A_∞-algebras and matrix factorisations

<u>Outline</u>:
1. A_∞-algebras, derived category
2. Examples from isolated hypersurface singularities $W \longmapsto A_\infty$-alg $\mathcal{C}_W$
3. Matrix factorisations $\longmapsto A_\infty$-modules

1. A_∞-algebras

(from algebraic topology, "homotopy algebras")
see [K] for background

<u>Def^N</u> An A_∞-algebra is a $\mathbb{Z}$-graded vector space

$$A = \bigoplus_{n\in\mathbb{Z}} A^n$$

with <u>operations</u> $m_n : A^{\otimes n} \longrightarrow A$, $n \geq 1$, k-linear, degree $2-n$.

$$\begin{array}{lll} m_1 : A \longrightarrow A & & \deg +1 \\ m_2 : A\otimes A \longrightarrow A & & \deg 0 \\ m_3 : A^{\otimes 3} \longrightarrow A & & \deg -1 \\ \quad \vdots & & \end{array}$$

such that for $n \geq 1$

$$\circledast \qquad \sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t}\left(\mathbb{1}^{\otimes r} \otimes m_s \otimes \mathbb{1}^{\otimes t}\right) = 0$$

$$\boxed{\begin{array}{c} A^{\otimes n} \\ \shortparallel \\ A^{\otimes r} \otimes A^{\otimes s} \otimes A^{\otimes t} \\ \downarrow 1\otimes m_s \otimes 1 \\ A^{\otimes r} \otimes A \otimes A^{\otimes t} \\ \downarrow m_{r+1+t} \\ A \end{array}}$$

⌜<u>strict unit</u> is $e \in A^0$, $m_1(e) = 0$, e a unit for m_2 and m_n vanishes for $n > 2$ as soon as any entry is e.
<u>homological unit</u> is a unit for H^*A, we say A is <u>h-unital</u>⌟

<u>Def^N</u> A is <u>minimal</u> if $m_1 = 0$.

<u>Example</u> $m_n = 0$, $n \geq 3$, $\circledast$ says (A, m_1, m_2) satisfies

write $ab = m_2(a \otimes b)$

$n=1$ $\quad m_1^2 = 0$

$n=2$ $\quad m_1(ab) = m_1(a)b + (-1)^{|a|} a\, m_1(b)$

$n=3$ $\quad m_2$ is associative.

$\therefore (A, m_1, m_2)$ is a DG-algebra

<u>In general</u> m_3 is a homotopy $m_2(\mathbb{1} \otimes m_2) \simeq m_2(m_2 \otimes \mathbb{1})$.

<u>Example</u> For $d>2$, $|\varepsilon| = 1$, $A^{(d)} = k[\varepsilon]/\varepsilon^2 = k \oplus k\varepsilon$

$$\left.\begin{aligned} &m_n = 0 \text{ for } n \notin \{2, d\} \\ &m_2 = \text{multiplication} \\ &m_d(\varepsilon \otimes \cdots \otimes \varepsilon) = 1 \end{aligned}\right\} A^{(d)} \text{ is a } \mathbb{Z}_2\text{-graded } A_\infty\text{-algebra}$$

<u>Where do A_∞-algebras come from?</u>

$$\begin{tikzcd}
\mathcal{T} \\
D \arrow[u, "H^0"'] \\
E \arrow[u, phantom, "\cup"] \arrow[d, mapsto] \\
\mathrm{End}_D(E) \arrow[d, mapsto, bend left, "\text{taking the minimal model}"'] \\
\left(H^*\mathrm{End}_D(E), \{m_n\}_{n \geq 2}\right)
\end{tikzcd}$$

$\mathcal{T}$: triangulated category of interest, e.g. $D^b(\mathrm{coh}\, X)$, $\mathrm{hmf}(W)$, $\mathrm{Fuk}(Y)$, ...

D: DG enhancement of $\mathcal{T}$, e.g. by K-injective res.

