete.*

Corollary 3.2. $\mathrm{HMF}(R, f)$ *is idempotent complete.*

Proof. $\mathrm{HMF}(R, f)$ admits all countable coproducts and the shift functor $X \mapsto X[1]$ induces a triangulated structure on $\mathrm{HMF}(R, f)$. □

Corollary 3.3. $\mathrm{hmf}(R, f)^\omega$ *is the full subcategory of direct summands of objects of* $\mathrm{hmf}(R, f)$.

Proof. See Corollary 2.3. □

Lemma 3.4. *Let $\mathcal{C}$ be a preadditive category with a zero object and $\mathcal{C}^\omega$ its idempotent completion. A functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}^\omega$ extends uniquely to a functor $\mathcal{C}^\omega \times \mathcal{C}^\omega \to \mathcal{C}^\omega$.*

Proof. We can embed $\mathcal{C}$ into $\mathcal{C}^\omega$ via the functor $C \mapsto (C, 0)$ or via the functor $C \mapsto (0, C)$. Using this we can show $(\mathcal{C} \times \mathcal{C})^\omega$ is $\mathcal{C}^\omega \times \mathcal{C}^\omega$ directly from the definition. □

References

[Bor94] Francis Borceux. *Handbook of categorical algebra 1: basic category theory.* Encyclopedia of mathematics and its applications v. 50. Cambridge [England] ; New York: Cambridge University Press, 1994. 345 pp. ISBN: 0-521-44178-1.

[Nee14] Amnon Neeman. *Triangulated Categories. (AM-148), Volume 148.* 2014. ISBN: 978-1-4008-3721-2.