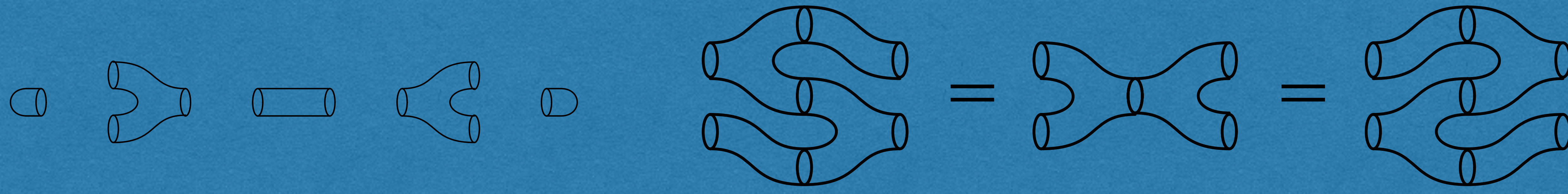
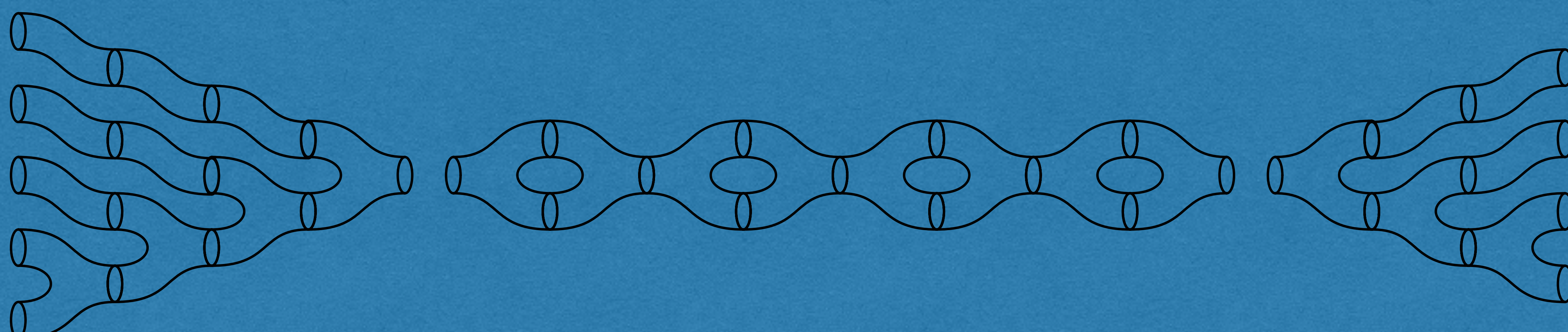


Topological Quantum Field Theory in two dimensions

Daniel Murfet



Q1: What is a TQFT?

Q2: Why do physicists care about TQFT?

Q3: Why do mathematicians care about TQFT?

- Atiyah “Topological quantum field theory” 1988
- Witten “Topological quantum field theory” 1988
- Witten “Quantum field theory and the Jones polynomial” 1989

TOPOLOGICAL QUANTUM FIELD THEORIES (1988)

by MICHAEL ATIYAH

To René Thom on his 65th birthday.

1. Introduction

In recent years there has been a remarkable renaissance in the relation between Geometry and Physics. This relation involves the most advanced and sophisticated ideas on each side and appears to be extremely deep. The traditional links between the two subjects, as embodied for example in Einstein's Theory of General Relativity or in Maxwell's Equations for Electro-Magnetism are concerned essentially with classical fields of force, governed by differential equations, and their geometrical interpretation. The new feature of present developments is that links are being established between *quantum* physics and *topology*. It is no longer the purely *local* aspects that are involved but their *global* counterparts. In a very general sense this should not be too surprising. Both quantum theory and topology are characterized by discrete phenomena emerging from a continuous background. However, the realization that this vague philosophical view-point could be translated into reasonably precise and significant mathematical statements is mainly due to the efforts of Edward Witten who, in a variety of directions, has shown the insight that can be derived by examining the topological aspects of quantum field theories.

