A_∞-category categories of matrix factorisations via A_∞-idempotents

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- D.M., "Constructing A_∞-categories of matrix factorisations" arXiv: 1903.07211.

Outline

1. Topological Landau-Ginzburg models
2. Connections and residues
3. Idempotent finite A_∞-models of singularities

Preliminaries

Potentials Let k be a commutative $\mathbb{Q}$-algebra, then $W \in R = k[x_1, \dots, x_n]$ is called a potential if

(i) $\partial_{x_1} W, \dots, \partial_{x_n} W$ is quasi-regular

(ii) $R/(\partial_{x_1} W, \dots, \partial_{x_n} W)$ is a f.g. free k-module

(iii) the Koszul complex of $\partial_{x_1} W, \dots, \partial_{x_n} W$ is exact outside deg. 0.

f.g. free $\mathbb{Z}_2$-graded free R-module

Def^n (Eisenbud) The DG-category $\mathcal{A} = mf(R, W)$ has

- **objects** f. rank matrix factorisations of W, i.e. $X \ni d_X^2 = W \cdot 1_X$.
- **morphisms** $\mathcal{A}(X, Y) = (\mathrm{Hom}_R(X, Y), \alpha \mapsto d_Y \alpha - (-1)^{|\alpha|} \alpha d_X)$.

This is a $\mathbb{Z}_2$-graded DG-category over R.

Preliminaries

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(iii) the Koszul complex of $\partial_{x_1} W, \dots, \partial_{x_n} W$ is exact outside deg. 0.

Example (i) Isolated hypersurface singularities, e.g. $W(x_1, \dots, x_n)$ over $k = \mathbb{C}$, s.t.

$$\dim\left(\mathbb{C}[\underline{x}] / (\partial_{x_1} W, \dots, \partial_{x_n} W) \right) < \infty$$

(ii) $k = \mathbb{R}[x_1, \dots, x_c]$, $R = k[x_{c+1}, \dots, x_n] = \mathbb{R}[x_1, \dots, x_c, \dots, x_n]$

$W = \sum_{i=c+1}^{n} x_i^2$ is a potential relative to $k \longrightarrow R$.

Topological Landau-Ginzburg models

potential $W \in \mathbb{C}[x_1, \dots, x_n]$

closed 2D TQFT
- Jacobi ring with residue pairing

open-closed 2D TQFT
- $H^* mf(R,W)$ is a Calabi-Yau triangulated category
- Kapustin-Li, Herbst-Lazaroiu, Polishchuk-Vaintrob, M, Dyckerhoff

Atiyah classes $[dx, \nabla]$

$\{$ all potentials $/ \mathbb{C} \}$ $\rightsquigarrow$ 2D defect TQFT
- pivotal superbicategory
- Carqueville-Runkel, Carqueville-M

$[dx, \nabla^{nc}]$

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$[dx, \nabla^{nc}]$

Beyond TQFT: A_∞-models of $mf(R,W)$

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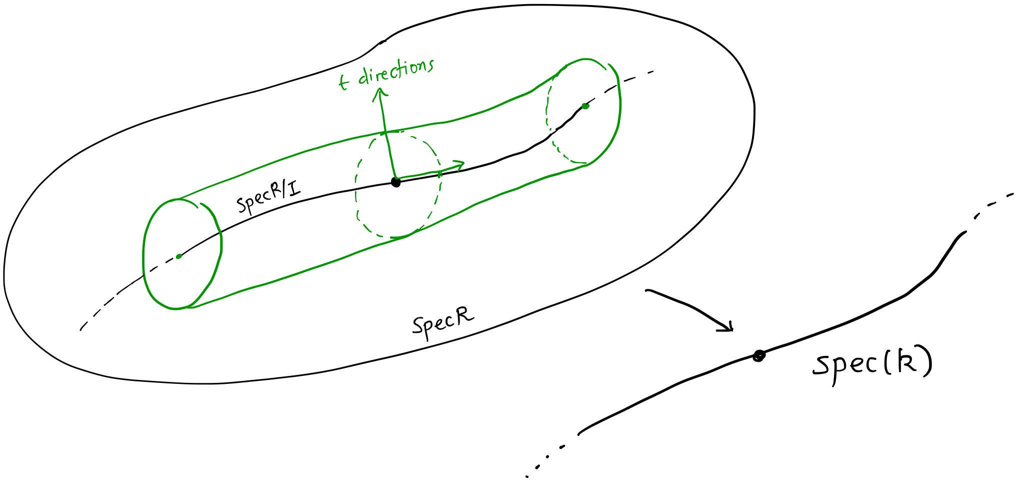
Connections and Residues

following J. Lipman "Residues and traces of differential forms via Hochschild homology", 1987.

Residues

Let $k \longrightarrow R$ be a morphism of commutative rings, and $t_1, \dots, t_n \in R$ be a quasi-regular sequence such that R/I is f.g. and projective over k, where $I = (t_1, \dots, t_n)$. Then the <u>Grothendieck residue</u> is

$$\operatorname{Res}_{R/k}\left[\frac{r\, dr_1 \cdots dr_n}{t_1, \dots, t_n}\right] \in k$$



Connections and Residues

Let k be a commutative $\mathbb{Q}$-algebra, R a k-algebra, and $t_1, \dots, t_n$ a quasi-regular sequence in R such that R/I is f.g. projective over k, $I = (t_1, \dots, t_n)$.

