The coborclism category 2Cob

The aim of today's talk is to complete the definition of the symmetric monoidal category 2Cob, and finally define 2D TQFTs as symmetric monoidal functors from 2Cob to vector spaces. The next lecture will prove the main theorem, that such functors are classified by commutative Frobenius algebras.

Outline

1. Reminder on cobordisms
2. Smooth structure on gluings
3. 2Cob is a category

1 Cobordisms (here, all manifolds are compact)

Let M be a smooth manifold with boundary ∂M. Recall that an orientation on M induces an orientation on ∂M. A cobordism from an oriented n-manifold S to an oriented n-manifold S' is an oriented $(n+1)$-manifold M together with a chosen decomposition of the boundary (as oriented manifolds)

$$\partial M = M_{in} \amalg M_{out}$$

and diffeomorphisms $\phi : S \longrightarrow M_{in}$, $\phi' : S' \longrightarrow M_{out}$ with ϕ orientation reversing and ϕ' orientation preserving. We write the data of the cobordism as a diagram

$$(M, \phi, \phi') : S \xhookrightarrow{\phi} M \xhookleftarrow{\phi'} S'.$$

Two cobordisms (M_1, ϕ_1, ϕ_1'), (M_2, ϕ_2, ϕ_2') are equivalent if there is an orientation preserving diffeomorphism $\psi : M_1 \longrightarrow M_2$ s.t. $\psi \circ \phi_1 = \phi_2$, $\psi \circ \phi_1' = \phi_2'$.

<u>Defn</u> The objects of <u>2Cob</u> are oriented closed 1-manifolds (including the empty space) and the morphism sets are

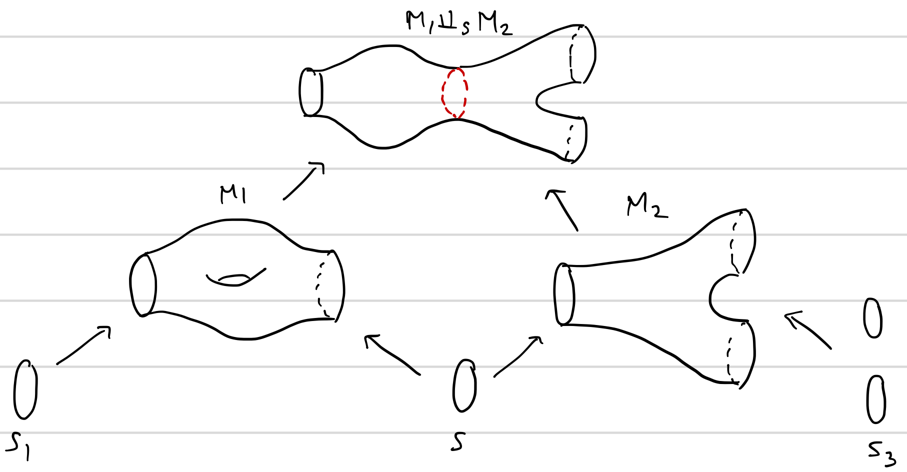
$$\mathrm{Hom}_{\underline{2Cob}}(S, S') = \{\text{cobordisms } S \longrightarrow S'\}/\text{equivalence}. \tag{2.1}$$

<u>Idea of composition</u> Given cobordisms $(M_1, \phi_1, \phi_1') : S_1 \longrightarrow S_2$ and $(M_2, \phi_2, \phi_2') : S_2 \longrightarrow S_3$ we consider the diagram in <u>Top</u> (the category of topological spaces) where the marked square is a pushout (in this case, a gluing)

$$\begin{tikzcd}
 & & M_1 \amalg_S M_2 & & \\
 & M_1 \arrow[ur, "\psi_1"] & & M_2 \arrow[ul, "\psi_2"'] & \\
S_1 \arrow[ur, "\phi_1"] & & S_2 \arrow[ul, "\phi_1'"'] \arrow[ur, "\phi_2"] \arrow[uu, phantom, "\circledast"] & & S_3 \arrow[ul, "\phi_2'"']
\end{tikzcd} \tag{2.2}$$

$$S_2 = S$$

For example,



(2.3)

