A_∞-algebras and minimal models

The aim of this talk is to describe the minimal model construction and give an example (see the end for references).

Outline

1. A_∞-algebras.
2. The minimal model theorem
3. Example from singularity theory

1. A_∞-algebras

(from algebraic topology, "homotopy algebras")
see [K] for background

Def^N An A_∞-algebra is a $\mathbb{Z}$-graded vector space

$$A = \bigoplus_{n\in\mathbb{Z}} A^n$$

with operations $m_n : A^{\otimes n} \longrightarrow A$, $n \geq 1$, k-linear, degree $2-n$.

$$\begin{aligned} m_1 &: A \longrightarrow A && \deg +1 \\ m_2 &: A\otimes A \longrightarrow A && \deg 0 \\ m_3 &: A^{\otimes 3} \longrightarrow A && \deg -1 \\ &\vdots \end{aligned}$$

such that for $n \geq 1$

$$(*) \quad \sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t}\left(\mathbb{1}^{\otimes r} \otimes m_s \otimes \mathbb{1}^{\otimes t}\right) = 0$$

$$\begin{tikzcd} A^{\otimes n} \arrow[d, phantom, "\shortparallel"] \\ A^{\otimes r}\otimes A^{\otimes s}\otimes A^{\otimes t} \arrow[d, "1\otimes m_s\otimes 1"] \\ A^{\otimes r}\otimes A\otimes A^{\otimes t} \arrow[d, "m_{r+1+t}"] \\ A \end{tikzcd}$$

Def^N A morphism $f : A \to B$ is $f_n : A^{\otimes n} \longrightarrow B$ s.t. $m_1 f = f m_1, \dots$

Example $m_n = 0$, $n \geq 3$, $\circledast$ says (A, m_1, m_2) satisfies

write $ab = m_2(a \otimes b)$

$n=1$ $\quad m_1^2 = 0$

$n=2$ $\quad m_1(ab) = m_1(a)b + (-1)^{|a|} a\, m_1(b)$

$n=3$ $\quad m_2$ is associative.

$\therefore (A, m_1, m_2)$ is a DG-algebra

[Ex: $(\Omega^*(M), d, \wedge)$
M manifold]

In general m_3 is a homotopy $m_2(\mathbb{1} \otimes m_2) \simeq m_2(m_2 \otimes \mathbb{1})$.

Example For $d > 2$, $|\varepsilon| = 1$, $A^{(d)} = k[\varepsilon]/\varepsilon^2 = k \oplus k\varepsilon$

$$\left.\begin{array}{l} m_n = 0 \text{ for } n \notin \{2, d\} \\ m_2 = \text{multiplication} \\ m_d(\varepsilon \otimes \cdots \otimes \varepsilon) = (-1)^{d-1} \cdot 1 \end{array}\right\} A^{(d)} \text{ is a } \mathbb{Z}_2\text{-graded } A_\infty\text{-algebra}$$

Where do A_∞-algebras come from?

$\mathcal{T}$	triangulated category of interest, e.g. $D^b(\mathrm{coh}\, X)$, $\mathrm{hmf}(W)$, $\mathrm{Fuk}(Y)$, ...
$\uparrow H^0$	
D	DG enhancement of $\mathcal{T}$, e.g. by K-injective res.
$\cup$	
E	Generator (all objects "built from" E)
$\downarrow$	
$\mathrm{End}_D(E)$	DG-algebra (usually ∞-dim $/\mathbb{C}$)
$\downarrow$ taking the minimal model	
$(H^*\mathrm{End}_D(E), \{m_n\}_{n \geq 2})$	A_∞-algebra quasi-iso to $\mathrm{End}_D(E)$. (f.dim $/\mathbb{C}$) Knows "everything" about $\mathcal{T}$.

Why find minimal models?