(Atiyah 1988 continued)

The best starting point is undoubtedly Witten's paper [11] where he explained the geometric meaning of super-symmetry. It is well-known that the quantum Hamiltonian corresponding to a classical particle moving on a Riemannian manifold is just the Laplace-Beltrami operator. Witten pointed out that, for super-symmetric quantum mechanics, the Hamiltonian is just the Hodge-Laplacian. In this super-symmetric theory differential forms are bosons or fermions depending on the parity of their degrees. Witten went on to introduce a modified Hodge-Laplacian, depending on a real-valued function f. He was then able to derive the Morse theory (relating critical points of f to the Betti numbers of the manifold) by using the standard limiting procedures relating the quantum and classical theories.

(Atiyah 1988 continued)

Perhaps a few further comments should be made to reassure the sceptical reader. The quantum field theories of interest are inherently *non-linear*, but the non-linearities have a natural origin, e.g. coming from non-abelian Lie groups. Moreover there is usually some scaling or coupling parameter in the theory which in the limit relates to the classical theory. Fundamental topological aspects of such a quantum field theory should be independent of the parameters and it is therefore reasonable to expect them to be computable (in some sense) by examining the classical limit. This means that such topological information is essentially robust and should be independent of the fine analytical details (and difficulties) of the full quantum theory. That is why it is not unreasonable to expect to understand these topological aspects before the quantum field theories have been shown to exist as rigorous mathematical structures. In fact, it may well be that such topological understanding is a necessary pre-requisite to building the analytical apparatus of the quantum theory.

(Atiyah 1988 continued)

My comments so far have been of a conventional kind, indicating that there may be interesting topological aspects of quantum field theories and that these should be important for the relevant physics. However, we can reverse the procedure and use these quantum field theories as a conceptual tool to suggest new mathematical results. It is remarkable that this reverse process appears to be extremely successful and has led to spectacular progress in our understanding of *geometry in low dimensions*. It is probably no accident that the usual quantum field theories can only be renormalized in (space-time) dimensions $\leqslant 4$, and this is precisely the range in which difficult phenomena arise leading to deep and beautiful theories (e.g. the works of Thurston in 3 dimensions and Donaldson in 4 dimensions).

It now seems clear that the way to investigate the subtleties of low-dimensional manifolds is to associate to them suitable infinite-dimensional manifolds (e.g. spaces of connections) and to study these by standard linear methods (homology, etc.). In other words we use quantum field theory as a refined tool to study low-dimensional manifolds.

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Communications in
Mathematical
Physics

Topological Quantum Field Theory

Edward Witten*

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Abstract. A twisted version of four dimensional supersymmetric gauge theory is formulated. The model, which refines a nonrelativistic treatment by Atiyah, appears to underlie many recent developments in topology of low dimensional manifolds; the Donaldson polynomial invariants of four manifolds and the Floer groups of three manifolds appear naturally. The model may also be interesting from a physical viewpoint; it is in a sense a generally covariant quantum field theory, albeit one in which general covariance is unbroken, there are no gravitons, and the only excitations are topological.

Commun. Math. Phys. 121, 351–399 (1989)

Communications in
Mathematical
Physics

Quantum Field Theory and the Jones Polynomial *

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Abstract. It is shown that $2+1$ dimensional quantum Yang-Mills theory, with an action consisting purely of the Chern-Simons term, is exactly soluble and gives a natural framework for understanding the Jones polynomial of knot theory in three dimensional terms. In this version, the Jones polynomial can be generalized from S^3 to arbitrary three manifolds, giving invariants of three manifolds that are computable from a surgery presentation. These results shed a surprising new light on conformal field theory in $1+1$ dimensions.