In her talk Michelle showed $M_1 \amalg_S M_2$ is a compact topological manifold (i.e. it is locally homeomorphic to an open ball in $\mathbb{R}^2$) but we did not yet put a smooth structure on $M_1 \amalg_S M_2$. Assuming we have done so, we will define composition in <u>2Cob</u> via the rule

$$[(M_2,\phi_2,\phi_2')]\circ[(M_1,\phi_1,\phi_1')] := [(M_1 \amalg_S M_2, \psi_1\circ\phi_1, \psi_2\circ\phi_2')] \tag{3.1}$$

It will then remain to show that this composition rule is well-defined, associative and has units; at which point we will have constructed $\underline{2Cob}$.

Remark In the case of 1-manifolds and 2-dimensional cobordisms we can use results special to surfaces (e.g. every topological surface admits a smooth structure) but we prefer an exposition using an approach which generalises to the higher-dimensional cases.

(2) Smooth structure on gluings

Our reference is Hirsch "Differential Topology" henceforth denoted [H], primarily [H, §8.1 – §8.2]. The basic results on cobordisms we need all rely on collars, the theory of which we review now.

Theorem [H, 8.2.1] Let N be a manifold with $\partial N \neq \emptyset$. There exists a C^∞-embedding

$$F : \partial N \times [0,\infty) \longrightarrow N$$

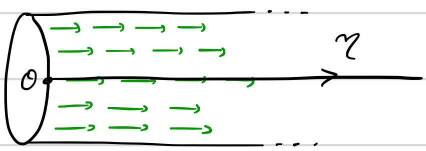
the "collar"

such that $F(x,0) = x$ for all $x \in \partial N$.

Sketch of proof There exists a C^∞ vector field X on a neighborhood U of ∂N in N which is nowhere tangent to ∂N and which points into N (cover ∂N with charts and use a partition of unity). For each $x \in \partial N$ we have an initial value problem $\gamma(0) = x$, $\gamma'(t) = X(\gamma(t))$ which by the usual yoga has a (maximal) solution on some interval $J(x) \subseteq [0,\infty)$, call it $\eta_x : J(x) \longrightarrow N$. Let $W \subseteq \partial N \times [0,\infty)$ be a neighborhood of ∂N on which the flow of X is defined, i.e. $\eta_x(t)$ is defined for all $(x,t) \in W$. Then by rescaling smoothly we get a C^∞ embedding $h : \partial N \times [0,\infty) \longrightarrow W$ leaving $\partial N \times 0$ fixed, and F is

$$\partial N \times [0,\infty) \xrightarrow{h} W \xrightarrow{\eta} N$$
$$(x,t) \longmapsto \eta_t(x). \qquad \square$$

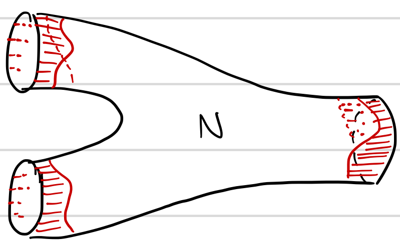
<u>Example</u> Let $N = S^1 \times [0,\infty)$ with $U = N$ and $X(\theta, x) = 1 \cdot \partial_x$.



The solutions are defined on $W = \partial N \times [0,\infty) = S^1 \times [0,\infty)$ (coordinates θ, t) and defined by $\eta_t(\theta) = (\theta, t)$. Hence F is the identity.

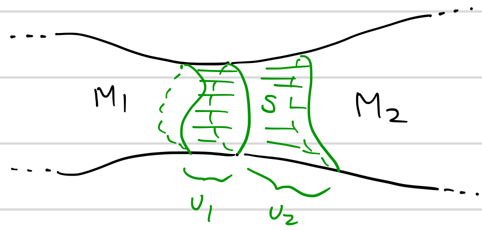
<u>Example</u> Same as above but with $X(\theta, x) = 2x \cdot \partial_x$, then $\eta_t(\theta) = (\theta, t^2)$ and $F(\theta, x) = (\theta, x^2)$ is the associated collar.

