b) The scaling hypotheses in themselves carry no information on the rescaling parameters p and q and therefore none of the results (13.1.2–5) and (13.1.12–13) can be used to compute the individual values of the critical exponents.

The renormalization program proposes to give a more fundamental explanation for the scaling hypotheses, one that would alleviate these criticisms.

13.2 The Renormalization Program

To present the basic idea in a simple form, we chose two didactic models where the program can be exactly implemented. The first is again the one-dimensional Ising model, for which the result is evidently trivial, but nevertheless instructive; see in particular the renormalization basic step (13.2.8), and its consequence (13.2.9). The second illustration is provided by the Dyson hierarchical model; here, the program requires serious analytic work, the main stages centering around: the basic iteration formula, (13.2.19); the discussion summarized in Fig. 13.3; and the values of the critical indices (13.2.40). Accordingly, the treatment of the model is broken up into three subsections: definitions (see Fig. 13.2), critical fixed points, and critical exponents.

Consider now the class of systems of the Ising-type, defined on a finite, d–dimensional lattice, with periodic boundary conditions, $\Lambda = (\mathsf{Z}_N)^d$; for convenience, we assume that $N = M^{m\lambda}$ with $\lambda = 2\kappa + 1$, M, m and κ positive integers, and m large. On each site k of this lattice sits a classical "spin" σ_k with possible values ± 1. Hence a configuration of this system is a function $\sigma : k \in \Lambda \to \sigma_k \in \{-1, +1\}$ so that the state-space of the system, i.e. the collection of all such configurations, is $\{-1, +1\}^\Lambda$. The Hamiltonian of any of these systems is taken to be of the form:

$$H : \sigma \in \{-1, +1\}^\Lambda \to H(\sigma) = \sum_A K_A \sigma_A \tag{13.2.1}$$

with A running over $\mathcal{P}(\Lambda)$, the collection of all subsets of Λ; and $\sigma_A = \prod_{k \in A} \sigma_k$; K_A is the strength of the interaction between the spins in A.

For convenience, we assume that the natural temperature $\beta = 1/k_{Boltz}T$ as been incorporated in the definition of K. For instance, for the ordinary Ising Hamiltonian – discussed in Sects. 12.1–2 for $d = 1, 2$, – the only $K_A \neq 0$ are $K_{\{k\}} = -\beta B$, the external magnetic field, *and* $K_{\{j,k\}} = -\beta J$ when j and k are nearest neighbors. Hence to specifiy a Hamiltonian in the class (13.2.1) is to specify the function $K : A \in \mathcal{P}(\Lambda) \to K_A \in \mathsf{R}$.

Let us now divide the original lattice in *cells* over which we take average. We thus introduce the *sublattice* $\Lambda' = (\lambda \mathsf{Z}_{N'})^d$ – with $N' = \lambda^{-1} N = M^{(m-1)\lambda}$ – and its complement $\Lambda'' = \{k \in \Lambda \,|\, k \notin \Lambda'\}$. We then denote by σ' [resp. σ'']

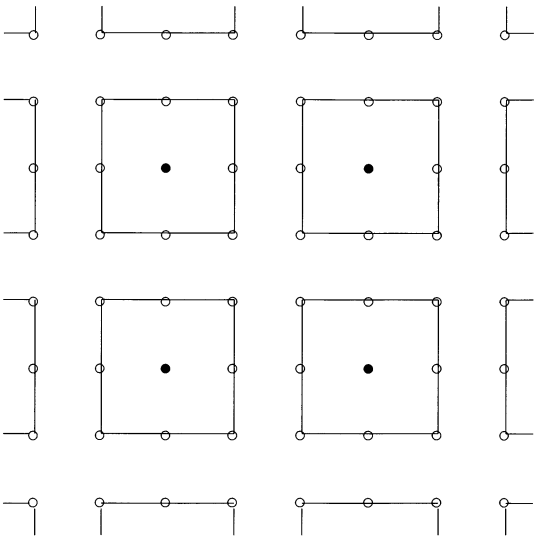

Fig. 13.1. Decimation and majority rules: Sites marked • belong to Λ' ; sites marked ○ belong to Λ''. Decimation retains only •. The majority rule attributes to • a value determined by the surrounding ○. Lines mark the boundaries of blocks

the restriction of the configuration σ to Λ' [resp. Λ''] , i.e. $\sigma' : k' \in \Lambda' \to \sigma'_{k'} = \sigma_{k'} \in \{-1, +1\}$; and similarly for σ'' .

The first step of the method known as *renormalization by decimation* (see Fig. 13.1) is to rewrite the partition function in the form:

$$\left.\begin{array}{l} Z = \sum_{\sigma} \mathrm{e}^{-H(\sigma)} = \sum_{\sigma'} \mathrm{e}^{-H(\sigma')} \\ \text{with} \\ \mathrm{e}^{-H(\sigma')} = \sum_{\sigma''} \mathrm{e}^{-H(\sigma' \cup \sigma'')} \end{array}\right\} \tag{13.2.2}$$

which defines the *coarse-grained* Hamiltonian

$$H(\sigma') = \sum_{A'} K'_{A'} \, \sigma'_{A'} \tag{13.2.3}$$

where A' runs over the subsets of Λ' ; and $\sigma'_{A'} = \prod_{k' \in A'} \sigma'_{k'}$.

On the space of the coupling constants $\mathcal{K}$, this defines a map $\mathcal{R}[K] = K'$. One iterates this procedure to produce successive Hamiltonians $H^{(n)}$ with coupling constants $K^{(n)}$, and the corresponding sequence of maps $\{\mathcal{R}^n \mid n \in \mathsf{Z}^+\}$ is called a *renormalization semigroup,* on account of the fact that the composition $\mathcal{R}^{n_1} \circ \mathcal{R}^{n_2} = \mathcal{R}^{n_1+n_2}$, defined for any pair of non-negative integers (n_1, n_2) , is an associative binary relation, with unit $\mathcal{R}^0 = Id$.

