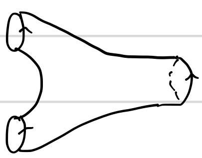
The computational content of Landau–Ginzburg models

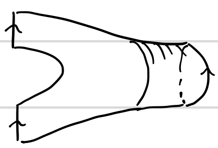
In this talk I would like to explain some of the computational or "logical" aspects of topological Landau–Ginzburg models. The primary references are

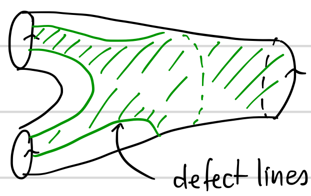
- D. Murfet "The cut operation on matrix factorisations" J. Pure and Applied Algebra 2018.
- N. Carqueville, D. Murfet "Adjunctions and defects in Landau Ginzburg models" Adv. Math 2016.

Flavours of 2D topological field theories (arising from algebraic geometry)

{closed 2D TFT} $\hookrightarrow$ { open-closed 2D TFTs } $\longrightarrow$ { 2D defect TFTs }





$M :=$ 

= commutative Frobenius algebras = Calabi-Yau categories "=" pivotal 2-categories

Landau-Ginzburg (B-twisted, topological) $\{W=0\}$ isolated sing.	$\mathbb{C}[\underline{x}]/(\partial W)$	$\mathrm{hmf}(W)$ (matrix factorisations)	$\mathcal{LG}$ (all potentials)
Sigma-models (B-twisted, topological) X smooth projective	$\bigoplus_{p,q} H^p(X, \Omega^q_X)$	$\mathbb{D}^b(\mathrm{coh}\, X)$	$\mathcal{V}ar$ (FM transforms)

The most important structure in $\mathcal{LG}$ is "defect fusion" or composition of 1-morphisms. This is quite a complicated operation, and the purpose of this talk is to examine this operation from a logical point of view (in a sense we will shortly elaborate).

Outline

1. A lightning introduction to logic
2. An overview of topological LG models
3. Theorem: an equivalence of bicategories.

1. Introduction to logic

One way to understand intuitionistic logic (i.e. logic without the law of excluded middle $P \vee \neg P$) is as the mathematical theory of algorithmic constructions, a connection which is made precise by the Curry-Howard correspondence, which relates proofs in various intuitionistic logics to algorithms in corresponding "programming languages". Perhaps the most important conceptual insight provided by logic is that there is a dynamical process lying behind function composition (which as mathematicians we are inclined to treat as primitive and "atomic"). This process is called computation: a "subatomic" theory of function composition.

For an explanation of the relation between TFT and quantum computation see

M.H. Freedman, A. Kitaev, Z. Wang "Simulation of Topological Field Theories by Quantum Computers" Commun. Math. Phys. 227, 587–603 (2002).

We can illustrate the general idea with the following simple example: consider functions

$$\mathbb{C} \xrightarrow{f} \mathbb{C} \xrightarrow{g} \mathbb{C}$$

$$f(x) = \zeta^2 x \qquad g(y) = \zeta^3 y \qquad \zeta = e^{2\pi i/n}$$

which from a set-theoretic point of view we identify with their <u>graphs</u>

$$\Gamma_f = \{ (x,y) \in \mathbb{C}^2 \mid \underbrace{y = \zeta^2 x}_{\mathcal{F}(x,y)} \} \qquad \Gamma_g = \{ (y,z) \in \mathbb{C}^2 \mid \underbrace{z = \zeta^3 y}_{\mathcal{G}(y,z)} \}.$$

From a set-theoretic point of view composition is

$$\Gamma_{g\circ f} = \{ (x,z) \in \mathbb{C}^2 \mid \exists y \in \mathbb{C} (y = \zeta^2 x \wedge z = \zeta^3 y) \}$$

substitution I

$$= \{ (x,z) \in \mathbb{C}^2 \mid \exists y \in \mathbb{C} (y = \zeta^2 x \wedge z = \zeta^3 (\zeta^2 x)) \}$$

elimination of y II

$$= \{ (x,z) \in \mathbb{C}^2 \mid z = \zeta^5 x \}$$

This process of substitution and elimination "rewrites" the predicate

$$\exists y (\mathcal{F}(x,y) \wedge \mathcal{G}(y,z)) \overset{\text{rewriting}}{\rightsquigarrow} z = \zeta^5 x$$