In a lecture at the Hermann Weyl Symposium last year [1], Michael Atiyah proposed two problems for quantum field theorists. The first problem was to give a physical interpretation to Donaldson theory. The second problem was to find an intrinsically three dimensional definition of the Jones polynomial of knot theory. These two problems might roughly be described as follows.

2. Axioms for Topological Quantum Field Theories

Before embarking on the axioms it may be helpful to make a few comparisons with standard homology theories. We can describe such a theory as a functor $\mathbf{F}$ from the category of topological spaces (or of pairs of spaces) to the category say of Λ-modules, where Λ is some fixed ground ring (commutative, with 1, e.g. $\Lambda = \mathbf{Z}, \mathbf{R}$ or $\mathbf{C}$). This functor satisfies various axioms including

(i) a *homotopy* axiom, described geometrically by using " cylinders " $X \times I$,

(ii) an *additive* axiom asserting that, for disjoint sums, $F(X_1 \cup X_2) = F(X_1) \oplus F(X_2)$.

Note that (ii) implies, for the empty set $\emptyset$,

(ii)′ $F(\emptyset) = 0$.

The theories we shall describe will be somewhat similar, but with the following significant differences:

a) they will be defined only for *manifolds* of *a fixed dimension*,
b) the homotopy axiom is *strengthened* by replacing cylinders with general *cobordisms*,
c) the additive axiom is replaced by a *multiplicative* axiom, and correspondingly the empty set has value Λ rather than 0.

Physically *b)* is related to *relativistic* invariance while *c)* is indicative of the *quantum* nature of the theory.

We come now to the promised axioms. A topological quantum field theory (QFT), in dimension d defined over a ground ring Λ, consists of the following data:

(A) A finitely generated Λ-module $Z(\Sigma)$ associated to each oriented closed smooth d-dimensional manifold Σ,
(B) An element $Z(M) \in Z(\partial M)$ associated to each oriented smooth $(d+1)$-dimensional manifold (with boundary) M.

These data are subject to the following axioms, which we state briefly and expand upon below:

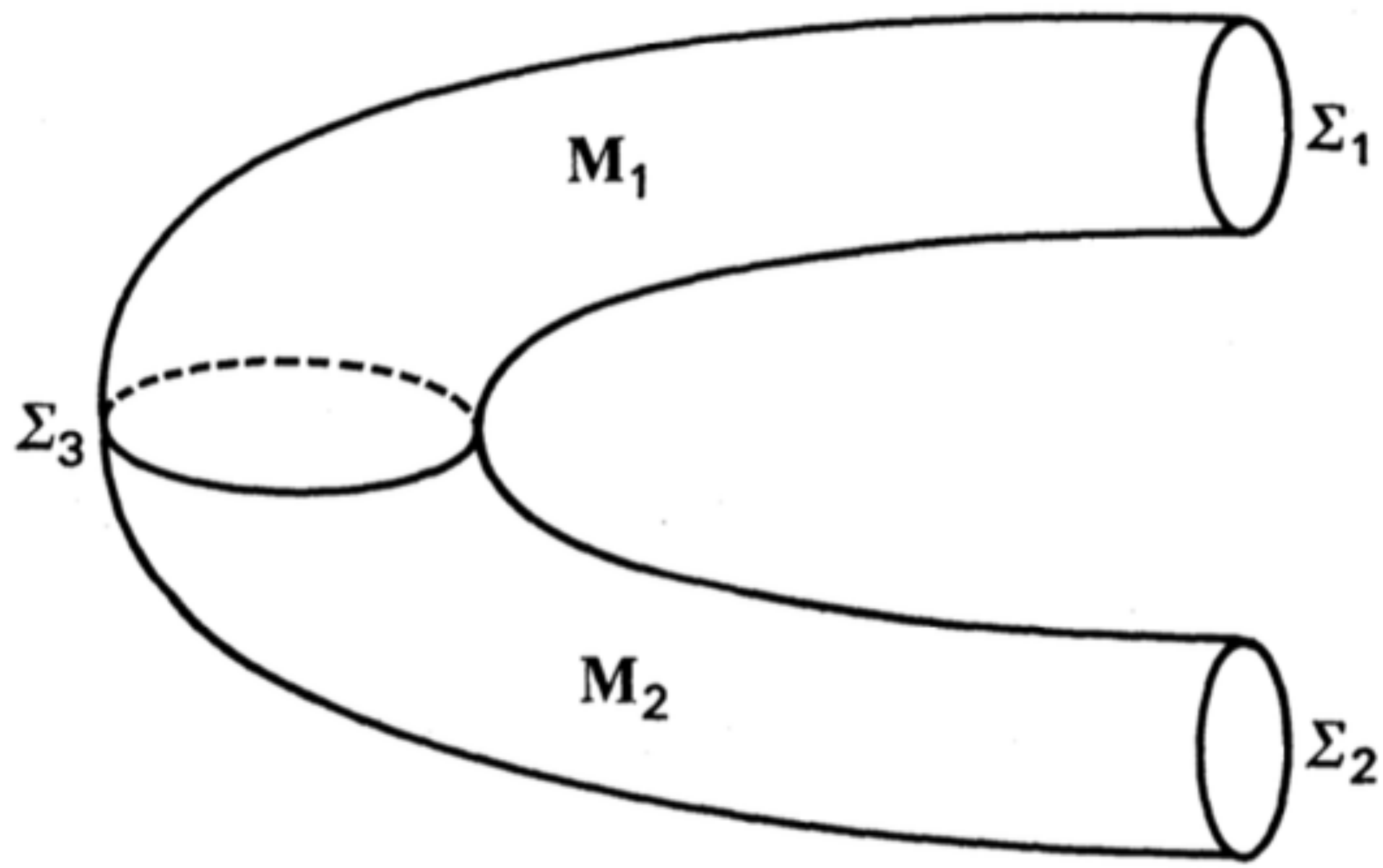
(1) Z is *functorial* with respect to orientation preserving diffeomorphisms of Σ and M,
(2) Z is *involutory*, i.e. $Z(\Sigma^*) = Z(\Sigma)^*$ where Σ^* is Σ with opposite orientation and $Z(\Sigma)^*$ denotes the dual module (see below),
(3) Z is *multiplicative*.

We now elaborate on the precise meaning of the axioms. (1) means first that an orientation preserving diffeomorphism $f: \Sigma \to \Sigma'$ induces an isomorphism $Z(f): Z(\Sigma) \to Z(\Sigma')$ and that $Z(gf) = Z(g)\,Z(f)$ for $g: \Sigma' \to \Sigma''$. Also if f extends to an orientation preserving diffeomorphism $M \to M'$, with $\partial M = \Sigma$, $\partial M' = \Sigma'$, then $Z(f)$ takes $Z(M)$ to $Z(M')$.

The multiplicative axiom (3) asserts first that, for disjoint unions,

(3a) $$Z(\Sigma_1 \cup \Sigma_2) = Z(\Sigma_1) \otimes Z(\Sigma_2).$$

Moreover if $\partial M_1 = \Sigma_1 \cup \Sigma_3$, $\partial M_2 = \Sigma_2 \cup \Sigma_3^*$ and $M = M_1 \cup_{\Sigma_3} M_2$ is the manifold obtained by glueing together the common Σ_3-component (see figure)