Lest the above looks too abstract, consider the very particular case where the initial Hamiltonian H is the one-dimensional Ising Hamiltonian with

nearest-neighbor interactions – $K_{\{j,k\}} = -\beta J$ when j and k are nearest neighbors, with all the other coupling constants being equal to zero, including the external magnetic field $K_{\{k\}} = -\beta B = 0$:

$$H(\sigma) = -K \sum_{k \in \Lambda} \sigma_k \sigma_{k+1} \tag{13.2.4}$$

and thus

$$\mathrm{e}^{-H(\sigma)} = \prod_{k \in \Lambda} \mathrm{e}^{K \sigma_k \sigma_{k+1}} = \prod_{k \in \Lambda} V(\sigma_k, \sigma_{k+1}) \tag{13.2.5}$$

where the transfer matrix V is rewritten in the form – compare with (12.1.7) – with the notation $\tilde{K} = \tanh[\beta J]$:

$$V = \begin{pmatrix} \mathrm{e}^{K} & \mathrm{e}^{-K} \\ \mathrm{e}^{-K} & \mathrm{e}^{K} \end{pmatrix} = \cosh K \begin{pmatrix} 1+\tilde{K} & 1-\tilde{K} \\ 1-\tilde{K} & 1+\tilde{K} \end{pmatrix} \quad . \tag{13.2.6}$$

Hence upon summing over the configurations attached to the sites that belong to Λ'' we obtain:

$$\left.\begin{array}{ll} & \sum_{\sigma''} \mathrm{e}^{-H(\sigma)} = \prod_{k' \in \Lambda'} V^{\lambda}(\sigma_{k'}, \sigma_{k'+1}) \\ \text{with} & \\ & V^{\lambda} = 2^{\lambda-1} (\cosh K)^{\lambda} \begin{pmatrix} 1+(\tilde{K})^{\lambda} & 1-(\tilde{K})^{\lambda} \\ 1-(\tilde{K})^{\lambda} & 1+(\tilde{K})^{\lambda} \end{pmatrix} \end{array}\right\} \tag{13.2.7}$$

which we can rewrite

$$\left.\begin{array}{ll} & \mathrm{e}^{-H(\sigma')} = \sum_{\sigma''} \mathrm{e}^{-H(\sigma' \cup \sigma'')} \\ \text{with} & \\ & H(\sigma') = -K' \sum_{k' \in \Lambda'} \sigma'_{k'} \sigma'_{k'+1} + C \\ \text{and} & \\ & K' = \beta J' = \tanh^{-1}\{[\tanh(\beta J)]^{\lambda}\} \quad \text{i.e.} \quad \tilde{K}' = \tilde{K}^{\lambda} \end{array}\right\} \tag{13.2.8}$$

where the constant C does not affect the computation of expectation values; it only betrays the presence of the multiplicative constant $2^{\lambda-1}(\cosh K)^{\lambda}$ in (13.2.7) rather than $\cosh K'$, and we can ignore it here.

We know – see Sect. 12.1 – that, even in the thermodynamical limit, the 1–d Ising model with nearest-neighbor interaction does not exhibit a phase transition at any finite temperature. The renormalization semi-group confirms this. Indeed, for this particular model, the map $\mathcal{R} : \tilde{K} \in [0,1] \to \mathcal{R}[\tilde{K}] = \tilde{K}' \in [0,1]$ has exactly two fixed points: 0 and 1, with $0 \leq \tilde{K} < 1 \models \lim_{n \to \infty} R^n[\tilde{K}] = 0$.

Upon recalling the notation $\tilde{K} = \tanh \beta J$, we see that the fixed point $\{0\}$ corresponds to the high temperature limit $T = \infty$, where the system behaves as if there were no interaction; and the fixed point $\{1\}$ corresponds to the low temperature limit $T = 0$, where everything is frozen: either all the spins are up, or all are down. Let us briefly examine what the renormalization analysis has to add.

We note: (i) the lucky fact that the coarse-grained Hamiltonian (13.2.8) and the original Hamiltonian (13.2.4) both are Ising model with nearest-neighbor interactions only; (ii) the coupling constants of these two systems are related by the third relation in (13.2.8); and (iii) λ is the ratio of the lattice spacing in the coarse-grained system over the lattice spacing of the original system. We then conclude that the correlation length ξ satisfies:

$$\xi(\tilde{K}') = \lambda^{-1}\xi(\tilde{K}) \quad \text{i.e.} \quad \xi \propto [\ln \tanh K\,]^{-1} \tag{13.2.9}$$

in agreement with (12.1.18) as $T \to 0$; for finite temperatures, the correlation decays exponentially fast, but the rate of decay ξ^{-1} slows down to 0 as $\mathrm{e}^{-\beta J}$ when the temperature approaches the absolute zero.

While not bringing new results yet, the approach is certainly more direct than that followed in Sect. 12.1.

Note that one can view renormalization by decimation as a selection process. Around each site $k' = k\,\lambda$ in the sublattice Λ', let us consider the box $B_{k'}$ of length $(\lambda - 1)$ centered on this site – recall that we chose λ to be odd, say, $\lambda = 2\kappa + 1$, so that $B_{k'} \equiv \{k \in \Lambda \,|\, k' - \kappa \leq k \leq k' + \kappa\}$. Note that we wrote this in one dimension only; if the lattice has more dimensions, these inequalities are required to hold for each components of the position vectors.

The idea of a representative selection process is now this: when the system is close to its critical point, the correlation length ξ is much larger than lattice spacing (chosen here to be $a = 1$); to focus on the behavior of the system in intermediate scales, one chooses λ such that $1 \ll \lambda \ll \xi$. The spins inside the same box $B_{k'}$ are therefore expected to be strongly correlated, and $\sigma_{k'}$ can therefore be considered as a "representative" of the box $B_{k'}$ in the middle of which it sits. This description suggests immediately other possible selection processes. For instance, a more "democratic" selection would be the *majority rule* by which one defines the spin variable $\sigma_{k'}$ the values ± 1 of which are determined by the condition $\sigma_{k'} \cdot \sum_{k \in B_{k'}} \sigma_k > 0 : \sigma_{k'}$ is $+1$ [resp. -1] when the majority of the spins in the box point up [resp. down]; this assignment is unambiguous, since we have assumed in effect that each box contains an odd number of sites, irrespectively of the dimensionality of the system. Still other assignments are possible, mostly chosen for computational convenience when searching for fixed points of the corresponding renormalization semigroup.

Yet another way to implement the idea of the renormalization method is to make it appear as a deviation from the Gaussian distribution of the central limit theorem – see Sect. 5.2 – when the random variables considered are *not* independent. Following [Collet and Eckmann, 1978], we present this approach in a case where it is implementable without approximation.

Definition of the hierarchical model. The original model was proposed by [Dyson, 1969]. For every positive integer N, let $\Lambda_N = \{1, 2, 3, \cdots, 2^N\}$ and $\{-1, +1\}^{\Lambda_N} = \{\sigma : k \in \Lambda_N \to \sigma_k \in \{-1, +1\}\}$. For each integer p with $0 \leq p \leq N$, consider the partition of Λ_N into 2^{N-p} blocks $B_{p,r}$ consisting of 2^p consecutive sites:

$$B_{p,r} = \{k \in \Lambda_N \mid (r-1)\,2^p + 1 \leq k \leq r\,2^p\}\ ;\ r = 1, 2, \cdots, 2^{N-p}\,. \quad (13.2.10)$$

The Hamiltonian then is defined as:

$$H_{\Lambda_N} = -\sum_{p=1}^{N} \frac{1}{2^{2p}+1} b_p \sum_{r-1}^{2^{N-p}} \Big(\sum_{k \in B_{p,r}} \sigma_k \Big)^2 \quad (13.2.11)$$

where the coupling constants b_p are positive numbers, defined for all non-negative integers p, and satisfying the condition that the following sum converges:

$$\lim_{N\to\infty} E_N = \sum_{p=1}^{N} 2^{p-1} E_N(p) \quad \text{with} \quad E_N(p) = \sum_{q=p}^{N} 2^{-2q} b_q \quad . \quad (13.2.12)$$