<u>Example</u> $W \in R = k[x_1, \dots, x_n]$ a potential and $I = (\partial_{x_1} W, \dots, \partial_{x_n} W)$, i.e. $t_i = \partial_{x_i} W$

<u>Lemma</u> (Formal tubular neighbourhood) Any k-linear section σ of $R \longrightarrow R/I$ induces an isomorphism of $k[[t_1, \dots, t_n]]$-modules

$$\sigma^* : R/I \otimes_k k[[t_1, \dots, t_n]] \longrightarrow \hat{R} \longleftarrow I\text{-adic}$$

$$r = \sum_{M \in \mathbb{N}^n} \sigma(r_M)\, t^M$$

<u>Upshot</u> There is a k-linear connection $\nabla : \hat{R} \longrightarrow \hat{R} \otimes_{k[t]} \Omega^1_{k[t]/k}$

$$\nabla(r) = \sum_{i=1}^{n} \underbrace{\sum_{M \in \mathbb{N}^n} M_i\, \sigma(r_M)\, t^{M - e_i}}_{\frac{\partial}{\partial t_i}(r)} dt_i$$

Connections and Residues

$$\sigma^* : R/I \otimes_k k[[t_1,\dots,t_n]] \xrightarrow{\cong} \hat{R}, \quad \nabla : \hat{R} \longrightarrow \hat{R} \otimes_{k[\underline{t}]} \Omega^1_{k[\underline{t}]/k}$$

Given $r, r_1, \dots, r_n \in R$

$$\begin{tikzcd}
\hat{R} \arrow[r, hook] \arrow[d, dashed, "{k[[\underline{t}]]\text{-linear}}"'] & \hat{R} \otimes_{k[\underline{t}]} \Omega^*_{k[\underline{t}]/k} \arrow[d, "{r[\nabla, r_1]\cdots[\nabla, r_n]}"] \\
\hat{R} \cong \hat{R} \otimes \Omega^n \arrow[r, hook] & \hat{R} \otimes_{k[\underline{t}]} \Omega^*_{k[\underline{t}]/k}
\end{tikzcd}$$

Connections and Residues

$$\sigma^* : R/I \otimes_k k[[t_1,\dots,t_n]] \xrightarrow{\cong} \hat{R}, \quad \nabla : \hat{R} \longrightarrow \hat{R} \otimes_{k[\underline{t}]} \Omega^1_{k[\underline{t}]/k}$$

Given $r, r_1, \dots, r_n \in R$

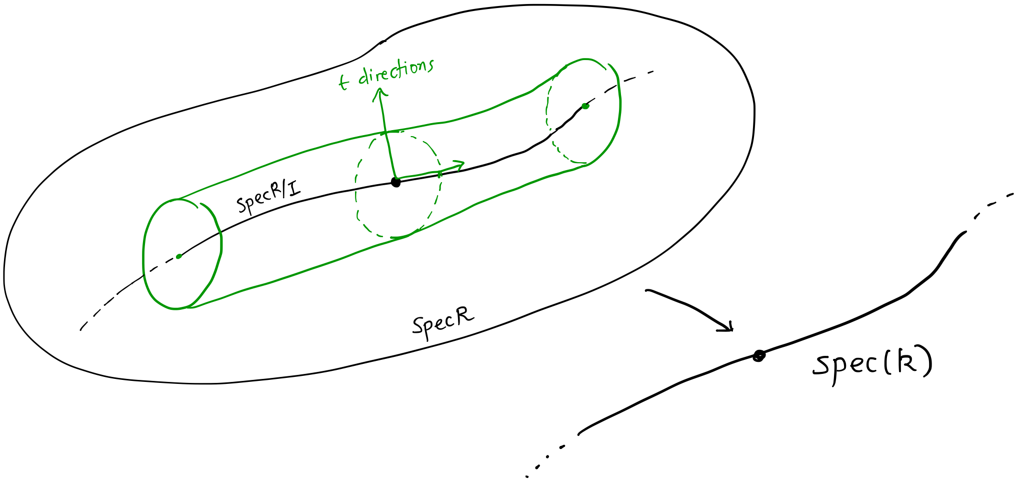
$$\begin{tikzcd}
R/I \cong \hat{R}/I\hat{R} \arrow[d, dashed, "k\text{-linear}"'] & \hat{R} \arrow[l] \arrow[r, hook] \arrow[d, dashed, "k[[\underline{t}]]\text{-linear}"'] & \hat{R} \otimes_{k[\underline{t}]} \Omega^*_{k[\underline{t}]/k} \arrow[d, "{r[\nabla, r_1]\cdots[\nabla, r_n]}"] \\
R/I \cong \hat{R}/I\hat{R} & \hat{R} \cong \hat{R} \otimes \Omega^n \arrow[l] \arrow[r, hook] & \hat{R} \otimes_{k[\underline{t}]} \Omega^*_{k[\underline{t}]/k}
\end{tikzcd}$$

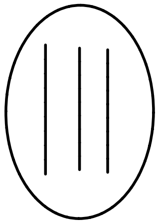
Connections and Residues

$$\sigma^* : R/I \otimes_k k[[t_1, \dots, t_n]] \xrightarrow{\cong} \hat{R}$$

$$\nabla : \hat{R} \longrightarrow \hat{R} \otimes_{k[t]} \Omega^1_{k[t]/k}$$

$$\mathrm{Res}_{R/k}\left[\frac{r\, dr_1 \cdots dr_n}{t_1, \dots, t_n}\right] = \mathrm{tr}_{R/I}\left(r[\nabla, r_1] \cdots [\nabla, r_n]\right) \in k$$