Obviously restricting F to some finite interval, e.g. $\partial N \times [0,1)$, gives a submanifold of N more recognisable as a collar, e.g.



Now let M_1, M_2 be n-manifolds together with a diagram

$$M_1 \xleftarrow{\phi_1'} S \xrightarrow{\phi_2} M_2$$



where ϕ_1', ϕ_2 are C^∞ embeddings identifying S with a connected component of ∂M_1 resp. ∂M_2.
We have defined $M_1 \amalg_S M_2$ as a topological n-manifold. Take collars

$$F_i : S \times [0,\infty) \longrightarrow M_i,$$

with images denoted $U_i \subseteq M_i$. Let $U := U_1 \amalg_S U_2$, an open neighbourhood of S in $M_1 \amalg_S M_2$.
We have a homeomorphism

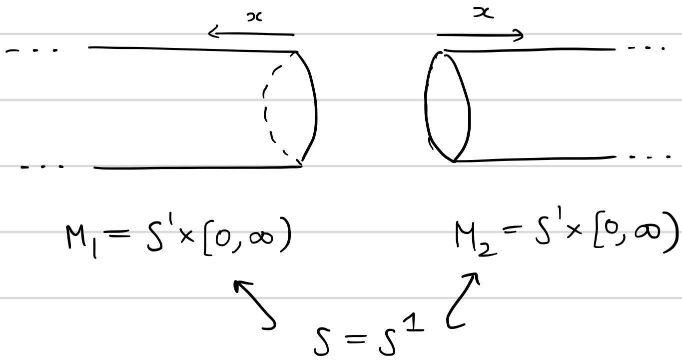
$$\begin{array}{c} \psi : S \times \mathbb{R} = (S \times (-\infty, 0]) \amalg_{S \times \{0\}} (S \times [0,\infty)) \\ F_1 \amalg_S F_2 \Big\downarrow \cong \\ U_1 \amalg_S U_2 = U \end{array}$$

identifying $S \times \{0\}$ with $S \subseteq M_1 \amalg_S M_2$.

Def^n Give $M_1 \amalg_S M_2$ the smooth structure given by all charts compatible with
(a) charts on $M_i \setminus \partial M_i$ coming from the smooth structure on M_i and
(b) charts on U induced by the smooth structure on $S \times \mathbb{R}$ via ψ.

Note that the charts of type (a), (b) are compatible by virtue of F_i being C^∞, and thus the canonical continuous maps $M_i \hookrightarrow M_1 \amalg_S M_2$ embed the M_i as submanifolds.

Remark If we choose our collars differently we may end up with different smooth structures on $M_1 \amalg_S M_2$. Consider



Let $F_1 = id : S^1 \times [0,\infty) \longrightarrow M_1$ and let $F_2^{(j)}$ be obtained from either of the vector fields in the earlier example, i.e. $F_2^{(1)}(\theta,t) = (\theta,t)$, $F_2^{(2)}(\theta,t) = (\theta,t^2)$. Then $U = S^1 \times \mathbb{R} = M_1 \amalg_S M_2$ and the homeomorphism $\psi^{(j)}$ is

$$\psi^{(1)} : S^1 \times \mathbb{R} \longrightarrow U \qquad \psi(\theta,x) = (\theta,x)$$

$$\psi^{(2)} : S^1 \times \mathbb{R} \longrightarrow U \qquad \psi(\theta,x) = \begin{cases} (\theta,x) & x \leq 0 \\ (\theta,x^2) & x > 0 \end{cases}$$

Clearly the smooth structure on $M_1 \amalg_S M_2 = S^1 \times \mathbb{R}$ induced by $\psi^{(1)}$ is the usual smooth structure, whereas $\psi^{(2)}$ induces a different (but clearly equivalent) smooth structure. Write $(M_1 \amalg_S M_2)^{\psi^{(j)}}$ for the two manifolds. The homeomorphism

$$M_1 \amalg_S M_2 \xrightarrow{id \amalg \sqrt{x}} M_1 \amalg_S M_2$$

clearly gives a diffeomorphism $(M_1 \amalg_S M_2)^{\psi^{(2)}} \longrightarrow (M_1 \amalg_S M_2)^{\psi^{(1)}}$.