To see the reason for the appelation "hierarchical" and the convergence condition, note that, for all integers p with $1 \leq p \leq N$: the block $B_{p,r}$ of length 2^p is the disjoint union of two consecutive blocks of length 2^{p-1}, namely $B_{p-1,2r-1}$ and $B_{p-1,2r}$. Consequently

$$\left.\begin{aligned} \Big(\sum_{k\in B_{p,r}} \sigma_k\Big)^2 = \Big(\sum_{k\in B_{p-1,2r-1}} \sigma_k\Big)^2 + \Big(\sum_{l\in B_{p-1,2r}} \sigma_l\Big)^2 \\ + \sum_{(k,l)\in B_{p-1,2r-1}\times B_{p-1,2r}} \sigma_k\sigma_l \end{aligned}\right\}\,. \quad (13.2.13)$$

Since the first and the second terms of the RHS are of the same form as the LHS, except for the fact that they involve blocks of length 2^{p-1} instead of a block of length 2^p, the Hamiltonian (13.2.11) can be rewritten as

$$H_{\Lambda_N} = \sum_{p=1}^{N} v_p \sum_{(k,l)\in V_p} \sigma_k\sigma_l \quad (13.2.14)$$

where V_p is the set of all pairs of sites (k,l) for which k and l belong to the same block of length 2^p but to different blocks of length 2^{p-1}. Hence the Hamiltonian can be viewed as the sum of interactions between consecutive blocks, at all possible levels p. This is illustrated in Fig. 13.2. Note then that for any pair of sites $(k,l) \in \Lambda \times \Lambda$ there is a smallest integer $p(k,l)$ such that k and l belong to the same block. With this picture in mind, one sees with [Dyson, 1969], that E_N – in (13.2.12) – is the sum of the interactions coupling an arbitrary, but fixed, spin to all the others. Hence condition (13.2.12) expresses that this bound remains finite so that one can define an infinite hierarchical model, as $N \to \infty$. Note also that the condition is satisfied for the particular choices $b_p = c^p$ with $1 < c < 2$.

The primary concern in [Dyson, 1969] was to use the model to explore the effect of interactions, decaying slowly with distance, on the occurrence of long-range order in one-dimension – see the closing remarks in Sect. 12.1.

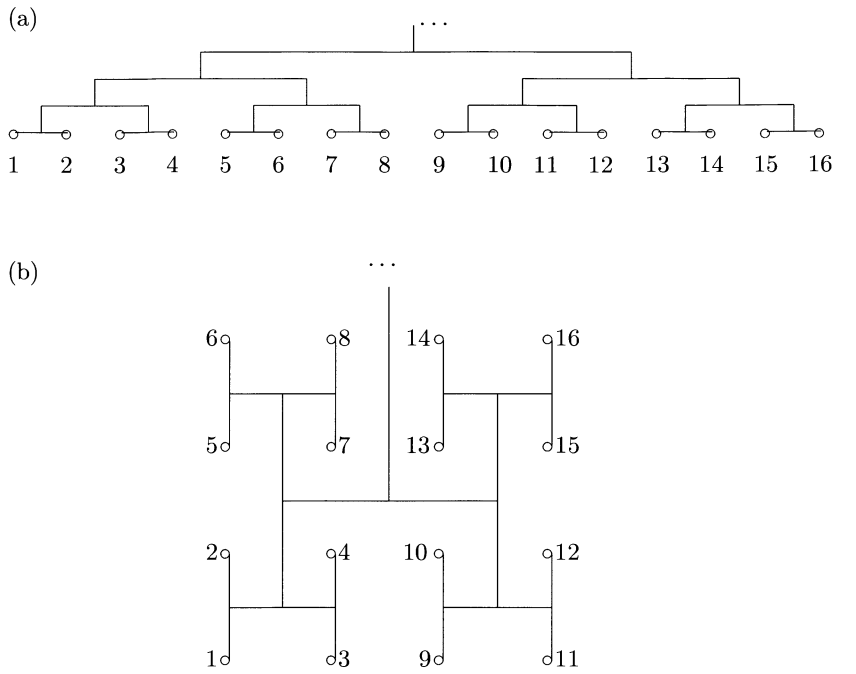

Fig. 13.2a,b. The one-dimensional hierarchical model. The successive blocks containing the site $\{1\}$ are shown: $B_{0,1} = \{1\}, B_{1,1} = \{1,2\}, B_{2,1} = \{1,2,3,4\}$, $B_{3,1} = \{1,2,\cdots,7\}$, $B_{4,1} = \{1,2,\cdots,16\}$ with the interactions binding each block indicated by a thin line.

The hierarchical model was re-invented by [Baker, Jr., 1972] for the explicit purpose of exhibiting an exactly solvable model to which the [Wilson, 1971] renormalization program did apply. The analysis of a class of models of this type was reworked with great mathematical care by [Bleher and Sinai, 1975, Gallavotti and Knops, 1975, Collet and Eckmann, 1978].

One has again a one-dimensional array Λ_N of 2^N sites, but the "spins" are assumed to take continuous rather than discrete values. The Hamiltonian is now:

$$\left.\begin{aligned} & H_{\{f,\Lambda_N\}}(\sigma) = H_{\{0,\Lambda_N\}}(\sigma) + \textstyle\sum_{k\in\Lambda_N} f(\sigma_k) \\ \text{with} \quad & H_{\{0,\Lambda_N\}}(\sigma) = -\textstyle\sum_{p=0}^{N} \frac{c^p}{2^{2p}+1} \sum_{r=1}^{2^{N-p}} (\sum_{k\in B_{p,r}} \sigma_k)^2 \end{aligned}\right\} \qquad (13.2.15)$$

where the value of the constant c and the form of the function f are still to be adjusted to ensure thermodynamical stability, in particular the existence of the thermodynamical limit. The hierarchical structure of the Hamiltonian is captured by performing in the Hamiltonian $H_{\{0,\Lambda_{N+1}\}}$ the change of variables

$$\left.\begin{array}{l} \tau_r = \lambda_c^{-1}\,(\sigma_{2r-1} + \sigma_{2r}) \\ \nu_r = \lambda_c^{-1}\,(\sigma_{2r-1} - \sigma_{2r}) \end{array}\right\} \quad \text{with} \quad r \in \Lambda_N \quad \text{and} \quad \lambda_c = 2\,c^{-\frac{1}{2}} \tag{13.2.16}$$

where the variable τ_r is proportional to the sum of the spins σ_k in the block $B_{1,r}$ of length 2 in the array Λ_{N+1}; the value of the scaling constant λ_c is chosen to ensure that (13.2.17) and (13.2.19) below are satisfied. We have:

$$H_{\{0,\Lambda_{N+1}\}}(\sigma) = H_{\{0.\Lambda_N\}}(\tau) - \frac{1}{2}\sum_{r\in\Lambda_N}(\tau_r)^2 \tag{13.2.17}$$

where the variables ν_k do not appear. The scalar counter-term in the full Hamiltonian $H_{\{f,\Lambda_{N+1}\}}(\sigma)$ – see (13.2.15) – contains sums of functions of the one-spin variables σ_k and requires more focussed attention. For this purpose, consider only the equilibrium expectation

$$\langle A\rangle_\beta = \int \mathrm{d}S\, P_{\{\beta,f,\Lambda_N\}}(S)\, A(S)$$

of the macroscopic variables of the form $A(\sum_{k\in\Lambda_N}\sigma_k)$, where A is any measurable function, β denotes the natural temperature and P is the measure corresponding to the Hamiltonian $H_{\{f,\Lambda_N\}}$; i.e.