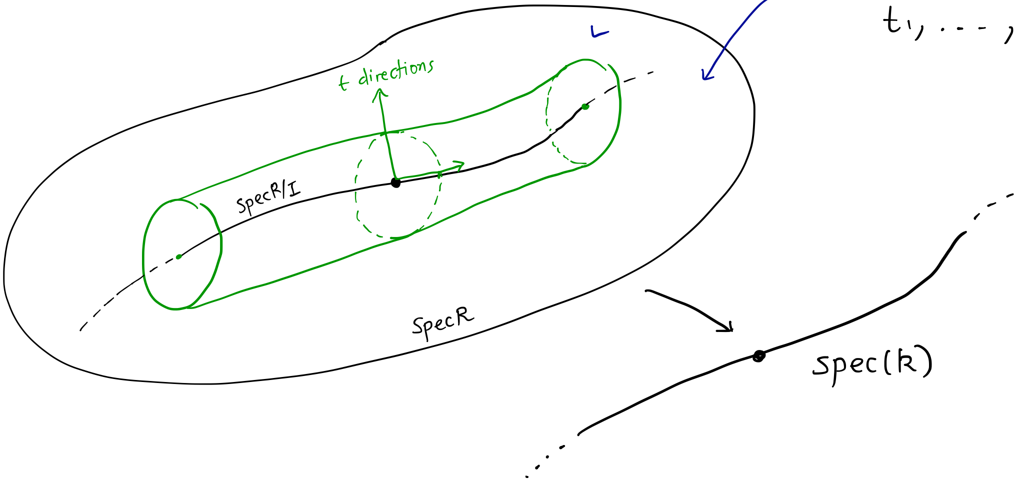


Idempotent A_∞-models of matrix factorisations

Analogy

$$\eta = \frac{r\,dr_1 \cdots dr_n}{t_1, \ldots, t_n}$$

$$\mathrm{Res}_{R/k}[\eta]$$

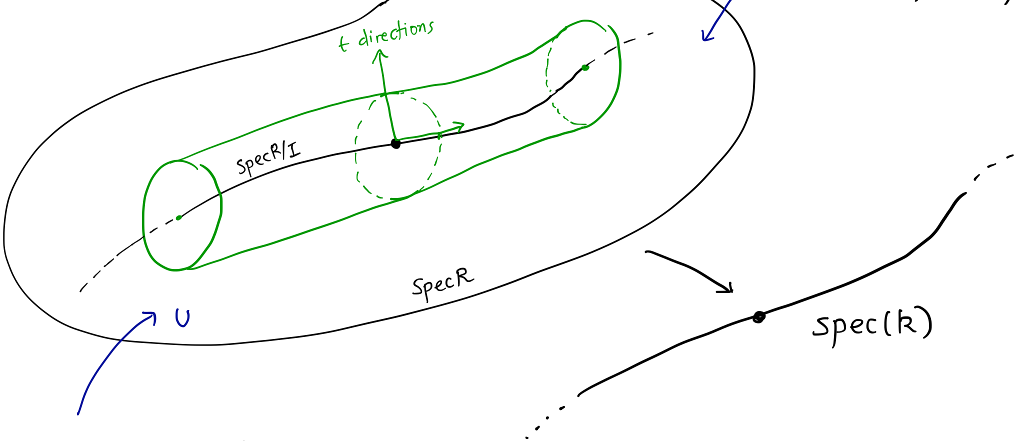


Analogy

$R = k[x_1, \dots, x_n]$, $W \in R$ a potential

$t_i = \partial_{x_i} W$, $I = (t_1, \dots, t_n)$

$$\eta = \frac{r\, dr_1 \cdots dr_n}{t_1, \dots, t_n}$$



$$\mathrm{Res}_{R/k}[\eta]$$

$$\mathcal{A} = mf(R, W)$$

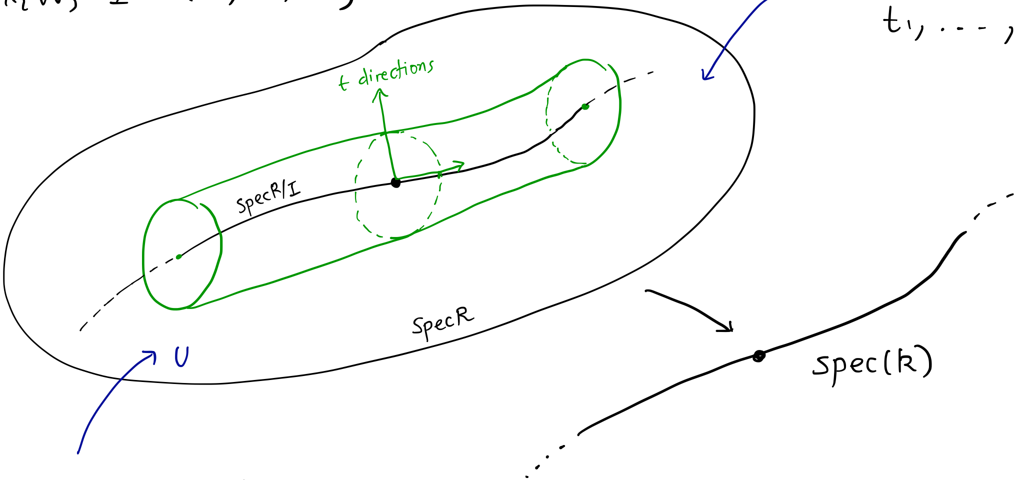
$$\mathcal{A}|_U \simeq 0$$

Analogy

$R = k[x_1, \dots, x_n]$, $W \in R$ a potential

$t_i = \partial_{x_i} W$, $I = (t_1, \dots, t_n)$

$$\eta = \frac{r\, dr_1 \cdots dr_n}{t_1, \dots, t_n} \longmapsto \mathrm{Res}_{R/k}[\eta]$$