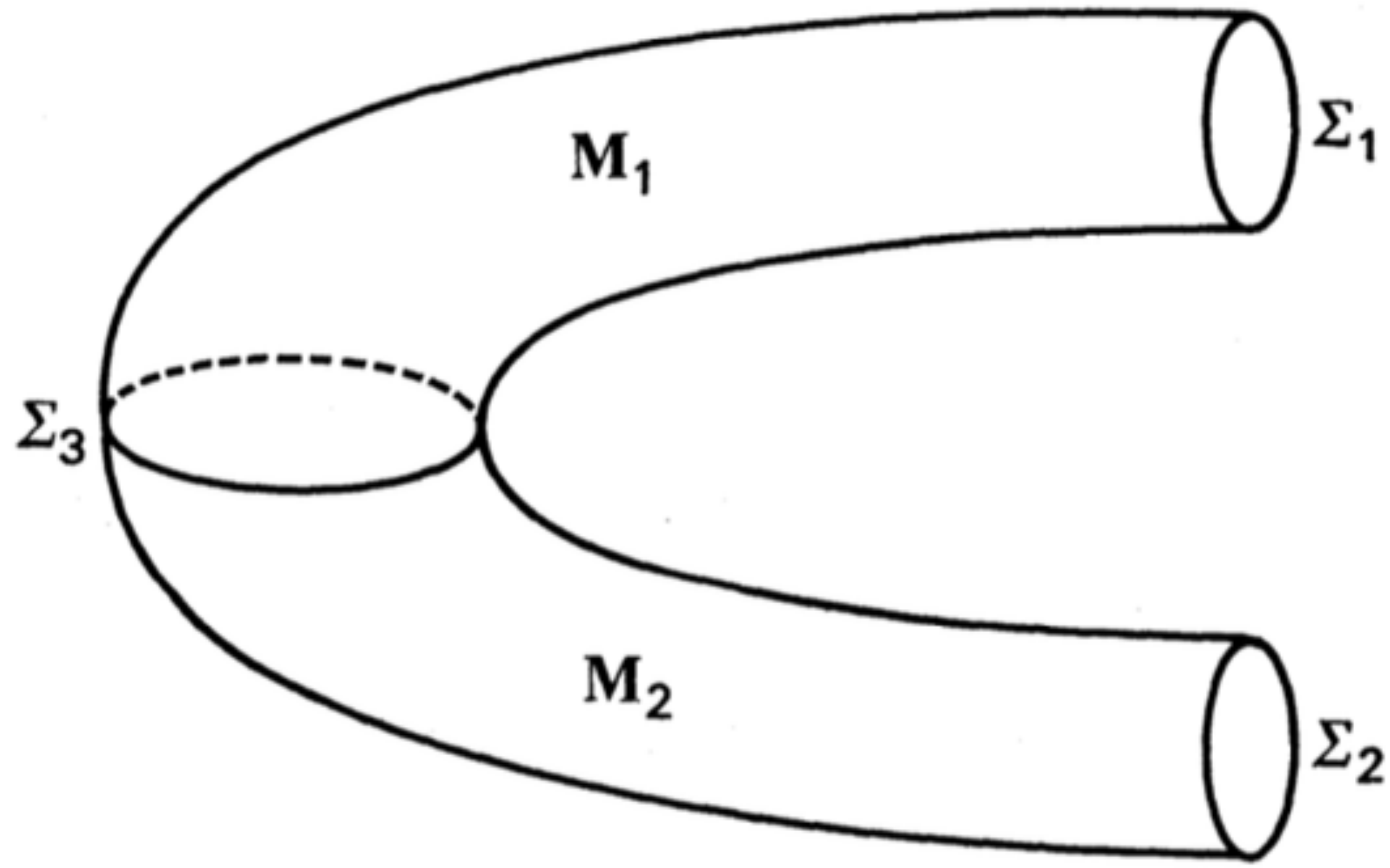


then we require:

(3b) $$Z(M) = \langle\, Z(M_1), Z(M_2) \,\rangle$$

where $\langle\ ,\ \rangle$ denotes the natural pairing

$$Z(\Sigma_1) \otimes Z(\Sigma_3) \otimes Z(\Sigma_3)^* \otimes Z(\Sigma_2) \to Z(\Sigma_1) \otimes Z(\Sigma_2).$$