$$\left.\begin{array}{l} P_{\{\beta,f,\Lambda_N\}}(S) = \\ Z_{\{\beta,f,\Lambda_N\}}{}^{-1}\ \int\cdots\int \mathrm{d}\sigma_1\cdots\mathrm{d}\sigma_{2^N}\ \mathrm{e}^{-\beta H_{N,f}(\sigma)}\ \delta(S - \sum_{k=1}^{2^N}\sigma_k) \\ \text{with} \quad Z_{\{\beta,f,\Lambda_N\}} = \int\cdots\int \mathrm{d}\sigma_1\cdots\mathrm{d}\sigma_{2^N}\ \mathrm{e}^{-\beta H_{N,f}(\tau)} \end{array}\right\}. \tag{13.2.18}$$

Upon using the definition (13.2.15) and the recursion relation (13.2.17), one obtains that the probability distribution (13.2.18) satisfies the *renormalized scaling property:*

$$\left.\begin{array}{ll} & \text{(a)}\ P_{\{\beta,f,\Lambda_{N+1}\}}(S) = \lambda_c^{-1}\,P_{\{\beta,\mathcal{N}_{\beta,c}[f],\,\Lambda_N\}}(\lambda_c^{-1}S) \\ \text{where} & \\ & \text{(b)}\ \mathcal{N}_{\beta,c}[f](\tau) = \\ & \quad -\frac{1}{\beta}\ln\{\lambda_c\,\mathrm{e}^{\frac{1}{2}\beta\tau^2}\int \mathrm{d}\nu\,\mathrm{e}^{-\beta[f(\frac{\lambda_c}{2}\tau+\nu)\,+f(\frac{\lambda_c}{2}\tau-\nu)]}\} \end{array}\right\}. \tag{13.2.19}$$

The most remarkable feature of the exact relation (13.2.19) implies that the Hamiltonian remains in the class defined by (13.2.15) and only involves tuning the one-spin function – f being replaced by $\mathcal{N}_{\beta,c}[f]$ – appearing in the scalar counter-term.

Remark: A motivating analogy with the central limit theorem. To bring up the sense in which the renormalization program can be viewed as a variation on the theme of the central limit theorem – Sect. 5.4 – suppose for an instant that the spin variables of our array of 2^N sites are *independent* and identically distributed, with one-spin density ϱ, with mean $m \equiv <\sigma>_\varrho$ and variance $v \equiv <(\sigma-m)^2>_\varrho^{\frac{1}{2}}$. ll Then (13.2.18) is replaced by

$$\left.\begin{array}{l} P_{\{0,\Lambda_N\}}(S) = \\ \int \mathrm{d}\sigma_1\, \varrho(\sigma_1) \int \mathrm{d}\sigma_2\, \varrho(\sigma_2) \cdots \int \mathrm{d}\sigma_{2^N}\, \varrho(\sigma_{2^N})\; \delta(S - \sum_{k=1}^{2^N} \sigma_k) \end{array}\right\} \tag{13.2.20}$$

and the central limit theorem asserts that the following limit exists (in the weak sense of convergence of expectation values):

$$\Phi_0(S) = \lim_{N\to\infty} P_{\{0,\Lambda_N\}} \left(2^{\frac{1}{2}N}(S + 2^N m)\right) \tag{13.2.21}$$

and is equal to the Gaussian probability distribution

$$\Phi_0(S) = \frac{1}{\sqrt{2\pi v^2}}\, \mathrm{e}^{-\frac{1}{2}(S/v)^2} \quad . \tag{13.2.22}$$

Note that (13.2.20) can be rewritten as the 2^N–fold convolution product

$$P_{\{0,2^N\}} = \varrho * \varrho * \cdots * \varrho \tag{13.2.23}$$

where for any two $\mathcal{L}_1$–functions f and g the $\mathcal{L}_1$–function $f * g$ is defined by:

$$f * g(s) \equiv \int_{-\infty}^{\infty} u\, f(s-u)\, g(u) = \int_{-\infty}^{\infty} \mathrm{d}v\, f(\frac{s}{2} + v)\, g(\frac{s}{2} - v) \quad .$$

In particular, when the one-spin distribution is Gaussian – as in (13.2,22) - the probability distribution for the array Λ_N is again a Gaussian, and does satisfy the central scaling property:

$$\left.\begin{array}{ll} P_{\{0,\Lambda_N\}}(S) = \frac{1}{\sqrt{2\pi v_N{}^2}}\, \mathrm{e}^{-\frac{1}{2}(S/v_N)^2} & \text{with} \quad v_N = |\Lambda_N|^{\frac{1}{2}} v\,, \\ P_{\{0,\Lambda_{N+1}\}}(S) = \lambda^{-1} P_{\{0,\Lambda_N\}}(\lambda^{-1} S) & \text{with} \quad \lambda = 2^{\frac{1}{2}} \quad . \end{array}\right\} \tag{13.2.24}$$

Compare this to (13.2.19); the comparison is pursued later on. In general, since the convolution product is associative, (13.2.23) entails

$$P_{\{0,\Lambda_{N+1}\}}(S) = [P_{\{0,\Lambda_N\}} * P_{\{0,\Lambda_N\}}](S) \tag{13.2.25}$$

or equivalently, for any $\lambda > 0$:

$$\begin{array}{l} \mathcal{R}_{\{0,\Lambda_N\}}(S) \equiv P_{\{0,\Lambda_N\}}(\lambda^N S) \quad \models \\ \mathcal{R}_{\{0,\Lambda_{N+1}\}}(S) = \lambda^N \int \mathrm{d}u\, \mathcal{R}_{\{0,\Lambda_N\}}(\frac{1}{2}\lambda S + u)\, \mathcal{R}_{\{0,\Lambda_N\}}(\frac{1}{2}\lambda S - u). \end{array} \tag{13.2.26}$$

The choice $\lambda = 2^{\frac{1}{2}}$ corresponds to the scaling that appears in the central limit theorem; hence (13.2.24) and (13.2.26) – with $\lambda = 2^{\frac{1}{2}}$ – can be viewed as recursion relations for which the limiting Gaussian distribution $P_{\{0,\Lambda_N\}}(2^{\frac{1}{2}N} S)$ is a fixed point.