$\mathcal{A} = mf(R, W)$

$\mathcal{A}|_U \simeq 0 \longmapsto$?

Preliminaries

<u>Def^n</u> A small $\mathbb{Z}_2$-graded A_∞-category $\mathcal{B}$ over k has a set $ob(\mathcal{B})$ of objects, and $\mathbb{Z}_2$-graded k-modules $\mathcal{B}(a,b)$ for all $a,b \in ob(\mathcal{B})$ equipped with <u>suspended forward compositions</u> which are odd linear maps

$$r_{a_0,\dots,a_n} : \mathcal{B}(a_0,a_1)[1] \otimes \cdots \otimes \mathcal{B}(a_{n-1},a_n)[1] \longrightarrow \mathcal{B}(a_0,a_n)[1]$$

r_n

satisfying the <u>A_∞-constraints</u> (without explicit signs)

$$\sum_{\substack{i\geq 0,\, j\geq 1\\ 1\leq i+j\leq n}} r_{a_0,\dots,a_i,a_{i+j},\dots,a_n} \circ \left(id_{a_0a_1} \otimes \cdots \otimes r_{a_i,\dots,a_{i+j}} \otimes \cdots \otimes id_{a_{n-1},a_n} \right) = 0$$

<u>Example</u> Any $\mathbb{Z}_2$-graded DG-category, $r_n = 0$ for $n \geq 3$.

Finite A_∞-model

Let $\varphi : k \longrightarrow R$ be a morphism of commutative rings, $\mathcal{A}$ a DG-category over R

Restriction of scalars gives a functor

$$\begin{tikzcd} A_\infty\text{-cat}(R) \arrow[d, "\varphi_*"] & \mathcal{A} \arrow[d, mapsto] & \\ A_\infty\text{-cat}(k) & \varphi_*(\mathcal{A}) \arrow[r, shift left, "F"] & \mathcal{B} \arrow[l, shift left, "G"] \end{tikzcd}$$

may have $r_1 \neq 0$

<u>Def^n</u> A <u>finite A_∞-model</u> of $\mathcal{A}$ over k is an A_∞-category $\mathcal{B}$ over k with all Hom-spaces f.g. projective$/k$, A_∞-functors F, G and A_∞-homotopies $F \circ G \overset{\infty}{\simeq} 1$, $G \circ F \overset{\infty}{\simeq} 1$.

Minimal A_∞-model

Let $\varphi : k \longrightarrow R$ be a morphism of commutative rings, $\mathcal{A}$ a DG-category over R

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<u>Def^n</u> A <u>minimal A_∞-model</u> of $\mathcal{A}$ over k is an A_∞-structure $\{r_n\}_{n \geq 1}$ on $H^*(\mathcal{A})$ with $r_1 = 0$, r_2 induced by composition, and A_∞-functors F, G and A_∞-homotopies $F \circ G \overset{\infty}{\simeq} 1$, $G \circ F \overset{\infty}{\simeq} 1$.

c.f. Remark 1.13 Seidel's book on Fukaya categories.

Idempotent finite A_∞-models

Let $\varphi: k \longrightarrow R$ be a morphism of commutative rings, $\mathcal{A}$ a DG-category over R

Restriction of scalars gives a functor

$$\begin{tikzcd} A_\infty\text{-cat}(R) \arrow[d, "\varphi_*"] \\ A_\infty\text{-cat}(k) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A} \arrow[d, mapsto] & & \\ \varphi_*(\mathcal{A}) \arrow[rr, shift left, "F"] & & \mathcal{B} \arrow[ll, shift left, "G"] \arrow[loop right, "E"] \end{tikzcd}$$

may have $r_1 \neq 0$.

<u>Def^n</u> An <u>idempotent finite A_∞-model</u> of $\mathcal{A}$ over k is an A_∞-category $\mathcal{B}$ with all Hom-spaces f.g. projective/k, A_∞-functors F, G, E as above and A_∞-homotopies $F \circ G \overset{\infty}{\simeq} E$, $G \circ F \overset{\infty}{\simeq} 1$. ($E = 1$ gives finite models)

Why finite models?

idempotent finite model	finite model	minimal model
$(\mathcal{B}, E_1, E_2, \ldots, r_1, r_2, r_3, \ldots)$	$(\mathcal{B}, r_1, r_2, r_3, \ldots)$	$(H^*(\mathcal{A}), r_2, r_3, \ldots)$
$\rotatebox{90}{$\simeq$}$ $A_\infty\text{-cat}(k)^\omega$	$\rotatebox{90}{$\simeq$}$	$\rotatebox{90}{$\simeq$}$
$(\mathcal{A}, 1, r_1, r_2)$	$(\mathcal{A}, r_1, r_2)$	$(\mathcal{A}, r_1, r_2)$

