Bar versus Koszul

Let k be a commutative ring, and $A = k[x_1, \dots, x_n]$ a polynomial ring. In the category of A-A-bimodules there are two natural projective resolutions of A as a bimodule (i.e. the diagonal)

Bar complex (noncommutative forms)

Koszul complex (commutative forms)

$$\begin{tikzcd} \mathbb{B} \arrow[rr, bend left, "\pi_B"] \arrow[d, bend right, "\Psi"'] & & A \\ \mathbb{K} \arrow[u, bend right, "\Phi"'] \arrow[urr, bend right, "\pi_K"'] & & \end{tikzcd}$$

By standard homological algebra there exist morphisms of complexes Φ and Ψ such that

$$\pi_K \circ \Psi = \pi_B, \quad \pi_B \circ \Phi = \pi_A, \quad \Phi \circ \Psi \simeq 1_{\mathbb{B}}, \quad \Psi \circ \Phi \simeq 1_{\mathbb{K}}.$$

The aim of this talk is, starting from only a knowledge of basic (homological) algebra, to first of all define $\mathbb{B}, \mathbb{K}$ and then describe explicitly Φ, Ψ. The map Φ is standard (e.g. from the proof of Hochschild-Kostant-Rosenberg's theorem) but the explicit description of Ψ as a chain map seems less well-known. On the latter point we are following the papers

[SW] A.V. Shepler and S. Witherspoon, "Quantum differentiation and chain maps of bimodule complexes" ANT (2011).

[CM] N. Carqueville and D. Murfet "Adjunctions and defects in Landau-Ginzburg models" Adv. Math (2016).

Our story begins, however, with the paper

[CQ] J. Cuntz and D. Quillen, "Algebra extensions and nonsingularity" JAMS (1995).

and the concept of

<u>Noncommutative differential forms</u> (over a commutative ring k)

A <u>differential graded algebra</u> (DGA) is a monoid in the monoidal category of $\mathbb{Z}$-graded complexes of k-modules $(Ch_{\mathbb{Z}}(k), \otimes, \mathbb{1}=k)$, that is, a DGA is a tuple (A, ∂, m, u) where $(A,\partial) \in Ch_{\mathbb{Z}}(k)$ and

$$m: A\otimes A \longrightarrow A, \quad u: k \longrightarrow A \qquad \left(A = \bigoplus_{i\in\mathbb{Z}} A^i\right)$$

are morphisms of complexes satisfying associativity and unit constraints

<u>Remarks</u> (1) $1_A := u(1_k)$ is a closed element of A^0. $\qquad (\partial^i : A^i \to A^{i+1})$

(2) A^0 is a k-algebra with $m^0|_{A^0\otimes A^0}$ and 1_A.

degree zero

(3) If (A, m, u) is a k-<u>algebra</u> then $(A, 0, m, u)$ is a <u>DGA</u>.

"<u>Lemma</u>" There is an adjoint pair of functors (I = inclusion)

$$\begin{tikzcd} DGA(k) \arrow[r, shift left, "(-)^0"] & Alg(k) \arrow[l, shift left, "I"] \end{tikzcd} \qquad (-)^0 \dashv I$$

differential graded k-algebras, and degree 0 maps

k-algebras (associative, unital, possibly not commutative)

"Proof" The unit is the identity $\eta_A : A \longrightarrow I(A)^0 = A$. This is natural, and given any algebra map $\alpha : A \longrightarrow B^0$ for a DGA B,

$$\tilde{\alpha} : I(A) \longrightarrow B \qquad \tilde{\alpha}_i = \begin{cases} 0 & i \neq 0 \\ \alpha & i = 0 \end{cases}$$

is the unique morphism of DGAs making

nope. This is not a correct proof (the Lemma is false)

$$\begin{tikzcd} A \arrow[r, "\eta_A = id"] \arrow[dr, "\alpha"'] & I(A)^0 \arrow[d, "(\tilde{\alpha})^0"] \\ & B^0 \end{tikzcd}$$

commute. "$\square$"

The problem with this "proof" is of course that $\tilde{\alpha}$ is <u>not</u> a morphism of DGAs as soon as $\mathrm{Im}(\alpha) \not\subseteq Z^0(B)$, since if $\partial_B \alpha(a) \neq 0$ then

$$\partial_B \tilde{\alpha}(a) \neq 0 = \tilde{\alpha} \partial_{I(A)}(a).$$

To fix this and find a right adjoint to $(-)^0$ we need "dA" fitting in a diagram

$$\begin{tikzcd} \boxed{0} & \boxed{1} & \cdots \\ A \arrow[d, equal] \arrow[dd, bend right=60, "\alpha"', to=3-1] & 0 & \\ A \arrow[r] \arrow[d, "\alpha"] & dA \arrow[d, dashed, "?"] & \cdots \\ B^0 \arrow[r, "\partial_B"'] & B^1 \arrow[r] & \cdots \end{tikzcd}$$

morphism of DGAs

<u>Def</u>

Def^N The DGA of <u>noncommutative differential forms</u> over an algebra A is, if it exists, a pair $(\Omega A, \eta)$ consisting of