E: Generator (all objects "built from" E)

$\mathrm{End}_D(E)$: DG-algebra (usually ∞-dim / $\mathbb{C}$)

$\left(H^*\mathrm{End}_D(E), \{m_n\}_{n \geq 2}\right)$: A_∞-algebra quasi-iso to $\mathrm{End}_D(E)$. (f.dim / $\mathbb{C}$)
Knows "everything" about $\mathcal{T}$.

Motivation

semantics of linear logic

gen. orb

- To study moduli (of $\mathcal{T}$ itself, or objects of $\mathcal{T}$),

e.g. Polishchuk's work on moduli of curves

deformation of singularities via hmf W

- Topological string theory (boundary sector)

= minimal, cyclic strictly unital A_∞-categories
(Herbst–Lazaroiu–Lerche, Costello, ...)

= axioms of open-closed TCFT.
(monoidal functors $\mathrm{Rie}_{clg} \longrightarrow \mathrm{Comp}_{\mathbb{C}}$)

- What are Atiyah classes? (of MFs. Physicists know!)

A_∞-modules An A_∞-module over an A_∞-algebra $(A, \{m_n\}_{n\geq 1})$ is a $\mathbb{Z}$-graded vector space M with operations $(n \geq 1)$

$$m_n^M : A^{\otimes(n-1)} \otimes M \longrightarrow M$$

of degree $2-n$ satisfying the same identities ⊛. A *morphism* of A_∞-modules $\varphi : M \longrightarrow N$ is a collection of linear maps $\varphi_n : A^{\otimes(n-1)} \otimes M \longrightarrow N$ of degree $1-n$ such that

$$(u = r+l+t) \qquad \sum_{r+s+t=n} \pm\, \varphi_u(\mathbb{1}^{\otimes r} \otimes m_s \otimes \mathbb{1}^{\otimes t}) = \sum_{r+s=n} \pm\, m_u^N(\mathbb{1}^{\otimes r} \otimes \varphi_s)$$

this is an eq. of maps $A^{\otimes(n-1)} \otimes M \longrightarrow N$

says $\varphi_1 m_1 = m_1 \varphi_1$ and φ_1 commutes with the action of A "up to htpy", etc...

The (ordinary) category of A_∞-modules and these morphisms is denoted $\mathrm{Mod}A$ (note H^*M is a H^*A-module).

<u>The derived category</u> A a h-unital A_∞-algebra.

- There is an A_∞-category of (h-unital) A_∞-modules $\mathrm{Mod}_\infty(A)$, such that $\mathrm{Mod}A = Z^0(\mathrm{Mod}_\infty A)$.

  ↑ this is a triangulated A_∞-cat (see Seidel [S])

  More concretely, $\hom^a(M,N)$ is the space of $\{t^n\}_{n\geq 1}$ with each

$$t^n : A^{\otimes(n-1)} \otimes M \longrightarrow N \qquad (\text{of degree } a-n+1)$$

  and only m_1, m_2 are nonzero in $\mathrm{Mod}_\infty(A)$ (i.e. this is a <u>DG-category</u>).

- The <u>perfect derived category</u> $\mathrm{per}(A)$ is the smallest triangulated subcategory of $H^0(\mathrm{Mod}_\infty A) = \mathrm{Mod}A/\sim$ containing A.

<u>Example</u> $A = A^{(d)}$ from above, $d>2$ (i.e. $m_n = 0$ $n \notin \{2,d\}$).
Given $2 \leq i \leq d-2$, $i < d-i$ we define an A_∞-module over $A^{(d)}$ by

↑ $\mathbb{Z}_2$-graded

$$M_{(i)} := \Lambda(k\xi) = k \oplus k\xi$$

with operations $\alpha_n = 0$ unless $n \in \{2, i+1, d-i+1\}$

$$\alpha_n : A^{\otimes(n-1)} \otimes M_{(i)} \longrightarrow M_{(i)}$$

$$\alpha_2(1,-) = \mathrm{id},$$
$$\alpha_{i+1}(\varepsilon,\varepsilon,\dots,\varepsilon,-) = \pm \xi^* \lrcorner (-) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$\alpha_{d-i+1}(\varepsilon,\varepsilon,\dots,\varepsilon,-) = \pm \xi \wedge (-) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

2. Isolated singularities

$$W \in (x_1, \dots, x_m)^3 \subseteq \mathbb{C}[x_1, \dots, x_m]$$ (actually works with minor changes for $W \in \mathfrak{m}^2$)

$$\dim_{\mathbb{C}} \mathbb{C}[x] / (\partial_{x_1} W, \dots, \partial_{x_m} W) < \infty$$

The scheme $X = Z(W) \subseteq \mathbb{C}^m$ has an isolated singularity at O.