- String field theory (A_∞) vs. topological field theory (Δed).
- The information in higher products is important (e.g. for studying moduli).
- Landau-Ginzburg / Conformal Field Theory correspondence (LG/CFT)

⌜Physics refs. Lazaroiu (JHEP 2001), Lazaroiu-Roiban (JHEP 2002), Lazaroiu (2006), Carqueville-Dowdy-Recknagel (JHEP 2012), Carqueville-Kay (CMP 2012), Baumgartl-Brunner-Gaberdiel (JHEP 2007), Baumgartl-Wood (JHEP 2009), Knapp-Omer (JHEP 2006).⌟

Why finite models?

idempotent finite model	finite model	minimal model
$(\mathcal{B}, E_1, E_2, \ldots, r_1, r_2, r_3, \ldots)$	$(\mathcal{B}, r_1, r_2, r_3, \ldots)$	$(H^*(\mathcal{A}), r_2, r_3, \ldots)$
$\wr\wr$ $A_\infty\text{-cat}(k)^\omega$	$\wr\wr$	$\wr\wr$
$(\mathcal{A}, 1, r_1, r_2)$	$(\mathcal{A}, r_1, r_2)$	$(\mathcal{A}, r_1, r_2)$

computable for special objects

- String field theory (A_∞) vs. topological field theory (Δed).
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References (A_∞ and matrix factorisations)

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- T. Dyckerhoff, "Compact generators in categories of matrix factorisations" Duke Math. J. 2011.

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- D. Shklyarov, "Calabi-Yau structures on categories of matrix factorisations" J. of Geometry and Physics 2017.

- J. Tu, "Categorical Saito theory I: a comparison result" arXiv: 1902.04596.

An idempotent finite A_∞-model of mf

- $W \in R = k[x_1, \dots, x_n]$ a potential, $\mathcal{A} = mf(R, W)$.
- $I = (t_1, \dots, t_n)$, $t_i := \partial_{x_i} W$.
- $\nabla : \hat{R} \longrightarrow \hat{R} \otimes_{k[\underline{t}]} \Omega^1_{k[\underline{t}]/k}$ a connection

Def^n The <u>Atiyah class</u> of $\mathcal{A}(X,Y)$ relative to $k \longrightarrow k[\underline{t}]$ is

k[t]-linear
& closed

$$[d_{\mathcal{A}}, \nabla] : \mathcal{A}(X,Y) \longrightarrow \mathcal{A}(X,Y) \otimes_{k[\underline{t}]} \Omega^1_{k[\underline{t}]}[1]$$

There are induced closed, odd, k-linear operators

$$At_i := [d_{\mathcal{A}}, \tfrac{\partial}{\partial t_i}] : R/I \otimes_R \mathcal{A}(X,Y) \longrightarrow R/I \otimes_R \mathcal{A}(X,Y).$$

An idempotent finite A_∞-model of mf

- $W \in R = k[x_1, \ldots, x_n]$ a potential, $\mathcal{A} = mf(R, W)$.
- $I = (t_1, \ldots, t_n)$, $t_i := \partial_{x_i} W$.

(i.e. $(\mathcal{B}, E) \overset{\infty}{\simeq} (\mathcal{A}, 1_{\mathcal{A}})$

<u>Theorem</u> (M, '19) There is an idempotent finite A_∞-model of $\mathcal{A} = mf(R, W)$ with

- $\mathrm{ob}(\mathcal{B}) = \mathrm{ob}(\mathcal{A})$, $\mathcal{B}(X, Y) = R/I \otimes_R \mathcal{A}(X, Y)$
- r_1, r_2 on $\mathcal{B}$ induced by r_1, r_2 on $\mathcal{A}$, $E : \mathcal{B} \to \mathcal{B}$ is identity on objects
- $E_1 \simeq \gamma_n \cdots \gamma_1 \gamma_1^{\dagger} \cdots \gamma_n^{\dagger}$ where $\gamma_i = At_i = [d_{\mathcal{A}}, \frac{\partial}{\partial t_i}]$ are Atiyah classes and, for homotopies $[\lambda_i, d_{\mathcal{A}}] = t_i \cdot 1_{\mathcal{A}}$,

$$\gamma_i^{\dagger} = -\lambda_i - \sum_{m \geq 1} \sum_{q_1, \ldots, q_m} \frac{1}{(m+1)!} [\lambda_{q_m}, [\cdots [\lambda_{q_1}, \lambda_i] \cdots] At_{q_1} \cdots At_{q_m}$$

An idempotent finite A_∞-model of mf

DG $\quad A = mf(R,W) \quad R = k[x_1,\dots,x_n] \quad t_1 = \partial_{x_1}W, \dots, t_n = \partial_{x_n}W$

DG $\quad A_\theta = \bigwedge F_\theta \otimes_k mf(R,W) \otimes_R \hat{R} \quad F_\theta = k\theta_1 \oplus \cdots \oplus k\theta_n$

$\Omega^1_{k[t]/k} \cong F_\theta \otimes_k k[t]$

$A_\infty \quad \mathcal{B} = R/I \otimes_R mf(R,W)$

A_∞-homotopy equivalence $/k$

$G \circ F \simeq 1$

$$\begin{tikzcd} A \arrow[r] & A \otimes_R \hat{R} \arrow[r, hook] & A_\theta \arrow[r, shift left, "F"] \arrow[loop below, "e"] & \mathcal{B} \arrow[l, shift left, "G"] \arrow[loop below, "E"] \end{tikzcd}$$

homotopy equiv.