- To study moduli (of $\mathcal{T}$ itself, or objects of $\mathcal{T}$),
- Topological string theory (boundary sector)

$= $ minimal, cyclic strictly unital A_∞-categories
(Herbst–Lazaroiu–Lerche, Costello)

A_∞, cyclicity, unit constraints for m's $=$ axioms of open-closed TCFT.
(monoidal functors $\mathrm{Rie} \longrightarrow \mathrm{Comp}_{\mathbb{C}}$)

Suspended forward compositions (technical point)

Let $(A, \{m_n\}_{n \geq 2})$ be an A_∞-algebra, define

$$s : A \longrightarrow A[1] \qquad s(a) = a$$

$$\pi_n : A^{\otimes n} \longrightarrow A, \quad \pi_n(a_1 \otimes \cdots \otimes a_n) = (-1)^{\sum_{i<j} |a_i||a_j|} m_n(a_n \otimes \cdots \otimes a_1)$$

$$c_n : A[1]^{\otimes n} \longrightarrow A[1], \qquad c_n = s \circ \pi_n \circ (s^{-1} \otimes \cdots \otimes s^{-1})$$

(degree $+1$) "suspended forward compositions"

⌜c_n are r_n in [L]⌟

Lemma The data $(A, \{c_n\}_{n \geq 2})$ satisfy for all n,

$$\sum_{r+s+t=n} c_{r+1+t} \circ (\mathbb{1}^{\otimes r} \otimes c_s \otimes \mathbb{1}^{\otimes t}) = 0. \qquad (*)$$

no signs!

These are the "suspended forward A_∞-relations"

2 The minimal model theorem. $\left(\overset{\infty.\text{d.}}{A} \underset{\text{qis}}{\cong} \overset{\text{f.d.}}{B} \right)$

Let (A, ∂, m) be a DG-algebra (suspended forward products).
i.e. s.t. $(A, \{\partial, m\})$ sat. $(*)$

A <u>strict homotopy retraction</u> of A is a $\mathbb{Z}$-graded vector space B and linear maps

$$H \circlearrowright A \underset{i}{\overset{p}{\rightleftarrows}} B$$

such that

(i) p, i are degree zero morphisms of cpxs (where B is given zero differential).

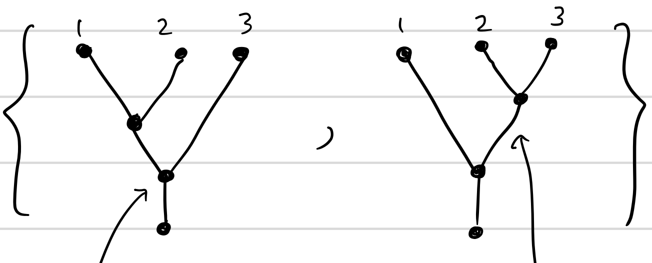
(ii) $p \circ i = 1_B$

(iii) $1_A - i \circ p = H\partial + \partial H$ (i.e. $i \circ p \simeq 1_A$)

$\Rightarrow$ $B \cong H^*(A, \partial)$, with a particular choice of how to project elements in A onto cocycles ($\partial ip(a) = i\partial p(a) = 0$).

$\mathcal{T}_n = \{$ oriented and connected planar trees, with $n+1$ leaves $\}$

inc. root

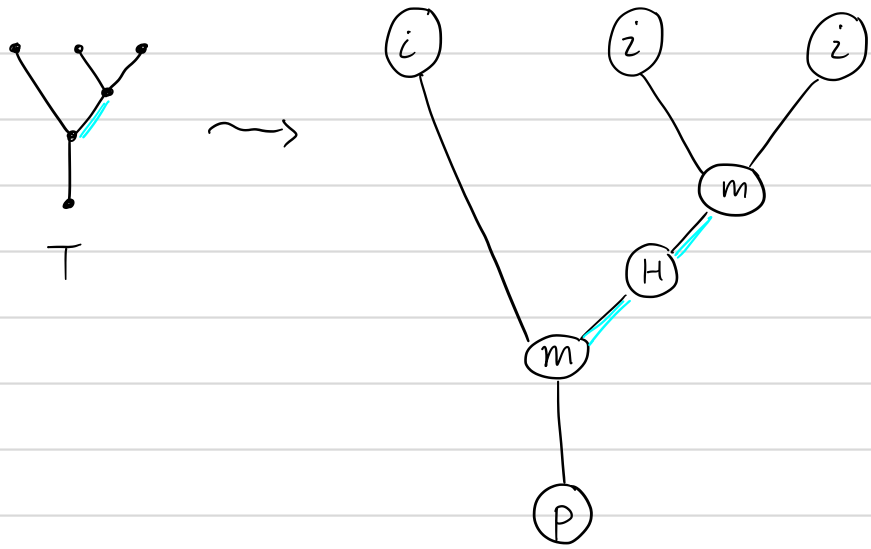
i.e. $\mathcal{T}_3 = \{$ 

$\}$

internal vertex (valency 3)

internal edge.