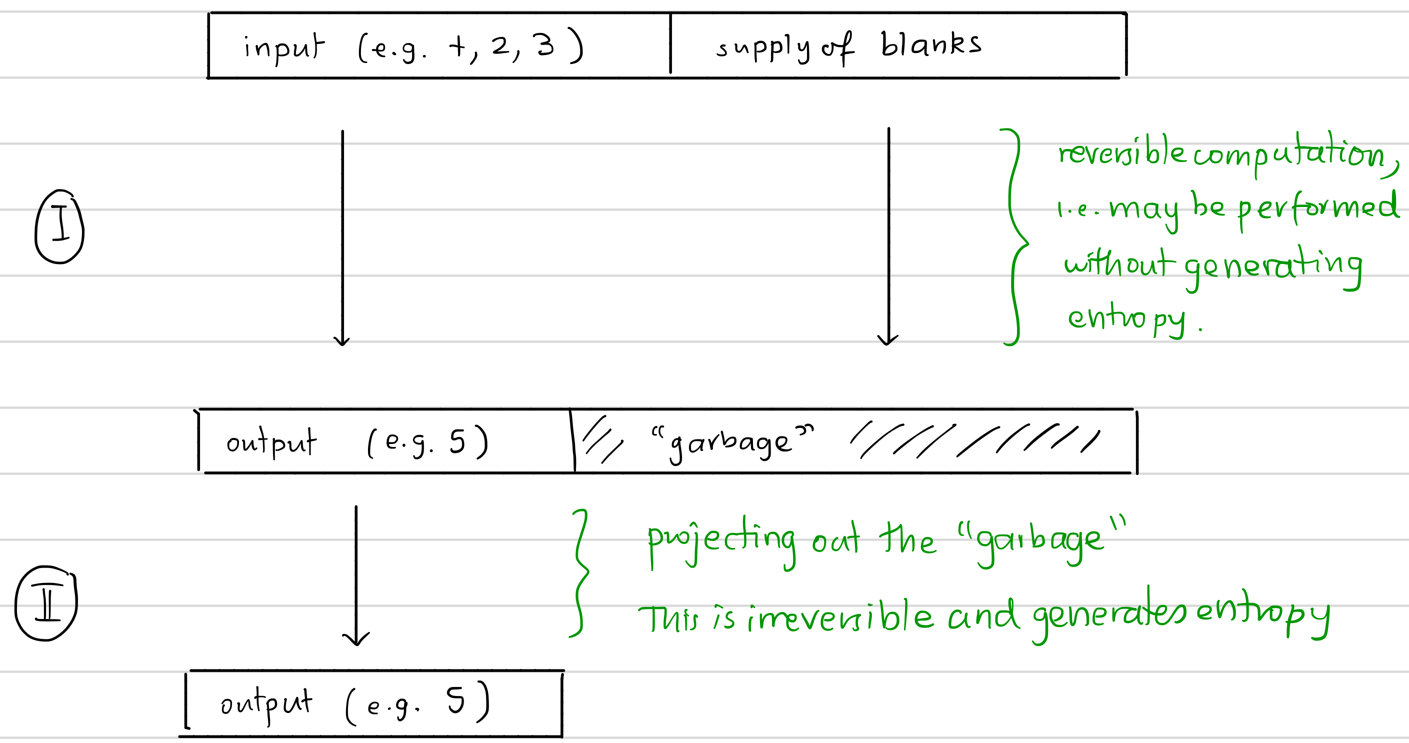
and this kind of substitution and rewriting is sufficient to describe <u>all computable functions</u>, as was proven by Church with his λ-calculus (in combination with Turing's work). From Church's point of view <u>computation</u> is this process of rewriting.

Subtle point Computation does not yield new information, as in the case of

$$2 + 3 = 5$$

the answer 5 is already implicit in the LHS.

A slightly different but related way of looking at computation will be familiar to anyone who has studied the relation between classical and quantum computation, which hinges on the notion of reversible computation (see e.g. Feynman's "Lectures on computation"). The upshot is that, by a Theorem of Bennett any computation (more precisely, any computation performed by a Turing machine) can without loss of generality viewed as a two-phase process