- a DGA ΩA,
- a morphism of algebras $\eta : A \longrightarrow (\Omega A)^\circ$

which is <u>universal</u>, in the sense that if B is a DGA and $\alpha : A \to B^\circ$ an algebra morphism, there is a unique morphism of DGAs $\tilde{\alpha} : \Omega A \longrightarrow B$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\eta"] \arrow[dr, "\alpha"'] & (\Omega A)^\circ \arrow[d, dashed, "(\tilde{\alpha})^\circ"] \\ & B^\circ \end{tikzcd}$$

⌜equivalently, unique s.t. $\tilde{\alpha} \circ \eta = \alpha$, viewing d as a map into ΩA⌟

<u>Theorem</u> (Cuntz-Quillen) $(\Omega A, \eta)$ exists for all A, so there is an adjunction

$$\begin{tikzcd} DGA(k) \arrow[r, shift left, "(-)^\circ"] & Alg(k) \arrow[l, shift left, "\Omega"] \end{tikzcd} \qquad (-)^\circ \dashv \Omega$$

That is, we have natural bijections

$$Hom_{Alg(k)}(A, B^\circ) \cong Hom_{DGA(k)}(\Omega A, B).$$

<u>Remark</u> The morphism of algebras $A \xrightarrow{1_A} I(A)^\circ = A$ induces a morphism of DGAs

$$\omega : \Omega A \longrightarrow I(A) \quad \text{s.t.} \quad \omega^\circ \circ \eta = 1_A,$$

i.e. $\eta : A \longrightarrow (\Omega A)^\circ$ has a <u>section</u> $\omega^\circ : (\Omega A)^\circ \longrightarrow A$, in $Alg(k)$.

<u>Proof</u> First observe that in any DGA (Ω, d) with $A \subseteq \Omega^0$ as a subalgebra

$$(1) \quad d(a_0 da_1 \cdots da_n) = da_0 da_1 \cdots da_n$$

$$(2) \quad (a_0 da_1 \cdots da_n)(a_{n+1} da_{n+2} \cdots da_k)$$

$$= (-1)^n a_0 a_1 da_2 \cdots da_k + \sum_{i=1}^{n} (-1)^{n-i} a_0 da_1 \cdots d(a_i a_{i+1}) \cdots da_k$$

The second formula results from multiple applications of

$$d(b)c = d(bc) - bd(c).$$

Thus Ω has a sub-DG-algebra spanned by expressions $a_0 da_1 \cdots da_n$. Since $\lambda \in k$ implies $d(\lambda \cdot 1_A) = 0$ (since $1_A = 1_\Omega$) this leads us to <u>define</u> a $\mathbb{Z}$-graded k-module $(\otimes = \otimes_k)$

$$\Omega A := \bigoplus_{n \geq 0} \underbrace{A \otimes \overline{A}^{\otimes n}}_{\text{denote this } \Omega^n A} \qquad \overline{A} = A/k$$

and make $(\Omega A, d)$ a complex with

$$d(a_0 \otimes \overline{a}_1 \otimes \cdots \otimes \overline{a}_n) = 1 \otimes \overline{a}_0 \otimes \overline{a}_1 \otimes \cdots \otimes \overline{a}_n.$$

Now there is a clever trick to induce the desired DGA structure. Define

$$\mathcal{E} = \operatorname{Hom}^*_k((\Omega A, d), (\Omega A, d)) \qquad \text{(DG-algebra with differential } [d,-])$$

$$[d, f] = d \circ f - (-1)^{|f|} f \circ d$$

($f : \Omega A \to \Omega A$ is a degree n map of $\mathbb{Z}$-graded modules)

Define k-linear maps

$$\ell : A \longrightarrow \mathcal{E}^0, \qquad \ell(a)(a_0 \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_n) = a a_0 \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_n$$

$$\ell_* : \Omega A \longrightarrow \mathcal{E}, \qquad \ell_*(a_0 \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_n) = \ell(a_0)[d, \ell(a_1)] \cdots \cdots [d, \ell(a_n)]$$

Note that

$$[d, \ell(a)](1 \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_n) = d(a \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_n) \qquad (*)$$
$$= 1 \otimes \bar{a} \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_n .$$