<u>Def^N</u> Given $T \in \mathcal{J}_n$ we define $\rho_T : B^{\otimes n} \longrightarrow B$ by example:



$$\begin{array}{c} B \otimes B \otimes B \\ \downarrow 1 \otimes i \otimes i \\ B \otimes A \otimes A \\ \downarrow 1 \otimes m \\ B \otimes A \\ \downarrow i \otimes H \\ A \otimes A \\ \downarrow m \\ A \xrightarrow[p]{} B \end{array}$$

$$\rho_T = (-1)^{\# \text{int. edges}} \; p \circ m \circ (i \otimes H) \circ (1_B \otimes m) \circ (1_B \otimes i \otimes i)$$

$$\boxed{\rho_n := \sum_{T \in \mathcal{J}_n} \rho_T : B[1]^{\otimes n} \longrightarrow B[1]}$$

<u>Theorem</u> (Minimal model) $(B, \{\rho_n\}_{n \geq 2})$ is an A_∞-algebra (with suspended forward products) and there is an A_∞-quasi-isomorphism

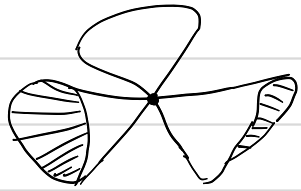
$$(A, m, \partial) \longrightarrow (B, \{\rho_n\}_{n \geq 2})$$

called the <u>minimal model</u>
(recall $B \cong H^* A$)

(3) Example

Let $W \in \mathfrak{m}^2 \subseteq \mathbb{C}[x_1, \dots, x_n]$ have an isolated singularity at O, i.e.

$$\dim_{\mathbb{C}} \mathbb{C}[\underline{x}] / (\partial_{x_1} W, \dots, \partial_{x_n} W) < \infty. \quad \left(\text{example: } W = x^2 y - y^3 + z^2 \quad (D4)\right)$$



Write $W = x_1 W^1 + \cdots + x_n W^n$, then

<u>Def^N</u> A_W is the $\mathbb{Z}_2$-graded DG-algebra with underlying module

$$A_W = \underbrace{\mathrm{End}_{\mathbb{C}}\left(\Lambda(\mathbb{C}\psi_1 \oplus \cdots \oplus \mathbb{C}\psi_n)\right)}_{\substack{\text{Clifford algebra, generated by } \psi_i = \psi_i \wedge -,\ \psi_i^* = \psi_i^* \lrcorner - \\ \text{satisfying } [\psi_i, \psi_j] = [\psi_i^*, \psi_j^*] = 0,\ [\psi_i, \psi_j^*] = \delta_{ij}.}} \otimes_{\mathbb{C}} \mathbb{C}[\underline{x}].$$

graded commutator

with the usual algebra structure, and differential

$$\partial(\alpha) = \Big[\underbrace{\textstyle\sum_i x_i \psi_i^* + \sum_i W^i \psi_i}_{\text{operator on } \Lambda(\mathbb{C}\psi_1 \oplus \cdots \oplus \mathbb{C}\psi_n) \otimes \mathbb{C}[x].} , \alpha \Big]$$

$$= \sum_i x_i [\psi_i^*, \alpha] + \sum_i W^i [\psi_i, \alpha].$$

<u>Example</u> $W = x^d \in \mathbb{C}[x]$, $d \geq 2$ $\qquad W = x \cdot x^{d-1}$, $W^1 = x^{d-1}$

$$\psi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \psi^* = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$A_W = \mathrm{End}_{\mathbb{C}}(\mathbb{C} \oplus \mathbb{C}\psi) \otimes_{\mathbb{C}} \mathbb{C}[x] \cong M_2(\mathbb{C}[x])$$

$$\partial(\alpha) = x[\psi^*, \alpha] + x^{d-1}[\psi, \alpha]$$

i.e. $\partial(\alpha) = x\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \alpha\right] + x^{d-1}\left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \alpha\right].$

<u>Theorem</u> (Dyckerhoff) There is a generator E for the category of matrix factorisations $\mathrm{hmf}(W)$ with DG-endomorphism algebra $\mathcal{A}_W$. Hence there is an equivalence

$$\underbrace{\mathrm{per}(\mathcal{A}_W)}_{\text{perfect } \mathcal{A}_W\text{-modules.}} \cong \mathrm{hmf}(W). \qquad \left(\cong \frac{D^b(\mathrm{coh}\, Z(W))}{\mathrm{Perf}(Z(W))} \right)$$