Next we prove that this is a general phenomenon: as a manifold $M_1 \amalg_S M_2$ depends on the choice of collars, but only up to diffeomorphism. This is not as trivial as Kock's book makes it seem, and while Hirsch proves it as Theorem 2.1 of §8.2 of his book, this relies on an earlier theorem (1.9) which implicitly was a result on isotopy of collars.

<u>Def^n</u> Let V, M be manifolds. An <u>isotopy</u> from V to M is a smooth map $F: V \times I \longrightarrow M$ such that for each $t \in I$ the map $F_t: V \to M$, $x \mapsto F(x,t)$ is an embedding. We say the embeddings F_0 and F_1 are <u>isotopic</u>. When $V = M$ and each F_t is a diffeomorphism we call F a <u>diffeotopy</u> (or ambient isotopy).

<u>Theorem</u> (A) Let M be a manifold with boundary and $f, g: \partial M \times [0, \infty) \longrightarrow M$ collars. Then f, g are isotopic by an isotopy which fixes the boundary.

<u>Proof</u> Follows from Theorem 5.3 of Chapter 4 of Hirsch. $\square$

This holds just as well for any <u>component</u> of the boundary.

<u>Theorem</u> (B) Let $U \subseteq M$ be open and $A \subseteq U$ compact. Let $G: U \times I \longrightarrow M$ be an isotopy with G_0 the inclusion, s.t. $\{(G(x,t), t) \in M \times I \mid (x,t) \in U \times I\}$ is open in $M \times I$. Then there is a diffeotopy of M having compact support, which agrees with G on a neighborhood of $A \times I$.

<u>Proof sketch</u> (see Theorem 1.4 of Chapter 8 of Hirsch). Define $\hat{G}: U \times I \longrightarrow M \times I$
$\hat{G}(x,t) = (G(x,t), t)$.

For each $x \in U$, $\hat{G}|_{x \times I}$ is a curve in $M \times I$ and the tangents define a vector field X on the open set $\hat{G}(U \times I) \in M \times I$. Define H by $X(y,t) = (H(y,t), 1)$, $H: \hat{G}(U \times I) \to TM$. By partition of unity extend to $Z: M \times I \longrightarrow TM$ and define a diffeotopy $F: M \times I \longrightarrow M$ by

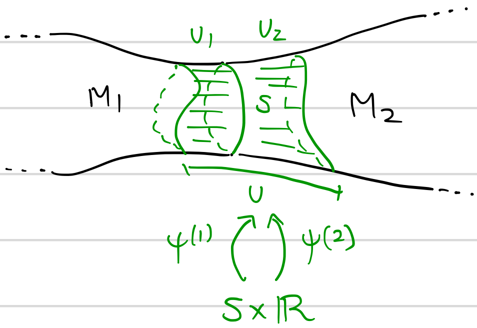
$$\frac{\partial F}{\partial t}(x,t) = Z(F(x,t), t). \quad \square$$

Corollary In the earlier notation, say we have collars

$$F_i^{(j)} : S \times [0,\infty) \longrightarrow M_i \qquad i,j \in \{1,2\}$$

with associated homeomorphisms

$$\psi^{(j)} : S \times \mathbb{R} \longrightarrow U = U_1 \amalg_S U_2$$



used to give the topological manifold $M_1 \amalg_S M_2$ smooth structures, denoted $(M_1 \amalg_S M_2)^{\psi(j)}$. There is a diffeomorphism $(M_1 \amalg_S M_2)^{\psi(1)} \cong (M_1 \amalg_S M_2)^{\psi(2)}$ which is the identity outside a neighborhood of the join S.

