LG seminar 1

(gsl)
13/1/22

A category has objects and morphisms between them (e.g. sets and functions). Two morphisms are either equal or they're not, but there is no possible deeper relation, because fundamentally morphisms form a set. A bicategory has objects, 1-morphisms and 2-morphisms (e.g. categories, functors and natural transformations), and since 1-morphisms form a category there is an expanded scope for interesting relations between 1-morphisms.

In this seminar series we will see a detailed construction of one particular bicategory, called the bicategory of Landau-Ginzburg models by mathematical physicists and the bicategory of isolated hypersurface singularities by algebraic geometers. It is denoted $\mathcal{LG}$ or $\mathcal{LG}_k$ if a base ring k is emphasised.

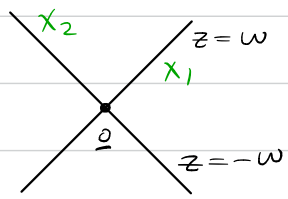
Rough definition The objects of $\mathcal{LG}_{\mathbb{C}}$ are complex polynomials $f(\underline{z}) = f(z_1, \ldots, z_n)$ in any number of variables, with the property that the zeros of the vector field

$$z \longmapsto \nabla f(z) = \left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n} \right)$$

are isolated points. Such polynomials are called potentials. A 1-morphism $X : f(\underline{z}) \longrightarrow g(\underline{w})$ between potentials is a subvariety of the zero set of $g - f$, that is, $X \subseteq \mathbb{V}(g-f) \subseteq \mathbb{C}^n \times \mathbb{C}^m$ if $\underline{z} = (z_1, \ldots, z_n)$, $\underline{w} = (w_1, \ldots, w_m)$. Recall a subvariety is defined by the vanishing of polynomial equations.

vanishing set

Example If $f = z^2$, $g = w^2$ we have



$$g - f = (w-z)(w+z)$$

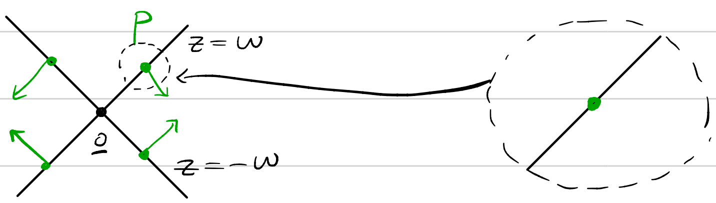
$$\therefore \mathbb{V}(g-f) = \underbrace{\mathbb{V}(w-z)}_{X_1} \cup \underbrace{\mathbb{V}(w+z)}_{X_2}$$

Both X_1 and X_2 determine morphisms $z^2 \to w^2$ (in fact X_1 is an isomorphism, since variable names don't matter).

Note that z^2, w^2 and w^2-z^2 are all potentials, and

$$\nabla(w^2-z^2) = (2w, -2z) \tag{2.1}$$

vanishes only at $\underline{0}$. Let us examine the geometric significance of this (non) vanishing. Pick a point $P = (a, b) \neq \underline{0}$ in $\mathbb{V}(w^2-z^2)$.



Then by assumption $h = w^2 - z^2$ vanishes at P. The vector $\nabla h(a,b)$ points in a direction normal to $\mathbb{V}(h)$ at (a,b) and together with (a,b) gives us a local coordinate system at P in which $\mathbb{V}(h)$ looks like the vanishing of a coordinate function. That is, $\mathbb{V}(h)$ is a <u>submanifold of $\mathbb{C}^2$ near P</u>.

But at $\underline{0}$ this is not true: $\mathbb{V}(h)$ does not look like a submanifold. A point like this, where both a function h <u>and</u> all its partial derivatives vanish, is called a <u>singularity of h</u>, and the pair (P, h) is called a <u>hypersurface singularity</u> (hypersurface because it is defined by one equation).

<u>Example</u> $(0, z^2)$, $(\underline{0}, w^2-z^2)$ are singularities.