<u>The minimal model</u> of $(\mathcal{A}_W, \partial)$

$$S = \Lambda(\mathbb{C}\theta_1 \oplus \cdots \oplus \mathbb{C}\theta_n) \qquad (\text{DG-alg with } \partial = 0)$$

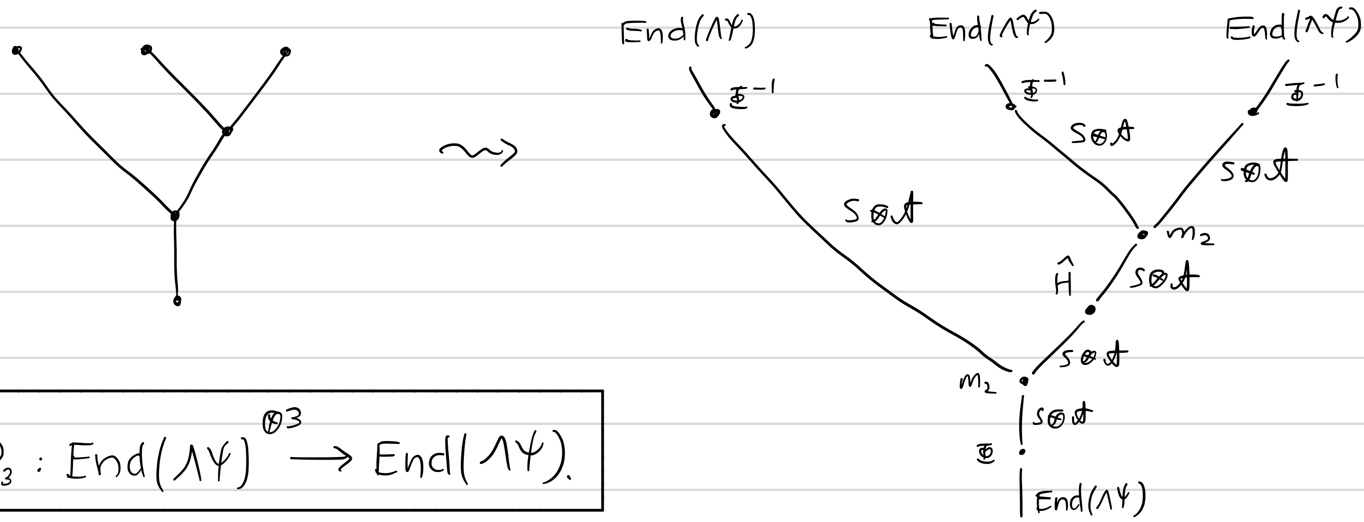
<u>Theorem</u> ([M]) There is a $\mathbb{C}$-linear strict homotopy retraction

$$\hat{H} \circlearrowright S \otimes_{\mathbb{C}} \mathcal{A}_W \underset{\Phi^{-1}}{\overset{\Phi}{\rightleftarrows}} \mathrm{End}_{\mathbb{C}}(\Lambda(\mathbb{C}\psi_1 \oplus \cdots \oplus \mathbb{C}\psi_n))$$

with $\Phi\Phi^{-1} = 1, \qquad \Phi^{-1}\Phi = 1 - [\partial, \hat{H}]$.

(precise form of $\Phi, \Phi^{-1}, \hat{H}$ from the perturbation lemma)

$\Longrightarrow$ the minimal model of $S \otimes_{\mathbb{C}} \mathcal{A}_W$ is $\mathrm{End}_{\mathbb{C}}(\Lambda\, \psi\text{'s})$ with higher multiplications computed by e.g. sums over trees like



$$\boxed{\rho_3 : \mathrm{End}(\Lambda\psi)^{\otimes 3} \longrightarrow \mathrm{End}(\Lambda\psi).}$$

$$\mathcal{M} = \Lambda(\mathbb{C}\psi_1^* \oplus \cdots \oplus \mathbb{C}\psi_n^*) \subseteq \mathrm{End}_{\mathbb{C}}(\Lambda(\mathbb{C}\psi_1 \oplus \cdots \oplus \mathbb{C}\psi_n))$$

embedded as contractions

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$[\psi_i, -]$ acts as contraction on $\mathcal{M}$.