Example $x_1^d + x_2^2 + x_3^2$ (A_{d-1}), x^d, $x_1(x_2^2 + x_1^{k-2}) + x_3^2$ $(D_k, k \geq 4)$

$\boxed{\text{AIM}}$ Explain a construction $W \longmapsto \mathbb{Z}_2$-graded minimal A_∞-alg $\mathcal{C}_W$ (f.d. / $\mathbb{C}$)

$$\operatorname{per}(\mathcal{C}_W) \simeq \operatorname{hmf}(W) \simeq D^b(\operatorname{coh} X) / \operatorname{Perf}(X)$$

- Dyckerhoff '09 DG alg $\operatorname{End}(k^{stab})$ generator
  + min. model thm + good choice of homotopy retract.

  makes explicit calculations actually feasible.

Def^N The underlying algebra of $\mathcal{C}_W$ is

$$\mathcal{C}_W = \bigwedge(k\xi_1 \oplus \cdots \oplus k\xi_m)$$

To define $m_n : \mathcal{C}_W^{\otimes n} \longrightarrow \mathcal{C}_W$ we introduce an auxiliary space

$$S := \mathcal{C}_W \otimes \bigwedge(k\theta_1 \oplus \cdots \oplus k\theta_m) \otimes \mathbb{C}[\underline{x}]$$

write m for the product in S

$$\begin{tikzcd} S \arrow[r, bend left, "p = \text{projection } (\theta, x \mapsto 0)"] & \mathcal{C}_W \arrow[l, bend left, "i = \text{embedding}"] \end{tikzcd}$$

Standard operations

$$S \circlearrowleft \underbrace{\overset{\text{wedge}}{\varphi_i}, \overset{\text{contraction}}{\varphi_i^*}, \theta_i, \theta_i^*}_{\substack{\text{fermion}\\ \text{creation / annihilation}}}, \underbrace{x_i, \partial_{x_i}}_{\substack{\text{boson creation /}\\ \text{annihilation}}}$$

$|\varphi_i| = |\varphi_i^*| = |\theta_i| = |\theta_i^*| = 1$

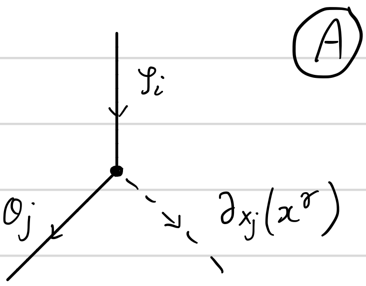
$|x_i| = |\partial_{x_i}| = 0$

skip

- $[\varphi_i, \varphi_j] = [\varphi_i^*, \varphi_j^*] = 0 \qquad [\varphi_i, \varphi_j^*] = \delta_{ij}$ (graded commutators)
  + same for θ_i, θ_i^*
- $[x_i, x_j] = [\partial_{x_i}, \partial_{x_j}] = 0 \qquad [x_i, \partial_{x_j}] = \delta_{ij}$

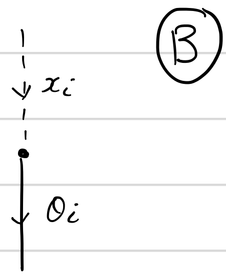
Interactions

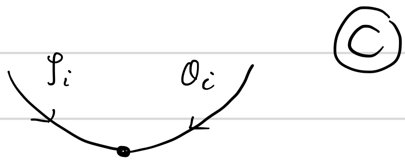
$W = \sum_i x_i W^i \qquad W^i = \sum_{\gamma \in \mathbb{N}^m} W^i(\gamma) x^\gamma \qquad W^i(\gamma) \in \mathbb{C}$