$e(\theta_i) = 0$

$E \leftarrow FeG$

<u>Theorem</u> $(\mathcal{B}, E)$ is an idempotent finite A_∞-model of $A \otimes_R \hat{R}$.

Proof sketch

$$\mathcal{A} = mf(R,W) \qquad R = k[x_1,\dots,x_n]$$
$$\mathcal{A}_\theta = \Lambda F_\theta \otimes_k mf(R,W) \otimes_R \hat{R}$$
$$\mathcal{B} = R/I \otimes_R mf(R,W)$$
$$\mathcal{A} \longrightarrow \mathcal{A} \otimes_R \hat{R} \hookrightarrow \mathcal{A}_\theta \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$$

Choose homotopies λ_i such that

$$[d_{\mathcal{A}}, \lambda_i] = t_i.$$

There is a strict homotopy retraction of complexes over k

$$\begin{tikzcd}
(\mathcal{A}_\theta(X,Y), d_{\mathcal{A}}) = \left(\Lambda F_\theta \otimes_k \mathrm{Hom}_R(X,Y) \otimes_R \hat{R},\ d_{\mathcal{A}}\right) \arrow[d, shift left, "e^{-\delta}"] \\
\left(\Lambda F_\theta \otimes_k \mathrm{Hom}_R(X,Y) \otimes_R \hat{R},\ d_{\mathcal{A}} + \sum_i t_i \theta_i^*\right) \arrow[u, shift left, "e^{\delta}"] \arrow[d, shift left, "\pi"] \\
(\mathcal{B}(X,Y), \overline{d_{\mathcal{A}}}) = \left(R/I \otimes_R \mathrm{Hom}_R(X,Y),\ \overline{d_{\mathcal{A}}}\right) \arrow[u, shift left, "\sigma_\infty"]
\end{tikzcd}$$

$$\delta = \sum_i \lambda_i \theta_i$$

by homological perturbation using connection ∇ $\longrightarrow \sigma_\infty$

$\pi \longleftarrow$ canonical projection

Proof sketch

$$\mathcal{A} = mf(R,W) \qquad R = k[x_1,\dots,x_n]$$
$$\mathcal{A}_\theta = \Lambda F_\theta \otimes_k mf(R,W) \otimes_R \hat{R}$$
$$\mathcal{B} = R/I \otimes_R mf(R,W)$$
$$\begin{tikzcd} \mathcal{A} \arrow[r] & \mathcal{A} \otimes_R \hat{R} \arrow[r, hook] & \mathcal{A}_\theta \arrow[r, shift left, "F"] & \mathcal{B} \arrow[l, shift left, "G"] \end{tikzcd}$$

$$\begin{tikzcd} \left(\mathcal{A}_\theta(X,Y),\, d_{\mathcal{A}}\right) \arrow[d, shift left, "\Phi", "\pi e^{-\delta}"'] \\ \left(\mathcal{B}(X,Y),\, \overline{d_{\mathcal{A}}}\right) \arrow[u, shift left, "\Phi^{-1}", "e^{\delta}\sigma_\infty"'] \end{tikzcd}$$

h.e.

$$\Phi \circ \Phi^{-1} = 1, \quad \Phi^{-1} \circ \Phi = 1 - [d_{\mathcal{A}}, H]$$

The A_∞-transfer (minimal model) theorem (Kadeshvili, Merkulov, Kontsevich–Soibelman and for our purposes Markl) constructs A_∞-products on $\mathcal{B}$ and A_∞-homotopy equivalences F, G

$$\begin{tikzcd} \mathcal{A}_\theta \arrow[r, shift left, "F"] & \mathcal{B} \arrow[l, shift left, "G"] \end{tikzcd}$$

$$F_1 = \Phi, \; G_1 = \Phi^{-1}, \; G \circ F \overset{\infty}{\simeq} 1$$

$r_1^{\mathcal{B}}, r_2^{\mathcal{B}}$ induced from $r_1^{\mathcal{A}}, r_2^{\mathcal{A}}$. $\square$

$$
\boxed{\begin{aligned}
\mathcal{A} &= mf(R, W) \qquad R = k[x_1, \dots, x_n] \\
\mathcal{A}_\theta &= \Lambda F_\theta \otimes_k mf(R, W) \otimes_R \hat{R} \\
\mathcal{B} &= R/I \otimes_R mf(R, W) \\
&\mathcal{A} \longrightarrow \mathcal{A} \otimes_R \hat{R} \hookrightarrow \mathcal{A}_\theta \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}
\end{aligned}}
$$

$$
\boxed{(\mathcal{A}_\theta(X, Y), d_{\mathcal{A}}) \underset{\Phi^{-1}}{\overset{\Phi}{\rightleftarrows}} (\mathcal{B}(X, Y), \overline{d_{\mathcal{A}}})}
$$

h.e.