By (1), (2) we have $\mathrm{Im}(\ell_*) \subseteq \mathcal{E}$ is the DG-subalgebra spanned by $\ell(A)$.
Now define $J : \mathcal{E} \longrightarrow \Omega A$ by $J(f) = f(1)$, this is k-linear and degree zero.
Moreover, using $(*)$,

$$\begin{aligned} J\ell_*(a_0 \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_n) &= \{ \ell(a_0)[d, \ell(a_1)] \cdots [d, \ell(a_n)] \}(1) \\ &= \ell(a_0)[d, \ell(a_1)] \cdots [d, \ell(a_{n-1})](1 \otimes \bar{a}_n) \\ &\ \ \vdots \\ &= \ell(a_0)(1 \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_n) \\ &= a_0 \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_n . \end{aligned}$$

Hence $J\ell_* = 1_{\Omega A}$ and ℓ_* is injective. Since $\ell_*(\Omega A) \subseteq \mathcal{E}$ was a sub-DGA this equips ΩA with a unique DG-structure s.t. ℓ_* is a map of DGAs.

<u>So far</u> we have defined a DGA $(\Omega A = \bigoplus_{n \geq 0} A \otimes \bar{A}^{\otimes n}, d)$, and in this DGA

$$a_0 da_1 \cdots da_n = a_0 \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_n . \qquad \text{(apply } \ell_* \text{ to both sides)}$$

$\therefore$ the structure of this DGA is completely determined by (1), (2).

It remains to show the algebra map

$$\eta = 1_A : A \longrightarrow (\Omega A)^0 = A$$

is universal : suppose B is a DGA and $u : A \to B^0$ an algebra map. Then

$$\tilde{u} : \Omega A \longrightarrow B, \qquad a_0 \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_n \longmapsto u a_0 \, \partial_B(u a_1) \cdots \partial_B(u a_n)$$

is a morphism of DGAs using (1), (2) and is clearly unique s.t. $\tilde{u}|_A = u$. $\square$

<u>Example</u> $A = k[x]$ $\quad \Omega A = \Big(k[x] \longrightarrow k[x] \otimes x k[x] \longrightarrow k[x] \otimes (x k[x])^{\otimes 2} \longrightarrow \cdots$

$$x^i \longmapsto 1 \otimes \overline{x^i}$$

$$\underset{j \geq 1}{x^i \otimes x^j} \longmapsto 1 \otimes \overline{x^i} \otimes \overline{x^j}$$

$$\therefore \qquad H^0(\Omega A) \cong k, \; H^n(\Omega A) = 0 \quad n > 0.$$

Kähler differentials

<u>Remark</u> If A is commutative $(\Omega A)_{sup} \cong (\Lambda_A \Omega^1_{A/k}, d)$ as DGAs.

quotient making it graded commutative and $r^2 = 0$

<u>The Bar complex $\mathbb{B}$</u> (normalised) is the complex

$$\mathbb{B} = \left(\Omega A \otimes A,\ b' \right)$$

$$b'(w\,da \otimes a') = (-1)^{|w|}(wa \otimes a' - w \otimes aa') \qquad a, a' \in A$$

More explicitly, $\mathbb{B}_n = \Omega^n A \otimes A = A \otimes \bar{A}^{\otimes n} \otimes A$ and using (2),

$$b'(a_0 \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_n \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \bar{a}_1 \otimes \cdots \otimes \overline{a_i a_{i+1}} \otimes \cdots \otimes a_{n+1}$$

<u>Lemma</u> The complex of A-A-bimodules

$$\underbrace{\cdots \longrightarrow A \otimes \bar{A}^{\otimes 2} \otimes A \xrightarrow{b'} A \otimes \bar{A} \otimes A \xrightarrow{b'} A \otimes A}_{\mathbb{B}} \xrightarrow{b' = m} A \longrightarrow 0$$

is exact, in fact <u>contractible</u> as a complex of <u>right</u> A-modules.

<u>Proof</u> $(\Omega A, d)$ is a DGA, and $d \otimes 1$ is a right A-linear degree -1 operator on $\mathbb{B} = \Omega A \otimes A$. One checks directly that

$$b'd + db' = 1 \quad \text{on } \mathbb{B}_n,\ n \geq 0$$

where we include $b' : A \otimes A \longrightarrow A$ for $n = 0$. $\square$

<u>QUESTION</u> How to see b' as an operator on $\Omega A \otimes A \cong \Omega_A(A \otimes A)$ purely from the universal property? I don't know.