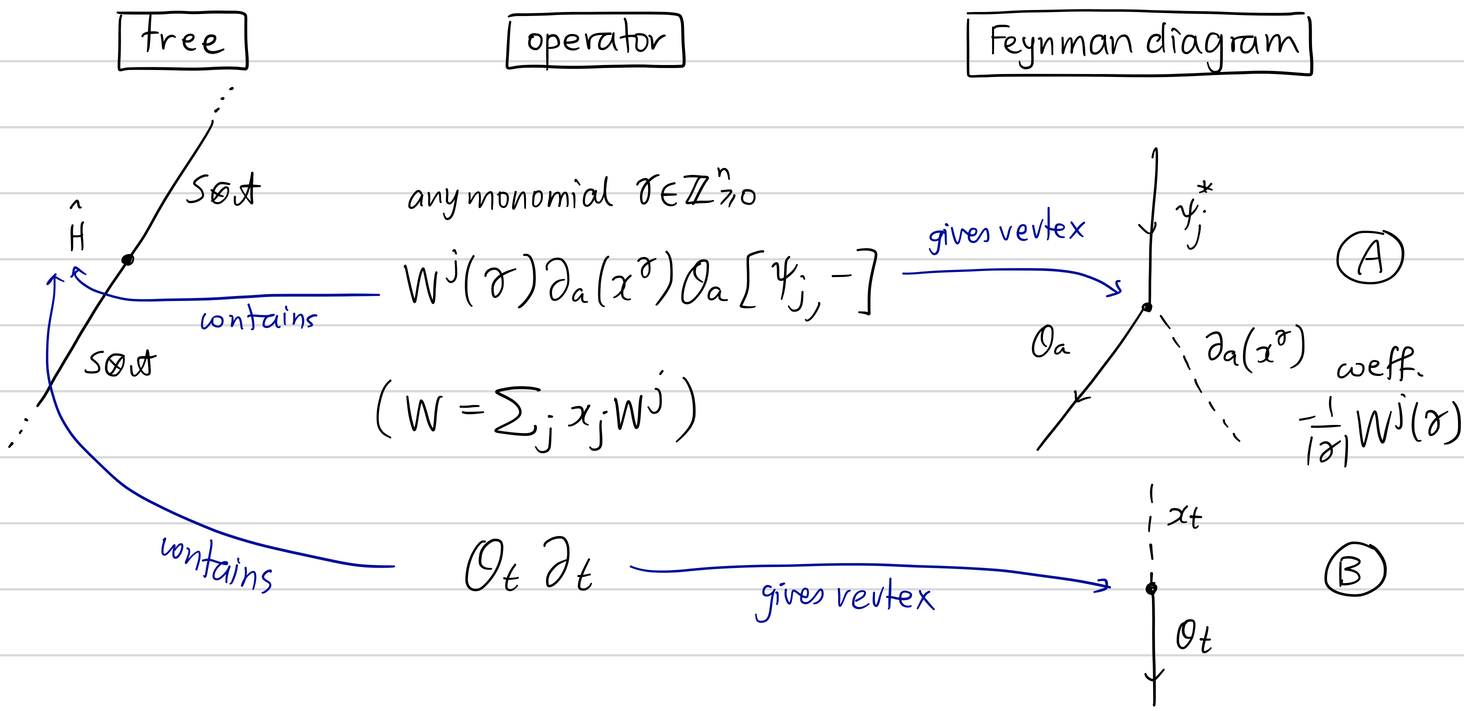
<u>Thm</u> All trees map $\mathcal{M}^{\otimes n} \subseteq \mathrm{End}(\Lambda\psi\text{'s})^{\otimes n}$ to $\mathcal{M} \subseteq \mathrm{End}(\Lambda\psi)$ and the induced A_∞-algebra $(\mathcal{M}, \{\rho_n|_{\mathcal{M}^{\otimes n}}\}_{n \geq 2})$ is the minimal model of $\mathcal{A}_W$ (for $W \in \mathfrak{m}^3$. If only $W \in \mathfrak{m}^2$ there are some small changes)