$-\frac{1}{|\gamma|} W^i(\gamma)$

(for all i, j and $\gamma \in \mathbb{N}^m$)





$-\frac{1}{|\gamma|} W^i(\gamma)\, \theta_j\, \partial_{x_j}(x^\gamma)\, \varphi_i^* \circlearrowleft S$

$\theta_i \partial_{x_i} \circlearrowleft S$

$\varphi_i^* \otimes \theta_i^* \circlearrowleft S^{\otimes 2}$

Feynman calculus for higher products

Describes $m_n : \mathcal{C}_W^{\otimes n} \longrightarrow \mathcal{C}_W$ via the structure constants $(\gamma_i, \delta \in \mathcal{C}_W)$

basis elt, i.e. $\psi_{i_1} \cdots \psi_{i_k}$ some $i_1 < \cdots < i_k$

coeff. of δ in $m_n(\gamma_1 \otimes \cdots \otimes \gamma_n)$

$$= \sum_{\substack{\text{binary} \\ \text{trees} \\ T}} \sum_{\substack{\text{Feynman} \\ \text{diagrams } D, \\ \gamma \text{ incoming} \\ \delta \text{ outgoing}}} \text{amplitude}(D)$$

"Def^N" A Feynman diagram D for a binary tree T (e.g.) is an oriented graph embedded in the thickening of T, with lines labelled ψ_i, θ_i, x_i $1 \le i \le m$ and nodes of type A, B, C, with the following constraints:

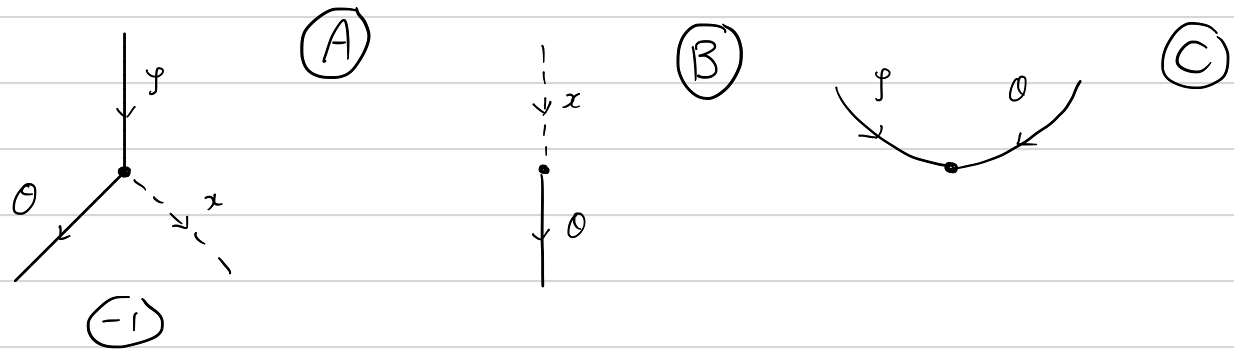
- A nodes may only occur along edges (not adjacent to root)
- B nodes " " " at internal vertices (i.e.)
- There is precisely one C node on every internal edge (and no other C nodes)
- The only lines incident at the boundary (of T) are ψ-lines.

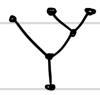
The amplitude of D is

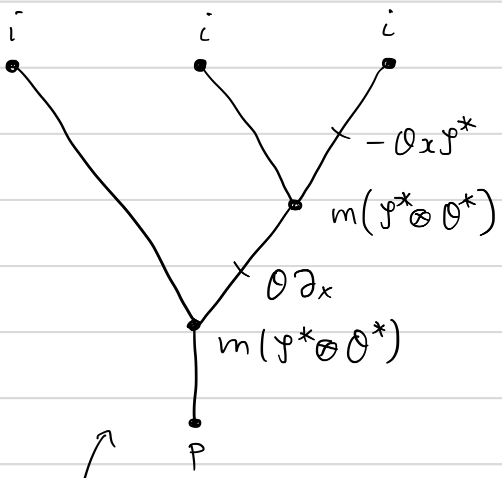
$$\text{amplitude}(D) = (-1)^{f(D)} \underbrace{T_D}_{\substack{\text{symmetry factor} \\ (\in \mathbb{Q})}} \prod_{A \text{ nodes}} \Big(- \underbrace{\frac{\gamma_j}{|\gamma|} W^i(\gamma)}_{j,\,i,\,\gamma \text{ depend on the node}} \Big)$$