The Koszul complex $\mathbb{K}$

Now we specialise to $A = k[x_1, \dots, x_n]$ so $\mathbb{B} \longrightarrow A$ is a <u>free</u> resolution. Another free resolution is the Koszul complex ($A^e = A \otimes A$, $|\theta_i| = 1$)

$$\mathbb{K} = \Big(\Lambda(k\theta_1 \oplus \cdots \oplus k\theta_n) \otimes A^e, \; \sum_{i=1}^{n} \theta_i^* \otimes [x_i \otimes 1 - 1 \otimes x_i] \Big)$$

$$\theta_i^*(\alpha\beta) = \theta_i^*(\alpha)\beta + (-1)^{|\alpha|}\alpha\,\theta_i^*(\beta)$$
$$\theta_i^*(\theta_j) = \delta_{ij}$$

[Think of θ_i as $\frac{\partial}{\partial x_i}$ so the differential is an Euler v. field]

The first few terms look like

$$0 \longrightarrow A\theta_1 \cdots \theta_n \longrightarrow \cdots \longrightarrow \bigoplus_{i=1}^{n} A^e \theta_i \xrightarrow{\begin{pmatrix} x_1 \otimes 1 - 1 \otimes x_1 \\ \vdots \\ x_n \otimes 1 - 1 \otimes x_n \end{pmatrix}} A^e \xrightarrow{m} A \longrightarrow 0$$

<u>Remark</u> With exterior multiplication, $\mathbb{K}$ is a DGA, $m : \mathbb{K} \longrightarrow A$ is a morphism of DGAs.

We have now <u>two</u> free resolutions of A as an A-A-bimodule.

$$\mathbb{B} \xrightarrow{\pi_{\mathbb{B}}} A, \quad \mathbb{K} \xrightarrow{\pi_{\mathbb{K}}} A$$

It is sometimes useful (in e.g. perturbation theory, which is the context of [CM]) to know <u>formulas</u> for chain maps $\Phi : \mathbb{K} \longrightarrow \mathbb{B}$, $\Psi : \mathbb{B} \longrightarrow \mathbb{K}$ lifting the identity on A. One reason to care about Ψ is that <u>lifting into $\mathbb{B}$</u> as in the diagram below, is often easier than lifting into $\mathbb{K}$, as noncommutative differential forms are better behaved than ordinary differential forms. One can obtain an explicit lifting to $\mathbb{K}$ by first lifting to $\mathbb{B}$ (i.e. F) and then composing to get $\Psi \circ F$.

$$\begin{tikzcd} & \mathbb{B} \arrow[d, "\Psi"] \arrow[dd, bend right] \\ X \arrow[ur, dashed, "F"] \arrow[dr, bend right, "f"'] & \mathbb{K} \arrow[d] \\ & A \end{tikzcd}$$

But only if one knows a formula for Ψ!

Now for the formulas

The induced homotopy equivalence $\Phi : \mathbb{K} \longrightarrow \mathbb{B}$ is

$$\Phi\big((r \otimes r')\theta_{i_1} \cdots \theta_{i_p} \big) = \sum_{\sigma \in S_p} (-1)^{|\sigma|} r\, dx_{i_{\sigma(1)}} \cdots dx_{i_{\sigma(p)}} \otimes r'$$

and its homotopy inverse is $\Psi : \mathbb{B} \longrightarrow \mathbb{K}$,

$$\Psi\big(r\, df_1 \cdots df_p \otimes r' \big) = \sum_{1 \le i_1 < \cdots < i_p \le n} (r \otimes r') \prod_{k=1}^{p} \partial_{[i_k]}(f_k)\, \theta_{i_1} \cdots \theta_{i_p}$$

where we use the divided difference operators

$$\partial_{[i]} : A = k[\underline{x}] \longrightarrow k[\underline{x}, \underline{x}'] = A^e$$

$$\partial_{[i]}(f) = \frac{f(x_1', \ldots, x_{i-1}', x_i, \ldots, x_n) - f(x_1', \ldots, x_{i-1}', x_i', \ldots, x_n)}{x_i - x_i'}$$

The formula for Φ is standard, Ψ is from [SW], [CM]. To prove Φ, Ψ are mutually inverse up to homotopy one just checks by hand the formulas give cochain maps, and that they induce 1_A.

Lemma Φ, Ψ are morphisms of DGAs between $\mathbb{K}$ and $(\mathbb{B}, b', \times)$ where $\times$ is the shuffle product.

Def^n Given an A-A-bimodule M,

$$\underline{\text{Hochschild homology}} \quad HH_n(A,M) := H_n(M \otimes_{A^e} \mathbb{B})$$

$$\underline{\text{Hochschild cohomology}} \quad HH^n(A,M) := H^n(\mathrm{Hom}_{A^e}(\mathbb{B}, M)).$$