For any model in which the spins *do* interact, and are thus *not independent* random variables, the central limit theorem cannot be expected to hold in the form just given. It is therefore remarkable that, for the hierarchical model (13.2.15), the following simple generalization of (13.2.26) holds:

$$\left.\begin{aligned} &\mathcal{R}_{\{\beta,f,\Lambda_N\}}(S) \equiv P_{\{\beta,f,\Lambda_N\}}(\lambda_c^N S) \quad \models \\ &\mathcal{R}_{\{\beta,f,\Lambda_{N+1}\}}(S) = \alpha_N \, \mathrm{e}^{\frac{1}{2}\beta S^2} \quad \times \\ &\qquad \lambda_c^N \int \mathrm{d}u \, \mathcal{R}_{\{\beta,f,\Lambda_N\}}(\tfrac{1}{2}\lambda_c S + u) \, \mathcal{R}_{\{\beta,f,\Lambda_N\}}(\tfrac{1}{2}\lambda_c S - u) \end{aligned}\right\} \tag{13.2.27}$$

where λ_c is the scaling defined in (13.2.16). Equivalently, (13.2.27) can be rewritten as:

$$P_{\{\beta,f,\Lambda_{N+1}\}}(S) = \alpha_N \, \mathrm{e}^{\frac{1}{2}\beta[\lambda_c^{-(N+1)}S]^2} \, [P_{\{\beta,f,\Lambda_N\}} * P_{\{\beta,f,\Lambda_N\}}](S) \quad . \tag{13.2.28}$$

Hence the only difference between the recursion relation (13.2.27) – valid for the hierarchical model – and the relation (13.2.26) – valid for the case where the spin variables are independent – is the factor $D(S) \equiv \mathrm{e}^{\frac{1}{2}\beta S^2}$; α_N only ensures that the LHS of (13.2.28) is still a probability distribution – specifically that its integral is still equal to 1. In particular, when the coupling constant $c = 0$, $\lambda_c^{-1} = 0$, and (13.2.28) reduces to (13.2.25) as it should. Moreover, in the high temperature limit $\beta \to 0$ the factor in $D(S)$ disappears confirming that the hierarchical model behaves properly: in this limit the thermal agitation prevails on the mechanical interaction, and the spins become independent of one another. Furthermore, as already pointed out, when f is a fixed point of the map $f \to \mathcal{N}_{\beta,c}[f]$ in (13.2.19b), the scaling relation (13.2.19a) is analogous to (13.2.24).

The analogy with the central limit theorem therefore suggests that in the study of the asymptotic behavior of the hierarchical model one should expect (13.2.21) to be replaced by

$$\Phi_\beta(S) = \lim_{|\Lambda_N| \to \infty} P_{\{\beta,f,\Lambda_N\}} \left(|\Lambda_N|^{\frac{1}{2}\tau} (S + |\Lambda| m) \right) \; ; \; \tau = 2 - \log_2 c \tag{13.2.29}$$

where we used the substitutions $|\Lambda| = 2^N$, and $\lambda_c = 2\, c^{-\frac{1}{2}}$ as defined in (13.2.16). When the limit (13.2.29) exists and is *not* Gaussian, it is called *critical*.

Critical fixed points for the hierarchical model. To complete the illustration of the renormalization method for the case of the hierarchical model, the existence and properties of fixed points have to be established. For this purpose, it is useful to replace the renormalization operator $\mathcal{N}_{\beta,c}$ – see (13.2.19b) – by an equivalent, but more manageable operator. Consider indeed the following two bijective transformations:

$$\left.\begin{array}{l} \mathcal{E}_\beta : f \to f_\beta \quad \text{and} \quad {\mathcal{S}_\beta}^{-1} : f_\beta \to \varphi \quad \text{defined by} \\ \\ f_\beta(s) = \mathrm{e}^{-\beta f(s)} \quad \text{and} \\ \\ \varphi(s) = [4\pi(2-c)/c\beta]^{\frac{1}{2}}\ \mathrm{e}^{\frac{1}{2}s^2}\ f_\beta([(2-c)/c\beta]^{\frac{1}{2}}\, s) \end{array}\right\} . \qquad (13.2.30)$$

One verifies that f is a fixed point of $\mathcal{N}_{\beta,c}$ if and only if f_β satisfies

$$f_\beta(s) = \lambda_c\ \mathrm{e}^{\frac{1}{2}\beta s^2}\ [f_\beta * f_\beta](\lambda_c\, s) \qquad (13.2.31)$$

and this happens if and only if φ is a fixed point of the new operator $\mathcal{N}_c$ defined by:

$$\mathcal{N}_c[\varphi](s) = \frac{1}{\pi^{\frac{1}{2}}} \int \mathrm{d}u\, \mathrm{e}^{-u^2}\, \varphi(\frac{1}{2}\lambda_c s + u)\, \varphi(\frac{1}{2}\lambda_c s - u) \quad . \qquad (13.2.32)$$

Two features of this operator govern the following discussion. First, the operator $\mathcal{N}_c$ depends only on the strength c of the interactions, and not on the temperature β. The latter is reintroduced into the picture by the inverse of the transformations (13.2.30), but only as a deviation from a critical temperature β_c determined by c. Second, however, $\mathcal{N}_c$ still inherits – through the convolution (13.2.31) – the non-linear character of the original renormalization operator $\mathcal{N}_{\beta,c}$. Clearly the constant function $\varphi(s) = 1$ is a fixed point of $\mathcal{N}_c$ for every c in the range of interest to us, namely $1 < c < 2$, which are conditions that come in handy when one established the existence of the thermodynamical limit [$c < 2$] and of a phase transition [$c > 1$].

For orientation purposes, let us tentatively limit our attention to the linear approximation $\mathcal{DN}_c(\varphi)$ of $\mathcal{N}_c$ around a solution φ of $\mathcal{N}_c[\varphi] = \varphi$ i.e. $\mathcal{N}_c[\varphi + \delta\varphi] \approx \varphi + \mathcal{DN}_c(\varphi)[\delta\varphi]$. Mathematically, $\mathcal{DN}_c(\varphi)$ is the tangent map to $\mathcal{N}_c$ at φ. Analytically, one verifies that $\mathcal{DN}_c(\varphi)[\delta\varphi](s) = 2\pi^{-\frac{1}{2}} \int \mathrm{d}u\, \mathrm{e}^{-u^2} \varphi(\frac{1}{2}\lambda_c s + u)\, (\delta\varphi)(\frac{1}{2}\lambda_c s - u)$; thus, in particular, for the solution $\varphi = 1$ of $\mathcal{N}_c[\varphi] = \varphi$:

$$\mathcal{DN}_c(1)[\delta\varphi](s) = 2\pi^{-\frac{1}{2}} \int \mathrm{d}u\, \mathrm{e}^{-(\frac{1}{2}\lambda_c s - u)^2} (\delta\varphi)(u) \quad . \qquad (13.2.33)$$