Proof By Theorem A there is an isotopy of $F_1^{(1)}$ with $F_1^{(2)}$. Precomposing with $(F_1^{(1)})^{-1} : U_1 \longrightarrow S \times [0,\infty)$ this is an isotopy $G : U_1 \times I \longrightarrow M_1$ of the inclusion $U_1 \hookrightarrow M_1$ with the map $F_1^{(2)} \circ (F_1^{(1)})^{-1} : U_1 \longrightarrow M_1$. By Theorem B (with $A = S \subseteq M_1$) there is a diffeomorphism $Q_1 : M_1 \longrightarrow M_1$ which agrees with $F_1^{(2)} \circ (F_1^{(1)})^{-1}$ on a neighborhood of S. Doing the same on the other side we produce a $Q_2 : M_2 \longrightarrow M_2$ and $Q_1 \amalg_S Q_2 : M_1 \amalg_S M_2 \longrightarrow M_1 \amalg_S M_2$ is a homeomorphism with the property that on some neighborhood of S it agrees with $(F_1^{(2)} \circ (F_1^{(1)})^{-1}) \amalg_S (F_2^{(2)} \circ (F_2^{(1)})^{-1})$.

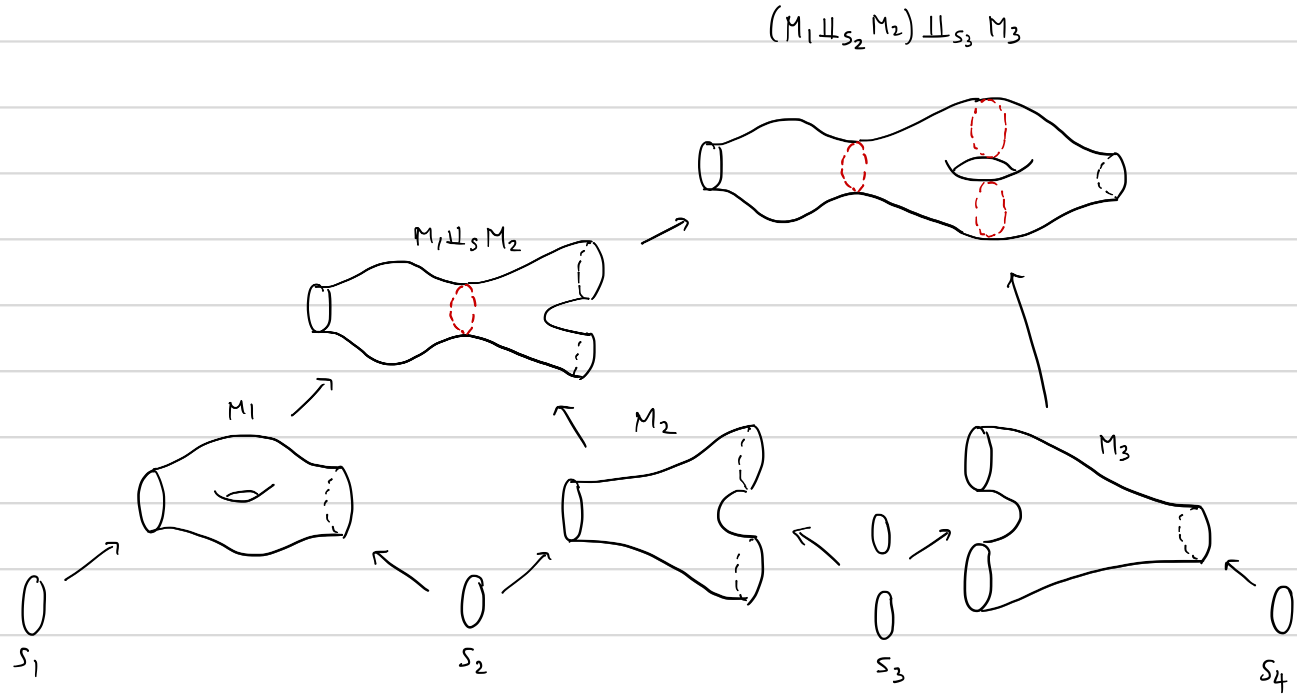
From

$$\begin{array}{ccc} & S \times \mathbb{R} = (S \times (-\infty, 0]) \amalg_{S \times \{0\}} (S \times [0,\infty)) & \\ \swarrow^{F_1^{(2)} \amalg_S F_2^{(2)}} & & \searrow^{F_1^{(1)} \amalg_S F_2^{(1)}} \\ U_1 \amalg_S U_2 = U & & U_1 \amalg_S U_2 = U \end{array}$$