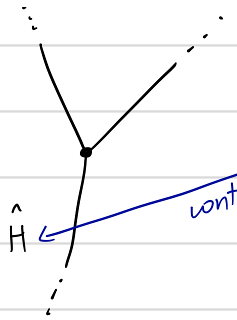
<u>Sketch of Feynman rules</u> (see [M2])

Higher products compute "scattering" of particles through trees. More precisely, Φ^{-1} and $\hat{H}$ can be written as sums of products of

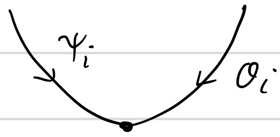
- bosonic creation and annihilation operators x, ∂_x
- fermionic creation and annihilation operators $\theta, \theta^*, [\psi_i, -]$

acting on $\Lambda(\mathbb{C}\theta\text{'s}) \otimes_{\mathbb{C}} \Lambda(\mathbb{C}\psi^*\text{'s}) \otimes_{\mathbb{C}} \mathbb{C}[x] \subseteq S \otimes \mathcal{A}$.



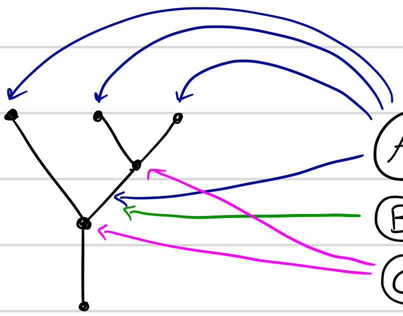


contains $[\psi_i, -] \otimes \theta_i^*$ gives vertex



C

Feynman rules computing $\rho_n : \mathcal{M}^{\otimes n} \longrightarrow \mathcal{M}$.



A vertices at inputs and internal edges (≥ 0)

B vertices – exactly one at each int. edge

C vertices at int. vertices (≥ 0)

To compute $\rho_n(\Psi_1 \otimes \cdots \otimes \Psi_n)$, sum over all trees T, and for each T compute scattering of $|\Psi\rangle_{in} = \Psi_1 \otimes \cdots \otimes \Psi_n$ with interaction vertices A, B, C (i.e. sum over all possible kinds of interaction).

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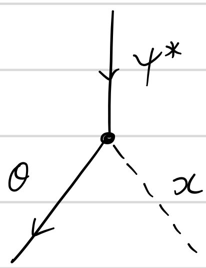
with vacuum bdry cond. for bosons x and fermions θ.

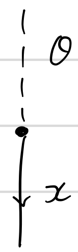
Example $W = x^3$. $\qquad W' = x^2$

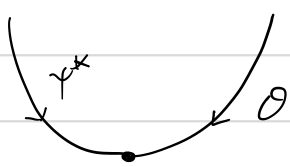
$$\mathcal{M} = \Lambda(\mathbb{C}\psi^*) = \mathbb{C} \oplus \mathbb{C}\psi^*$$

A $\qquad$ B $\qquad$ C

$\gamma = 2$
(i.e. x^2)