It is then straightforward to verify that for every $n \in \mathsf{Z}^+$:

$$\mathcal{DN}_c(1)[\psi_c^{(n)}](s) = \lambda_c^{(n)}\, \psi_c^{(n)}(s) \qquad (13.2.34)$$

where the eigenfunctions $\psi_c^{(n)}$ are the normalized Hermite polynomials

$$\psi_c^{(n)}(s) = h_n((1-c^{-1})^{\frac{1}{2}}\, s) \quad \text{with} \quad \left\{\begin{array}{l} h_n(s) = (2^n\, n!)^{\frac{1}{2}} H_n \\ \\ H_n(s) = (-1)^n \mathrm{e}^{s^2} \frac{\mathrm{d}^n}{\mathrm{d}^n s} \mathrm{e}^{-s^2} \end{array}\right. ;$$

and where the eigenvalues are $\lambda_c^{(n)} = 2\, c^{-\frac{1}{2}n}$. In particular, for all values of c : $\lambda_c^{(o)} = 2$ and $\lambda_c^{(1)} = 2c^{-\frac{1}{2}} = \lambda_c$, where λ_c is precisely the scaling factor

introduced in (13.2.16). Since $\{\psi_c^{(n)} \mid n \in \mathsf{Z}^+\}$ is an orthonormal basis in the Hilbert space $\mathcal{H}_c \equiv \mathcal{L}^2(\mathsf{R}, d\mu_c)$ with $d\mu_c(s) = \mathrm{e}^{(1-c^{-1})s^2}\, ds$, $\mathcal{DN}_c(1)$ can be viewed as a self-adjoint operator, mapping $\mathcal{H}_c$ into itself, and with simple discrete spectrum

$$\mathrm{Sp}\,[\mathcal{DN}_c(1)] = \{\lambda_c^{(n)} = 2\,c^{-\frac{1}{2}n} \mid n \in \mathsf{Z}^+ \cdots\} \subset (0,2] \quad . \tag{13.2.35}$$

For any $c \in (1,2)$, the spectrum (13.2.35) contains finitely many eigenvalues larger than 1 and infinitely many eigenvalues smaller than 1, $\varphi = 1$ is hyperbolic, with finitely many unstable directions and infinitely many stable directions.

In addition, there may exist one zero eigenvalue: this happens exactly when the coupling constant c hits one of the special values $\{2^{\frac{2}{n}} \mid n = 3, 4, \cdots\}$. For instance, if $c = 2^{\frac{2}{n}}$, there are n unstable directions, corresponding to $2 = \lambda_c^{(o)} > \lambda_c^{(1)} > \cdots > \lambda_c^{(n-1)} > 1$, while $\lambda_c^{(n)} = 1$, and $\forall\, k > n : \lambda_c^{(k)} < 1$.

The analysis of $\mathcal{DN}_c(1)$ therefore sugests that, at a particular value of the coupling constant c for which one of the $\lambda_c^{(n)}$ vanishes, a new solution of $\mathcal{N}_c[\varphi] = \varphi$ may bifurcate away from the trivial fixed point $\varphi = 1$. The proof that this really happens – namely that such a solution exists and has smooth analytic properties – requires some hard analysis that goes beyond the elementary perturbation theory of bifurcations; this analysis is provided in [Collet and Eckmann, 1978]. We summarize their results.

First of all, a disclaimer: the above conjecture is borne out only when n is even, i.e. for $n = 2k$ and thus $c \in \{2^{\frac{1}{k}} \mid k = 2, 3, \cdots\}$. The first putative point where a new solution may branch off is therefore $n = 4$ *i.e.* $c = 2^{\frac{1}{2}}$; since it turns out to be typical, we can lighten the notation, and restrict our attention to this branching point.

Second, the existence and uniqueness of a non-trivial fixed point can be proven only in a finite, but small, neighborhood of our branching point, and the notation $c(\varepsilon) = 2^{\frac{1}{2}(1-\varepsilon)}$ reminds us of this. We denote by φ_o the trivial fixed point $\varphi_0 = 1$.

The main result is that for all ε positive and sufficiently small, there exists a function $\varphi_\varepsilon : s \in \mathsf{R} \to \varphi_\varepsilon(s) \in \mathsf{R}$ smooth in ε and s, such that $\mathcal{N}_{c(\varepsilon)}[\varphi_\varepsilon] = \varphi_\varepsilon$ and $\varphi_\varepsilon \neq \varphi_o$, with φ_ε branching off smoothly from φ_o at $c(0) = 2^{\frac{1}{2}}$. φ_ε decreases like $\mathrm{e}^{-\frac{1}{2}\varepsilon\theta s^4}$ as $|s| \to \infty$, for some constant $\theta \in (0,1)$ the precise value of which is known, but not essential beyond the central claim, namely that φ_ε is critical. Upon recalling the defining relations (13.2.30), the physicist as well as the probabilist will be interested in noting with [Collet and Eckmann, 1978] that the distribution density $\mathrm{e}^{-\frac{1}{2}s^2}\varphi_\varepsilon$ is *not* infinitely divisible; thus it *cannot* be written as a limit of sums of *indenpendent* random variables – for the definition and properties of infinitely divisible distributions in the context of the techniques associated to the central limit theorem, the Reader may consult [Feller, 1968, 1971].

Pursuing the analysis one step further, one can show that the spectrum of $\mathcal{DN}_{c(\varepsilon)}(1)$ changes controllably little when one passes from the trivial

solution $\varphi = 1$ with $c = 2^{\frac{1}{2}}$ to the bifurcated solution φ_ε corresponding to $c(\varepsilon) = 2^{\frac{1}{2}(1-\varepsilon)}$ with $\varepsilon > 0$; the spectrum remains discrete, and there are still four eigenvalues larger than 1, corresponding thus again to four stable directions; perturbation computations provide these eigenvalues: $\{\lambda^{(n)}_{c(\varepsilon)} \approx 2c(\varepsilon)^{-\frac{1}{2}n} \mid n = 0,1,2,3\}$. In the sequel, we will need to know that, for the first two eigenvalues, this approximation turns out to be exact: $\lambda^{(0)}_{c(\varepsilon)} = 2$, and $\lambda^{(1)}_{c(\varepsilon)} = \lambda_{c(\varepsilon)}$. For the eigenvalue $\lambda^{(2)}_{c(\varepsilon)}$ the approximation is good only for small ε; indeed while this eigenvalue is smooth in ε, starts exactly at $\lambda^{(2)}_{c(0)} = 2c(0)^{-1} = 2^{\frac{1}{2}}$, and at first increases for a little while, it soon reaches a maximum from which it decreases monotonically the limiting value $1 \neq 2 = 2c(1)^{-1}$ as ε approaches 1. Moreover, the fifth eigenvalue is now strictly smaller than 1, and thus all – but the first four – eigenvalues correspond to stable directions.

Additional – precise and technically important – information on the global properties of the flow of the renormalization map $\mathcal{N}_c$ is also presented in [Collet and Eckmann, 1978].