transfer A_∞-structure

$$
\begin{aligned}
\mathcal{A}_\theta(X, Y) &= \Lambda F_\theta \otimes_k \mathrm{Hom}_R(X, Y) \otimes_R \hat{R} \\
&\cong \Lambda F_\theta \otimes_k \mathrm{Hom}_k(\tilde{X}, \tilde{Y}) \otimes_k \hat{R} \\
&\cong \Lambda F_\theta \otimes_k \mathrm{Hom}_k(\tilde{X}, \tilde{Y}) \otimes_k R/I \otimes_k k[[t_1, \dots, t_n]] \\
&\cong \Lambda F_\theta \otimes_k \mathcal{B}(X, Y) \otimes_k k[[t_1, \dots, t_n]] \supset \mathcal{B}(X, Y)
\end{aligned}
$$

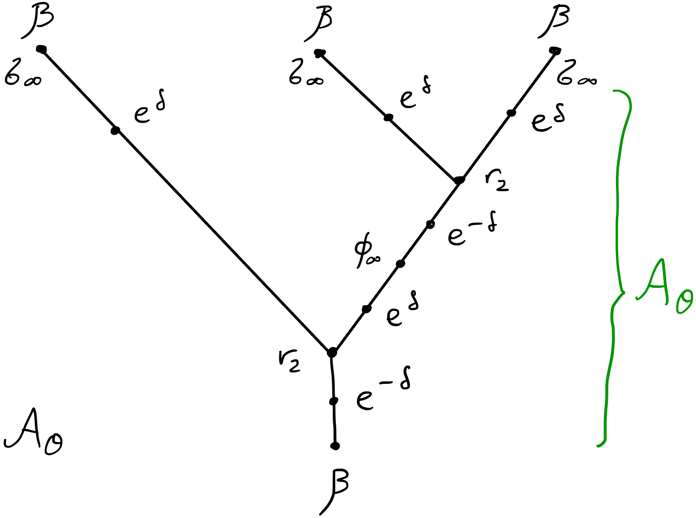
(choose bases for X, Y i.e. $X \cong \tilde{X} \otimes_k R$)

$$
\nabla = \sum_i \theta_i \frac{\partial}{\partial t_i} \qquad \zeta(\omega \otimes \alpha \otimes f) = \frac{1}{|\omega| + |f|} \omega \otimes \alpha \otimes f.
$$

$$\boxed{\begin{array}{l} \mathcal{A} = mf(R,W) \quad R = k[x_1,\dots,x_n] \\ \mathcal{A}_\theta = \bigwedge F_\theta \otimes_k mf(R,W) \otimes_R \hat{R} \\ \mathcal{B} = R/I \otimes_R mf(R,W) \\ \mathcal{A} \longrightarrow \mathcal{A} \otimes_R \hat{R} \hookrightarrow \mathcal{A}_\theta \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B} \end{array}}$$

$$At_{\mathcal{A}} := [\nabla, d_{\mathcal{A}}]$$

(Atiyah class of $\mathcal{A}$)



$$\sigma_\infty = \sum_{m \geq 0} (-1)^m (\zeta At_{\mathcal{A}})^m \sigma : \mathcal{B} \longrightarrow \mathcal{A}_\theta$$

$$\phi_\infty = \sum_{m \geq 0} (-1)^m (\zeta At_{\mathcal{A}})^m \zeta \nabla : \mathcal{A}_\theta \longrightarrow \mathcal{A}_\theta$$

$$\delta = \sum_{m \geq 0} \lambda_i \theta_i^* : \mathcal{A}_\theta \longrightarrow \mathcal{A}_\theta$$

$$r_3 : \mathcal{B}[1]^{\otimes 3} \longrightarrow \mathcal{B}[1]$$

$At_{\mathcal{A}}$, δ rewritten using σ^*

Feynman diagrams

Suppose $X = \Lambda F_\xi \otimes_k R$, $Y = \Lambda F_\eta \otimes_k R$ are Koszul-type MFs.

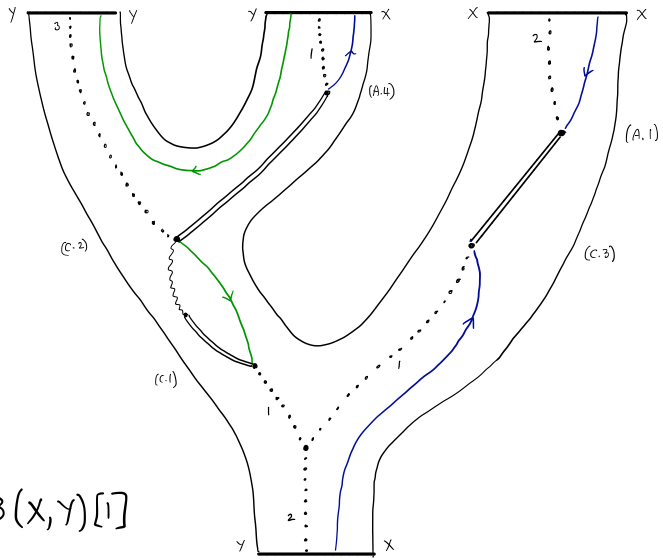
$$\underbrace{\Lambda(F_\theta \oplus F_\xi^* \oplus F_\eta) \otimes_k R/I \otimes_k k[1 \pm 1]}_{A_\theta(X,Y),\ \text{interior of trees}} \supset \underbrace{\Lambda(F_\xi^* \oplus F_\eta) \otimes_k R/I}_{\mathcal{B}(X,Y),\ \text{exterior}}$$

- Apart from ζ all operators involved in computing A_∞-products can be written as polynomials in creation and annihilation operators.

- Feynman diagrams organise reduction of such trees to normal form.