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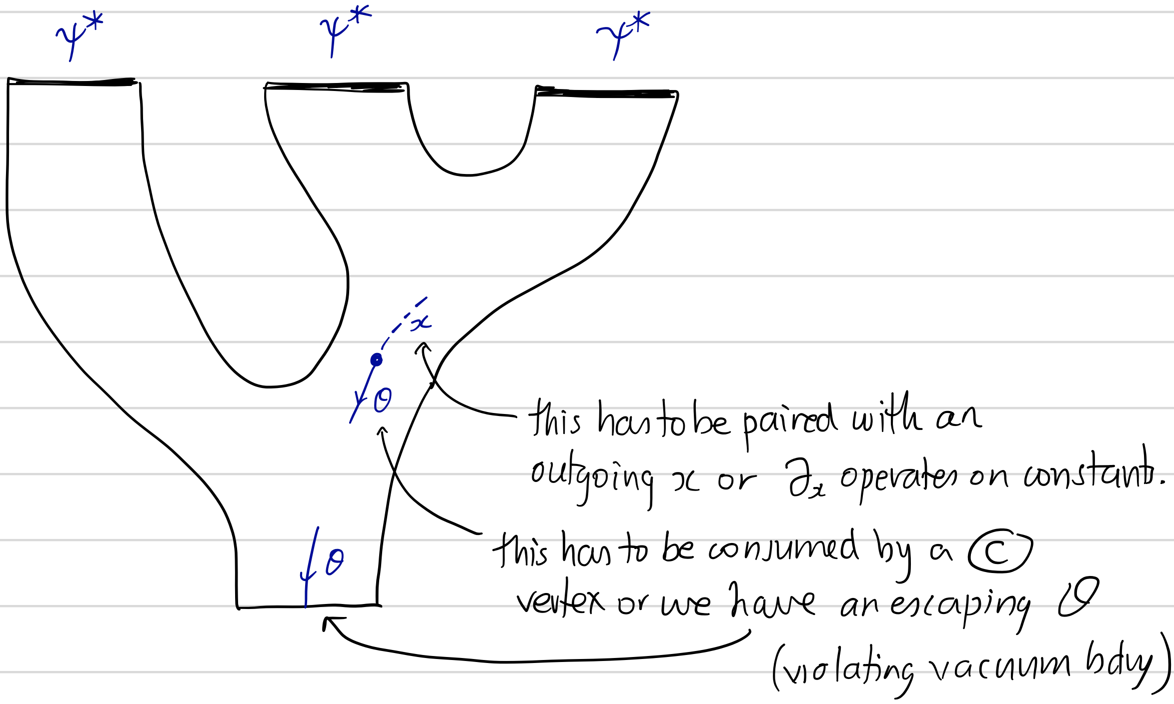
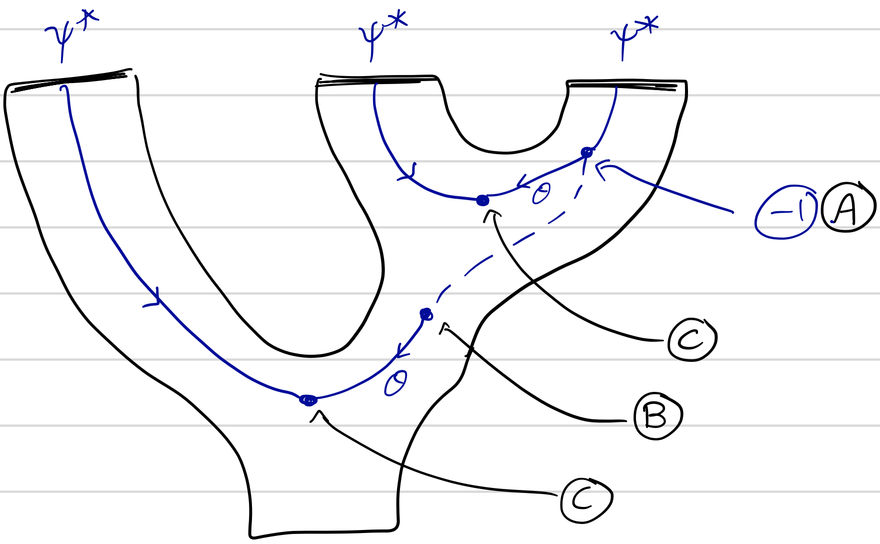
" $-\frac{1}{|\gamma|} W_j(\gamma)$

To compute $\rho_3(\psi^* \otimes \psi^* \otimes \psi^*) \in \mathcal{M}$,



Hence the only contributing collection of interaction vertices is



$\therefore$

$$\rho_3(\psi^* \otimes \psi^* \otimes \psi^*) = -1.$$

One can check only ρ_2, ρ_3 are nonzero and determines ρ_3. So the minimal model of A_W for $W = x^3$ is $A^{(3)}$ from earlier.

Conclusion

$$\begin{array}{lll}
\mathcal{T} = \mathrm{hmf}(W) & & \text{triangulated category} \\
\uparrow & & \\
\mathcal{D} = \mathrm{mf}(W) & & \text{DG-category} \\
\rotatebox{90}{$\in$} & & \\
E & & \text{generator} \\
\mathcal{A}_W = \mathrm{End}_{\mathcal{D}}(E) & & \text{DG-algebra} \\
\downarrow{\scriptstyle \text{minimal model}} & & \\
(\mathcal{M}, \{\rho_n\}_{n \geq 2}) & \longleftarrow & \text{for } W = x^d \text{ we find the}
\end{array}$$

A_∞-algebra $A^{(d)} = \left(k[\varepsilon]/\varepsilon^2, \rho_2, \rho_3\right)$ as the minimal model.

References

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[M2] D. Murfet work-in-progress github.com/dmurfet/ainfmf