The critical exponents of the hierarchical model. The temperature has to be brought back into the picture. This is done in a somewhat roundabout way by tuning the parameters c and f that define the Hamiltonian of the hierachical model (13.2.15) in such a manner that it has a preassigned critical temperature. To this effect, choose $\varepsilon > 0$ sufficiently small, and let $c(\varepsilon) = 2^{\frac{1}{2}(1-\varepsilon)}$; this fixes the first parameter in (13.2.15). Next choose a positive number $\beta_o > 0$ which will play the role of a critical temperature for the Hamiltonian still to be specified by a properly tuned f. One single restriction is imposed on the choice of β_o, namely that it be different from $4\pi e\,(2-c(\varepsilon))\,c(\varepsilon)^{-2}$. To complete the specification of f, consider a sufficiently small neighborhood $\mathcal{W}$ of the fixed point φ_ε. This manifold – the elements of which are functions φ on which the renormalization map acts – splits into a finite dimensional unstable manifold $\mathcal{U}$ and an infinite dimensional stable manifold $\mathcal{S}$; in particular, any function $\varphi_{\mathcal{S}} \in \mathcal{S}$ gets closer and closer to φ_ε under the successive iterations of the renormalization map. One last technicality: it is possible to choose $\varphi_{\mathcal{S}} \in \mathcal{S}$ such that it satisfies the following five conditions: (i) $\varphi_{\mathcal{S}} > 0$, (ii) $\varphi_{\mathcal{S}}$ is continuously differentiable, (iii) and (iv) the functions $x \to x\frac{d}{dx}[\varphi_{\mathcal{S}}](x)[\varphi_{\mathcal{S}}(x)]^{-\frac{1}{2}}$ and $x \to \log[\varphi_{\mathcal{S}}](x)x\frac{d}{dx}[\varphi_{\mathcal{S}}](x)[\varphi_{\mathcal{S}}(x)]^{-\frac{1}{2}}$ are measurable and essentially bounded, (v) the ess-sup norm (see Example (D.2.3) $\| \frac{d}{dx}[\varphi_{\mathcal{S}} - \varphi_\varepsilon \|_\infty$ is small. Provided these conditions are satisfied, the ultimate results do not depend on the choice of $\varphi_{\mathcal{S}}$, one of the nice features of the situation, known as *universality*. The function $f_{\mathcal{S}}$ is then defined by inverting the bijective transformations (13.2.30):

$$f_{\mathcal{S}} = {\mathcal{E}_{\beta_o}}^{-1} \circ \mathcal{S}_{\beta_o}[\varphi_{\mathcal{S}}] \quad . \tag{13.2.36}$$

The Hamiltonian is specified by inserting in (13.2.15) $c(\varepsilon)$ for c and $f_{\mathcal{S}}$ for f.

The introduction of the temperature β with $\beta \in V_o$ where V_o is a small neighborhood of β_o changes $e^{-\beta_o f_S}$ to $e^{-\beta f_S}$ i.e. $\mathcal{E}_{\beta_o}[f_S]$ to $\mathcal{E}_\beta[f_S]$; this transformation is traced in the φ–function space by the transformation

$$\left.\begin{aligned} \varphi_S \to \varphi_\beta \equiv \mathcal{S}_\beta{}^{-1} \circ \mathcal{E}_\beta[f_S] &= \mathcal{S}_\beta{}^{-1} \circ \mathcal{E}_\beta \circ \mathcal{E}_{\beta_o}{}^{-1} \circ \mathcal{S}_{\beta_o}[\varphi_S] \\ &= \mathcal{S}_\beta{}^{-1} \left(\mathcal{S}_{\beta_o}[\varphi_S]\right)^{\frac{\beta}{\beta_o}} \end{aligned}\right\} . \quad (13.2.37)$$

The curve traced in the φ–function space by $\beta \to \varphi_\beta$ can be shown to be differentiable and to intersect the stable manifold *transversally* at φ_S. Hence β can be used as a local coordinate in the φ–function space.

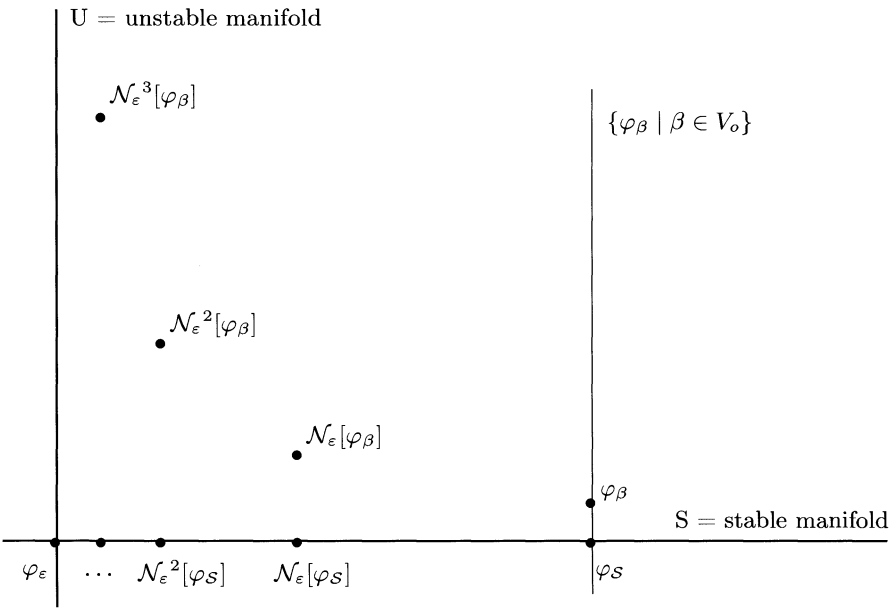

Fig. 13.3. The renormalization flow: straightened picture of the flow under $\mathcal{N}_\varepsilon$ around φ_ε ; only one quadrant shown: similar patterns occur in all four quadrants depending on the choice of φ_o and the sign of $\beta - \beta_o$.