An equivalent way of formulating (3*b*) is to decompose the boundary M into two components (possibly empty) so that

$$\partial M = \Sigma_1 \cup \Sigma_0^*;$$

then $Z(M) \in Z(\Sigma_0)^* \otimes Z(\Sigma_1) = \mathrm{Hom}(Z(\Sigma_0), Z(\Sigma_1))$. We can therefore view any *cobordism* M between Σ_0 and Σ_1 as inducing a linear transformation

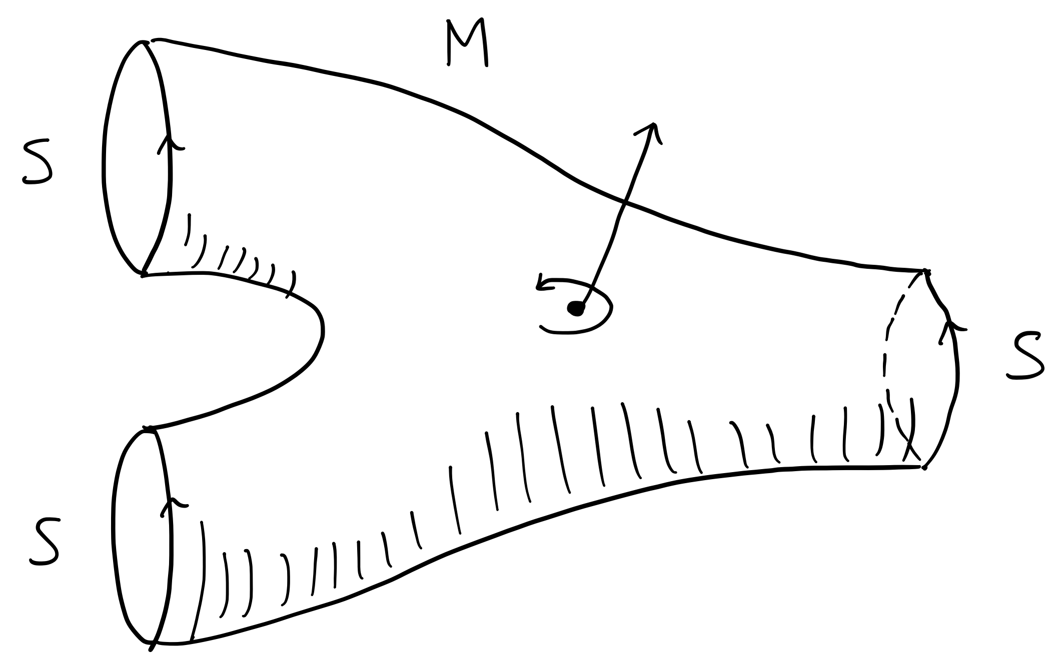
$$Z(M) : Z(\Sigma_0) \to Z(\Sigma_1).$$

Axiom (3*b*) asserts that this is *transitive* when we compose bordisms.

Σ is meant to indicate the physical space (e.g. $d = 3$ for standard physics) and the extra dimension in $\Sigma \times I$ is "imaginary" time. The space $Z(\Sigma)$ is the Hilbert space of the quantum theory and a physical theory, with a Hamiltonian H, will have an evolution operator e^{itH} or an "imaginary time" evolution operator e^{-tH}. The main feature of *topological* QFTs is that $H = 0$, which implies that there is no real dynamics or propagation, along the cylinder $\Sigma \times I$. However, there can be non-trivial "propagation" (or tunneling amplitudes) from Σ_0 to Σ_1 through an intervening manifold M with $\partial M = \Sigma_0^* \cup \Sigma_1$: this reflects the topology of M.

The reader may wonder how a theory with zero Hamiltonian can be sensibly formulated. The answer lies in the Feynman path-integral approach to QFT. This incorporates relativistic invariance (which caters for general $(d+1)$-dimensional "space-times") and the theory is formally defined by writing down a suitable Lagrangian—a functional of the classical fields of the theory. A Lagrangian which involves only first derivatives in time formally leads to a zero Hamiltonian, but the Lagrangian itself may have non-trivial features which relate it to the topology.