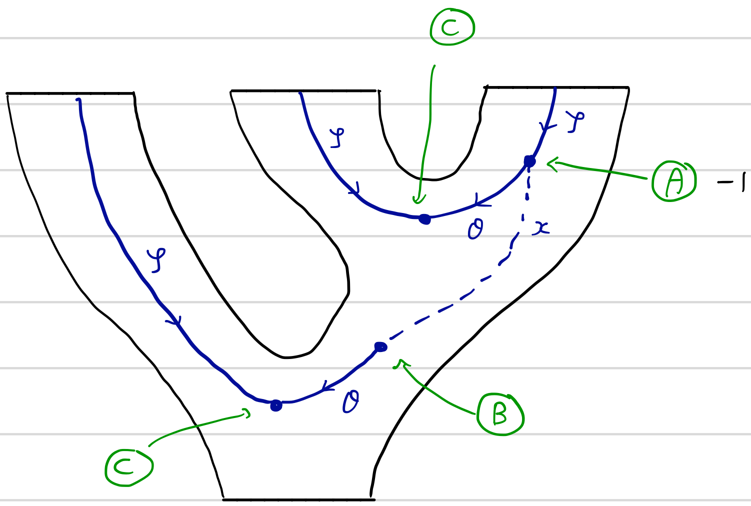
Example In the special case $W = x^3 = x \cdot x^2$, so $W' = x^2$

$$\mathcal{C}_W = \Lambda(k\psi) = k \oplus k\psi \qquad S = \Lambda(k\psi) \otimes \Lambda(k\theta) \otimes \mathbb{C}[x]$$



A Feynman diagram for $T =$  is:





in $= \psi, \psi, \psi$, out $= 1$

amplitude $= -1$

denotes the linear map $\mathcal{C}_W^{\otimes 3} \longrightarrow \mathcal{C}_W$ defined by

$$p\, m(\psi^* \otimes \theta^*)\Big(i(-) \otimes \theta\partial_x m(\psi^* \otimes \theta^*)\big(i(-) \otimes (-\theta x \psi^*) i(-) \big) \Big)$$

$$\psi \otimes \psi \otimes \psi \longmapsto -1$$

In fact this is the only nontrivial Feynman diagram, so $\mathcal{C}_W = \Lambda(k\psi)$ has $m_2 =$ usual product, $m_3(\psi \otimes \psi \otimes \psi) = -1$ otherwise zero, $m_n = 0$ $n \notin \{2, 3\}$.

Lemma For $d>2$, $\mathcal{C}_{x^d} = A^{(d)}$ defined earlier (i.e. $m_n = 0$ $n \notin \{2,d\}$).

Theorem (M) Thus defined $\left(\mathcal{C}_W = \Lambda(k\mathfrak{I}_1 \oplus \cdots \oplus k\mathfrak{I}_m), \{m_n\}_{n\geq 2}\right)$ is a (minimal) $\mathbb{Z}_2$-graded A_∞-algebra and

$$\mathrm{per}(\mathcal{C}_W) \cong \mathrm{hmf}(W)$$

minimal model of $\mathrm{End}(k^{stab})$

Proof Dyckerhoff's $\mathrm{End}(k^{stab})$ + Keller-Lefevre-Hasegawa "generator theorem" + minimal model thm + homotopy retract of arXiv: 1402.4541. $\square$

Notes
- Calculating $\mathcal{C}_W$ is feasible (e.g. $W = y^3 - x^3$).
- Special cases: Herbst-Lazaroiu-Lerche 2004, Efimov 2011.
- Approach generalises to compute $\mathrm{End}(X)$, any MF X.
- Next: cyclic structures / TCFT from W (Shklyarov 1604.05365)