The argument leading to (13.2.37) allows to rewrite the renormalization map in the physical space of the f–functions as:

$$\left.\begin{aligned} &f \to \mathcal{N}^\beta[f] \\ \text{with} \quad &\mathcal{N}^\beta[f] = \mathcal{S}_\beta{}^{-1} \circ \mathcal{E}_\beta \circ \mathcal{N}_{c(\varepsilon)} \circ \mathcal{E}_\beta{}^{-1} \circ \mathcal{S}_\beta[f] \end{aligned}\right\} \quad (13.2.38)$$

which the following diagram describes:

$$\begin{array}{ccc} \varphi_\beta & \xrightarrow{\mathcal{N}_\varepsilon} & \mathcal{N}_\varepsilon[\varphi_\beta] \\ {\scriptstyle \mathcal{S}_\beta{}^{-1}\mathcal{E}_\beta}\uparrow & & \downarrow{\scriptstyle \mathcal{E}_\beta{}^{-1}\mathcal{S}_\beta} \\ f_{\mathcal{S}} & \xrightarrow{\mathcal{N}^\beta} & \mathcal{N}^\beta[f_{\mathcal{S}}] \end{array}$$

A pattern similar to that depicted in Fig. 13.3 holds then in the space of the structure functions f defining the Hamiltonian. Note that this plays here the role played elsewhere (see e.g. Sect. 13.1) by the space referred to as the space of coupling constants. As the renormalization map is repeatedly applied, the system grows larger and thermodynamical functions computed at a temperature β near β_o approach corresponding values moving away on the unstable manifold. To capture them back, one let simultaneously $\beta \to \beta_o$. Clearly different thermodynamical functions involve different directions in the unstable manifold, and thus different rates as measured by the different eigenvalues $\lambda^{(n)} > 1$ in the spectrum of $\mathcal{D}\mathcal{N}_{c(\varepsilon)}(1)$. With the critical exponents defined generically for an observable A by

$$\lim_{\beta\to\beta_o} [\log \lim_{N\to\infty} \langle A\rangle_{\beta,f,\Lambda_N}][\log|\beta-\beta_o|]^{-1} \tag{13.2.39}$$

where the limit as $N \to \infty$ is controlled by the renormalization map; [Collet and Eckmann, 1978] compute the following values for some of the traditional critical exponents

$$\boxed{\begin{aligned} \beta &= [\log c(\varepsilon)]/[2\,\log\lambda^{(2)}(\varepsilon)] \\ \gamma &= [\log 2 - \log c(\varepsilon)]/[\log\lambda^{(2)}(\varepsilon)] \\ \delta &= [2\,\log\lambda^{(1)}(\varepsilon)]/[\log c(\varepsilon)] \\ \eta &= 1 + [\log c(\varepsilon)]/[\log 2] \end{aligned}} \quad . \tag{13.2.40}$$

Since $\lambda^{(1)}$ and $\lambda^{(2)}$ are computable functions of the coupling constant $c(\varepsilon)$, (13.2.40) gives formula for each of the critical exponents *separately*.

To make the connection with the scaling laws obtained in Sect. 13.1, we can eliminate $\lambda^{(1)} = 2c(\varepsilon)^{-\frac{1}{2}}$ and $\lambda^{(2)}$ from (13.2.40) to express two of these coefficients in terms of the other two, e.g.:

$$\boxed{\eta = \frac{4\beta+\gamma}{2\beta+\gamma} \quad \text{and} \quad \delta = \frac{\beta+\gamma}{\beta}} \quad . \tag{13.2.41}$$

The relations (13.2.41) are not new, but they are precisely on target: indeed, they also result from the elimination of α and ν from the general scaling

relations (13.1.6) and (13.1.13) with $d = 1$. Moreover, in the limit $\varepsilon \to 0$ and thus $c \to 2^{\frac{1}{2}}$ – but in this limit only – the critical exponents listed above reduce to the classical values: $\beta = \frac{1}{2}, \gamma = 1, \delta = 3$ which are the classical values of these exponents – see (11.3.17–19) – and $\eta = \frac{3}{2}$ which is not, a reflection of the price one must expect to pay ultimately for limiting one's attention to the 1−d hierarchical model.

With these results we close our overview of the renormalization idea as they apply without uncontrolled approximation to the one-dimensional hierarchical model.

Nevertheless, we should mention that much more is done in [Collet and Eckmann, 1978], in particular the analysis in the large – rather than the local analysis reviewed here – of the global flow under the renormalization map; this bears among other things on the existence and properties of the thermodynamical limit and distinct thermodynamical phases. For an updated discussion, see [Bleher and Major, 1987].

The Reader will also find in [Baker, Jr., 1990] and references therein, results pertaining to the extension of the method to *higher-dimensional* versions of the hierarchical model. One may certainly argue that the results obtained for the critical exponents of the one-dimensional hierarchical model do not match the empirical values obtained in real materials. This however is not the point of our argument. Indeed we wanted to *illustrate the method with a particular model* which can be analyzed in all mathematical rigor; which does not involve any prescription beyond the specification (13.2.15) of the Hamiltonian; and which provides values for individual exponents rather than just putative relations between them, thus providing a check that these relations pertain genuinely to the realm of statistical mechanics.

The usefulness of hierachical couplings is not limited to the analysis of classical models; for instance, a generalization to quantum oscillators is presented in [Albeverio, Kondratiev, and Kozitsky, 1997].

The value of the renormalization program itself extends beyond this particular exactly solvable model. Other, more realistic models have been amenable through various approximations to a treatment by renormalization methods to provide empirically reliable information [Wilson, 1971, Wilson and Fisher, 1972, Wilson and Kogut, 1974, Wilson, 1975, Fisher, 1974]. A personal perspective on that line of research is offered in [Fisher, 1998]. The probabilistic approach to the renormalization program – which we chose to follow here – was initiated by [Di Castro and Jona Lasinio, 1969]; a synthesis of its developments is presented in [Benfatto and Gallavotti, 1995], the broad sweep of which informs a wide array of related ideas and techniques.

The more traditional physics literature offers a somewhat bewildering range of more or less pragmatic texts, such as [Pfeuty and Toulouse, 1977, Patashinskii and Pokrovskii, 1979, Amit, 1984, Parisi, 1988, Goldenfeld, 1992, Cardy, 1996]. These either build – or are built on – analogies between the process of iterating the renormalization map, which we reviewed in this section to

account for the divergences that appear near the critical point, and a program originally invented for the purpose of dealing with the divergences of quantum field theory in four dimensions. Although the divergences themselves have different physical origins, the importance of formal relations between these two domains – as well as others, such as KAM theory – for a primary understanding of either of them is still debated; contrast for instance [Fisher, 1998], [Benfatto and Gallavotti, 1995] and the rich crop of supportive references cited there.

In order to place succintly these in a historical perspective, one traces back the idea of a renormalization "group" in QFT to [Stuekelberg and Petermann, 1953, Gell-Mann and Low, 1954] via [Bogoliubov and Shirkov, 1959][Chap. VIII]; for a vernacular presentation of the original formulations of QFT renormalization, see [Schweber, 1994]; and for an overview of the related mathematical issues there – just before the cross-over to the study of critical exponents in classical statistical mechanics – see [Hepp, 1969].

The scope of renormalization methods has become so encompassing during the last thirty years, that it seems bound to stay as an integral part of our understanding of the coexistence of the different scales on which the world operates ... and thus must be apprehended and understood. In the lattice spin systems we considered, there are at least three such scales: the microscopic scale given by the lattice spacing, the mesoscopic scale characterized by the correlation length, and the macroscopic scale on which the naked eye usually operates. For the hierarchical model of phase transitions and critical phenomena, this coexistence is manifestly built in the Hamiltonian itself; however, for most physical systems, the focusing on a particular scale requires insights and the mediation of approximations that are more difficult to control. The renormalization program can be viewed as providing the "coarse graining" necessary to bridge from the micro- to the meso-scopic scales.