- **Definition:** A 2d (closed) topological field theory Z is a symmetric monoidal functor from the category of cobordisms 2Cob to vector spaces



$$M : S^{\otimes 2} \longrightarrow S \qquad S \cup S \overset{-}{\hookrightarrow} \partial M \hookleftarrow S$$

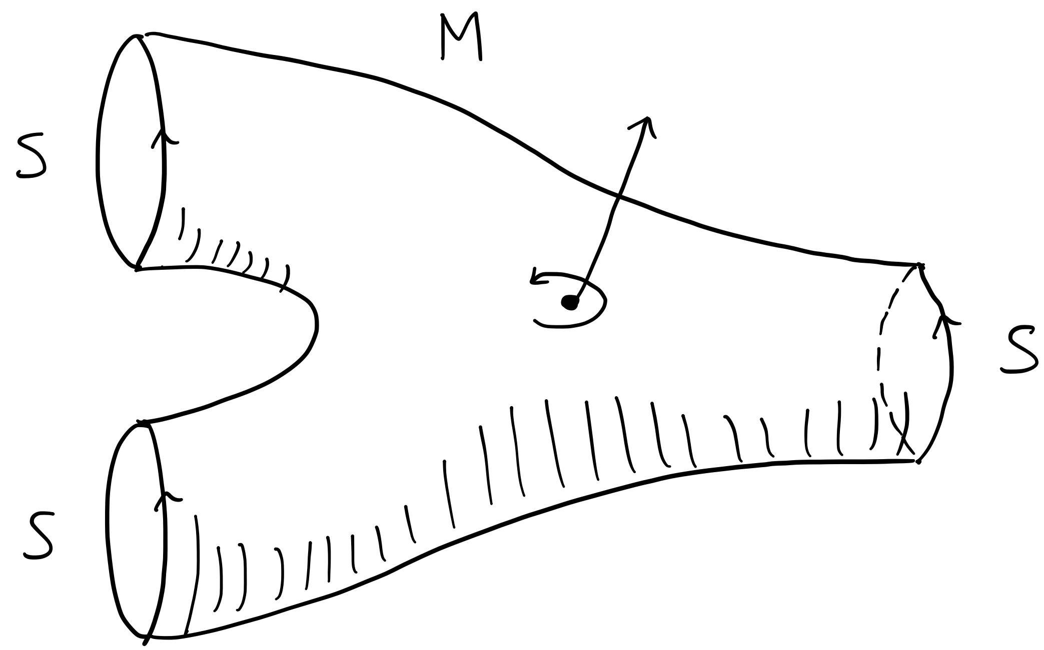
$$M : S \longrightarrow S^{\otimes 2} \qquad S \overset{-}{\hookrightarrow} \partial M \hookleftarrow S \cup S$$

$$Z(M) : V^{\otimes 2} \longrightarrow V$$

$$Z(M) : V \longrightarrow V^{\otimes 2}$$

$$V = Z(S)$$

- **Theorem:** there is a “bijection” between 2d closed TQFTs and commutative Frobenius algebras, given by sending Z to the algebra Z(S).



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$Z(M) : V^{\otimes 2} \longrightarrow V$

$Z(M) : V \longrightarrow V^{\otimes 2}$

$$V = Z(S)$$

- 3+1 things to understand
    - Bordism category (Michelle + …)
    - Frobenius algebras (Patrick + …)
    - Relevant category theory (…)
    - Examples in physics (…)

Beyond 2d closed TQFTs

- Higher dimensions (= higher categories)
- Enriched 2d bordisms:
  - Open/closed 2D TQFTs (=Calabi-Yau categories)
  - Defect 2D TQFTs (=Pivotal 2-categories)
  - Fully extended 2D TQFTs