3. minimal models for MFs:

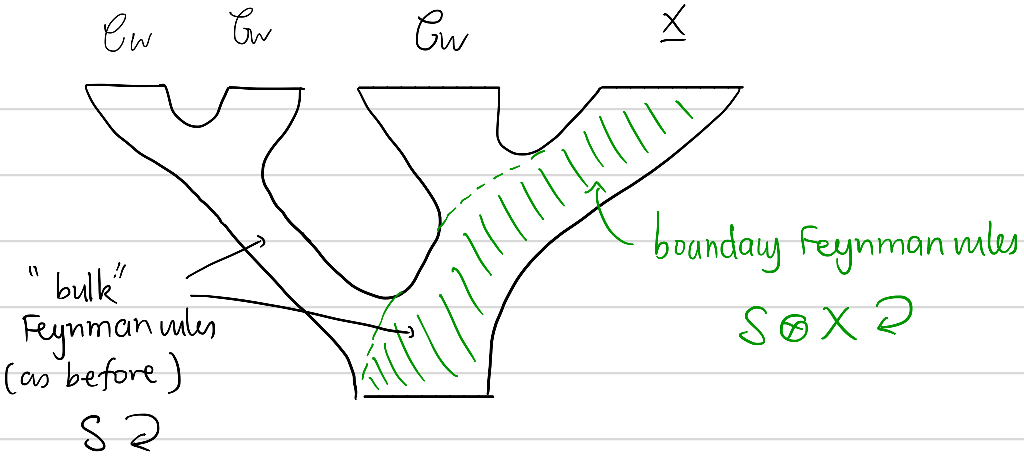
Q/ What is the A_∞-module corresponding to $X \in \mathrm{hmf}(W)$?

Assume for simplicity that $d_X(X) \subseteq \mathfrak{m}^2 X$, then the underlying v.space is

$$\underline{X} := X \otimes_{\mathbb{C}[x]} \mathbb{C}[m]$$

$$\alpha_n : \mathcal{C}_W^{\otimes(n-1)} \otimes \underline{X} \longrightarrow \underline{X}$$

is computed by Feynman rules on diagrams of operators on $S \otimes_{\mathbb{C}[x]} X$, e.g.

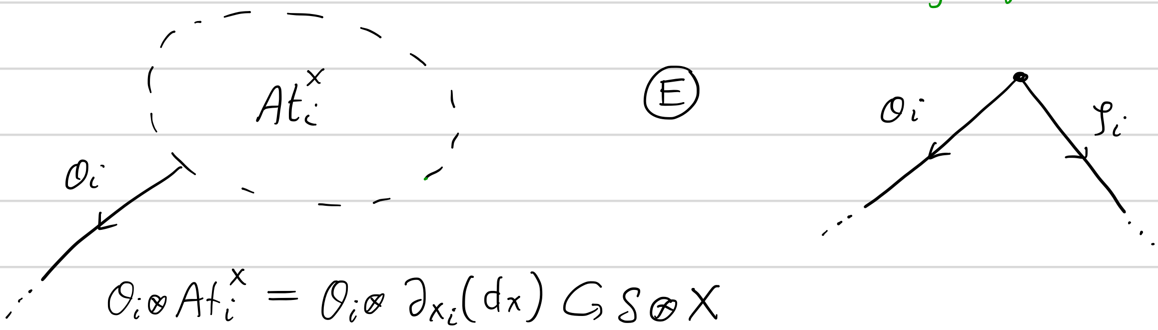


<u>Boundary Feynman rules</u> (in addition to Ⓐ, Ⓑ, Ⓒ)

Ⓓ, Ⓔ vertices allowed on any edge

Ⓓ

Ⓔ



$$\mathcal{O}_i \otimes At_i^X = \mathcal{O}_i \otimes \partial_{x_i}(d_X) \circlearrowright S \otimes X$$

"Atiyah class"

Note that the Atiyah classes completely determine the A_∞-module structure.