it is clear that $Q_1 \amalg_S Q_2$ is the required diffeomorphism. □

③ 2Cob is a category

We have just shown the composition operation is well-defined, and it remains to prove it is associative and has identities. Associativity is trivial, as the gluings (and associated smooth structures) are disjoint, as in the diagram

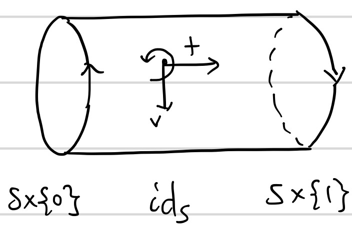


Lemma Given $(S, \omega_S) \in ob(\underline{2Cob})$ we define the cobordism (ω_S is the orientation)

$$id_S := S \times [0,1] \text{ with } \begin{array}{ccccc} S & \hookrightarrow & S\times[0,1] & \hookleftarrow & S. \\ x \mapsto (x,0) & & & (x,1) \leftarrow\!\shortmid x & \end{array}$$

Here id_S is given the orientation $\tilde{\omega}(v,+) = -\omega_S(v)$ for $v \in T_sS$, $s \in S$. Then for any morphism $M : S' \longrightarrow S$ or $N : S \longrightarrow S''$ in $\underline{2Cob}$ we have $id_S \circ M = M$ and $N \circ id_S = N$ (obviously as morphisms, i.e. up to diffeo).

Example $S = S^1$

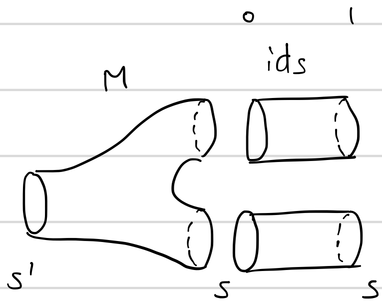


Note that the two boundaries receive opposite orientations from id_S.

$$\tilde{\omega}(v_{out}, v) = \tilde{\omega}(-,v) = +\tilde{\omega}(v,+) = -\omega_S(v)$$
$\Rightarrow$ induced orientation on $S\times\{0\}$ is $-\omega_S$.

<u>Proof of Lemma</u> We have

$$\mathrm{id}_S \circ M := (S \times [0,1]) \amalg_S M$$



We have to show there is a diffeomorphism Φ making the following diagram commute

$$\begin{tikzcd} & (S \times [0,1]) \amalg_S M & \\ S' \arrow[ur] \arrow[dr] & & S \arrow[ul] \arrow[dl] \\ & M \arrow[uu, dashed, "\Phi"'] & \end{tikzcd} \qquad (10.1)$$

embeds as $S \times \{1\}$

Let $f : S \times [0,\infty) \longrightarrow M$ be the collar chosen to define the gluing, and write C for the submanifold $f(S \times [0,1])$ of M, and $M' = M \setminus C$. There are <u>diffeo</u>morphisms

$$\partial M' \cong S \qquad C \cong S \times [0,1] \qquad M \cong C \amalg_S M'$$

and hence using the obvious diffeomorphism $[-1,0] \amalg_0 [0,1] \cong [-1,1] \cong [0,1]$ and the proof of associativity already given,

$$\begin{aligned} (S \times [0,1]) \amalg_S M &\cong (S \times [0,1]) \amalg_S (C \amalg_S M') \\ &\cong ((S \times [0,1]) \amalg_S C) \amalg_S M' \\ &\cong S \times ([0,1] \amalg_0 [-1,0]) \amalg_S M' \\ &\cong (S \times [0,1]) \amalg_S M' \\ &\cong M. \end{aligned}$$

and this clearly makes (10.1) commute. $\square$

We conclude that <u>2Cob</u> is a category.