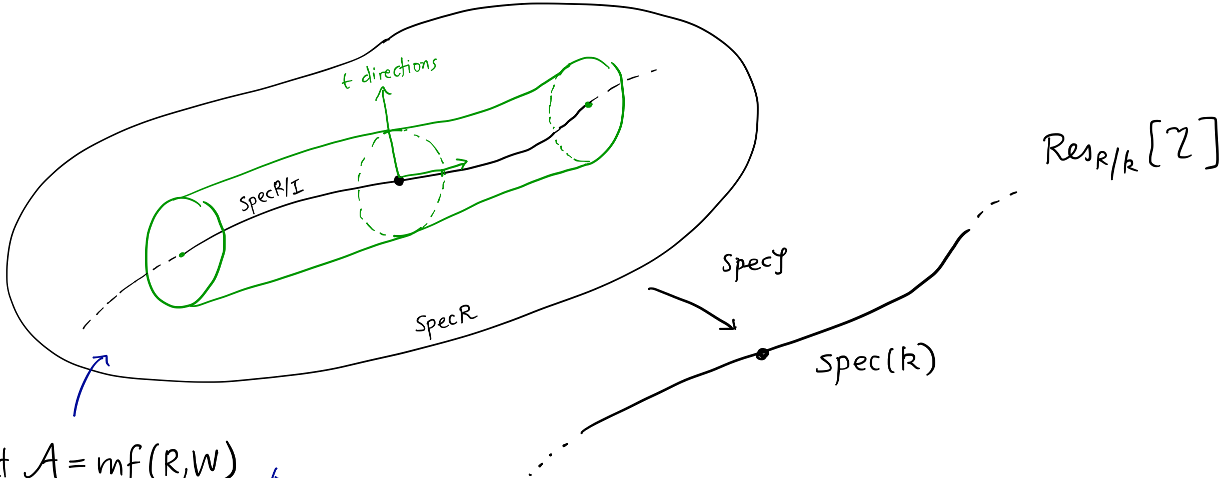
<u>Example</u> One contribution for $W = \frac{1}{5}x^5$ to

$$r_3 : \mathcal{B}(X,X)[1] \otimes \mathcal{B}(X,Y)[1] \otimes \mathcal{B}(Y,Y)[1] \to \mathcal{B}(X,Y)[1]$$



$r_3(x^2\xi \otimes x\eta\xi^* \otimes x^3\eta^*)$

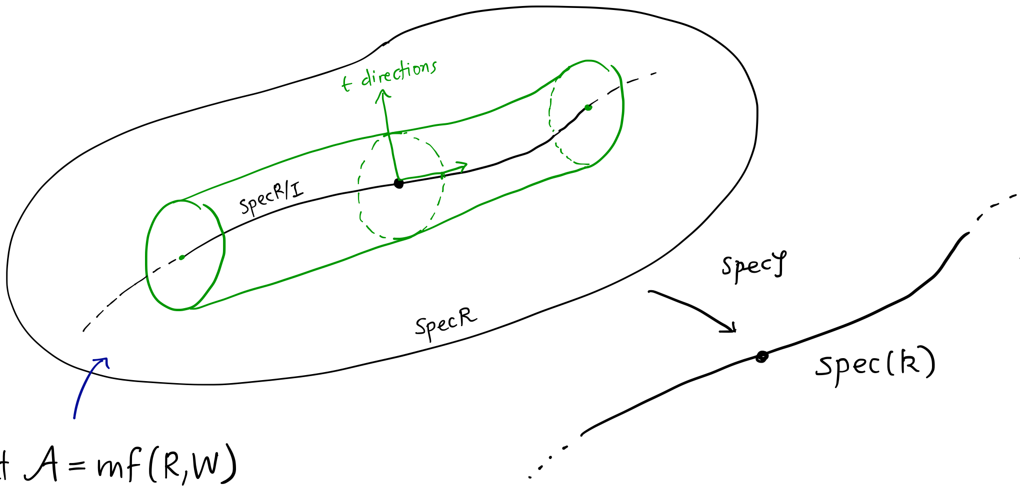
$$\left[\frac{r\,dr_1 \cdots dr_n}{t_1, \ldots, t_n}\right] \in H^n_I(\Omega^n_{R/k}) \cong H^{n-1}(U, \Omega^n_{R/k})$$



DG-cat $\mathcal{A} = mf(R,W) \longmapsto (\mathcal{B}, E)$

$$\begin{gathered}(\mathcal{B}, E)\\ \cong_\infty\\ (\iota_*\mathcal{A}, 1)\end{gathered}$$

$$\left[\frac{r\, dr_1 \cdots dr_n}{t_1, \ldots, t_n}\right] \in H^n_I(\Omega^n_{R/k}) \cong H^{n-1}(U, \Omega^n_{R/k})$$



$$\mathrm{Res}_{R/k}[\eta]$$
$$\|$$
$$\mathrm{tr}(r[\nabla, r_1]\cdots[\nabla, r_n])$$
$$\circlearrowright$$
$$R/I$$

DG-cat $\mathcal{A} = mf(R, W)$ $\longleftrightarrow$ (β, E)

$$\wr\wr\ \infty$$

$$(\varphi_* \mathcal{A}, 1)$$

r_n, E_n are functions of $[\nabla, d_{\mathcal{A}}], \lambda_1, \ldots, \lambda_n$

$$\circlearrowright$$

$$R/I \otimes \mathrm{Hom}_R(X, Y)$$