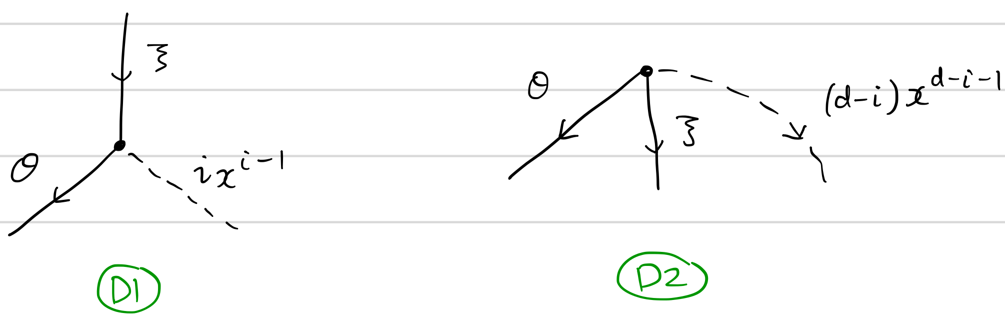
<u>Example</u> Consider $W = x^d$, $d \geq 3$ and $\mathcal{C}_W = (\Lambda(k\rho), m_2, m_d)$

$$X = (\Lambda(k\xi), x^i \xi^* + x^{d-i}\xi) \qquad |\xi| = 1$$

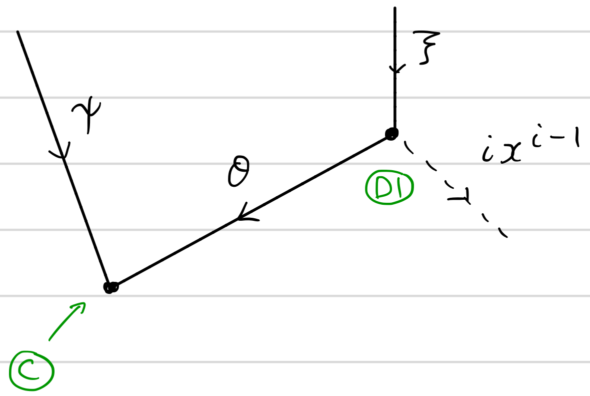
and assume $2 \leq i \leq d-2$. Then

$$\underline{X} = (k \oplus k\xi)[1] \quad \text{and} \quad \partial_x(d_X) = i x^{i-1}\xi^* + (d-i)x^{d-i-1}\xi$$

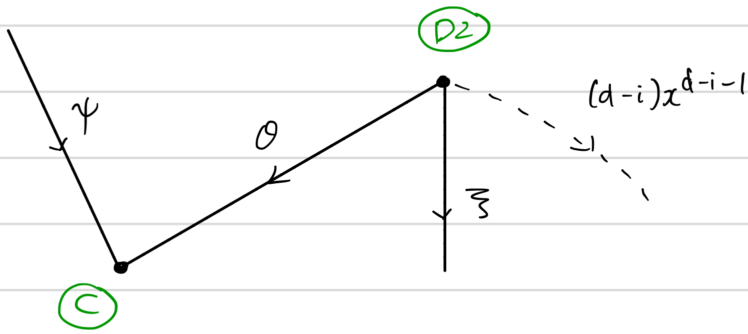
Hence the "Atiyah" interaction is actually two interactions:



Since $\underline{X}$ is an A_∞-module over $\mathcal{C}_W = \Lambda(k\varphi)$ we want to know how φ "acts" on ξ. The only interactions are the ones mediated by a θ:



(11.1)



(11.2)

__Lemma__ The A_∞-module $\underline{X}$ for $X = \begin{pmatrix} 0 & x^i \\ x^{d-i} & 0 \end{pmatrix}$ is $M_{(i)}$ from earlier.

__Proof__ (11.1) gives rise to the operation $\varphi^{\otimes i+1} \longmapsto \xi^*$, (11.2) to $\varphi^{\otimes d-i+1} \longmapsto \xi$. $\square$

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