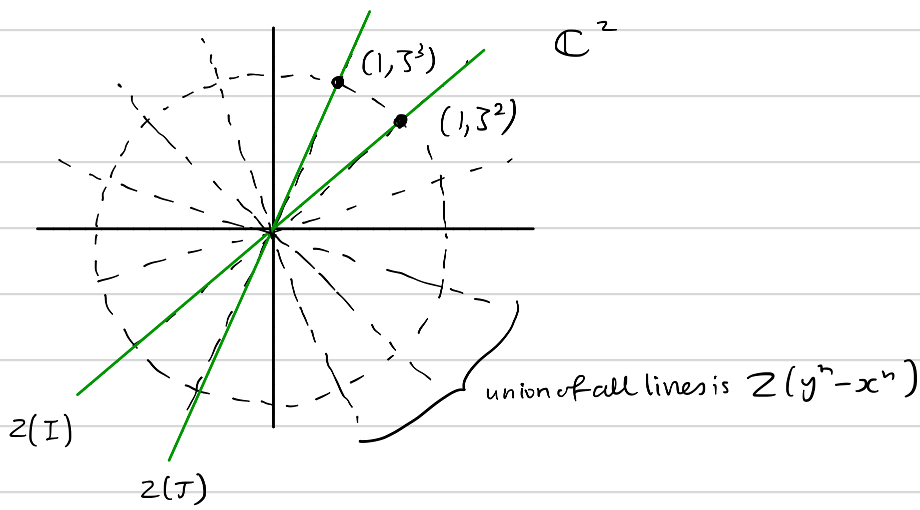
Informally, any computation can be structured as first rearranging the information into a more useful form and then erasing part of that rearranged information.

2. Topological LG models

Let us now reinterpret this as a story about correspondences between (zero-dimensional) isolated hypersurface singularities:

$$I = (y - \zeta^2 x) \subseteq \mathbb{C}[x,y], \qquad J = (z - \zeta^3 y) \subseteq \mathbb{C}[y,z]$$

$$(y^n - x^n) \subseteq I \qquad\qquad (z^n - y^n) \subseteq J$$



$$\underbrace{y^n - x^n = (y - \zeta^2 x)\cdot \prod_{i \in \mathbb{Z}_n \setminus \{2\}} (y - \zeta^i x)}_{\text{this factorisation may be packaged as a matrix}} \qquad \underbrace{z^n - y^n = (z - \zeta^3 y)\cdot \prod_{i \in \mathbb{Z}_n \setminus \{3\}} (z - \zeta^i y)}_{\text{packaged as}}$$

<u>Def^n</u> Consider the matrices

$$D_{\{2\}} = \begin{pmatrix} 0 & y - \zeta^2 x \\ \prod_{i \in \mathbb{Z}_n \setminus \{2\}} (y - \zeta^i x) & 0 \end{pmatrix}, \quad D_{\{3\}} = \begin{pmatrix} 0 & z - \zeta^3 y \\ \prod_{i \in \mathbb{Z}_n \setminus \{3\}} (z - \zeta^i y) & 0 \end{pmatrix}$$

Then

$$D_{\{2\}}^2 = \begin{pmatrix} y^n - x^n & 0 \\ 0 & y^n - x^n \end{pmatrix}, \quad D_{\{3\}}^2 = \begin{pmatrix} z^n - y^n & 0 \\ 0 & z^n - y^n \end{pmatrix}.$$

<u>Def^n</u> A <u>potential</u> is $W \in \mathbb{C}[x_1, \dots, x_n]$ such that $\dim_{\mathbb{C}} \mathbb{C}[\underline{x}] / (\partial_{x_1} W, \dots, \partial_{x_n} W) < \infty$.

<u>Def^n</u> (Eisenbud) A <u>matrix factorisation</u> of a potential W is a block matrix over $\mathbb{C}[\underline{x}]$,

$$D = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \quad \text{s.t.} \quad D^2 = W \cdot I_{2k} \iff AB = BA = W \cdot I_k$$

which may be viewed as an odd $\mathbb{C}[\underline{x}]$-linear operator $X \circlearrowleft D$, on a $\mathbb{Z}_2$-graded free $\mathbb{C}[\underline{x}]$-module of finite rank. These form a triangulated category $\mathrm{hmf}(\mathbb{C}[\underline{x}], W)$.

As was elaborated by Kapustin–Li, Herbst–Brunner, Lazaroiu and then by Orlov, matrix factorisations are the appropriate homological objects to describe boundary conditions and defects in topological B–twisted Landau–Ginzburg models. The <u>defect</u> part of this story is captured by the idea that the appropriate notion of "correspondence" between isolated singularities $W \in \mathbb{C}[\underline{x}]$ and $V \in \mathbb{C}[\underline{y}]$ are matrix factorisations of $V - W$. This theory may be organised into a bicategory $\mathcal{LG}$:

$$\mathrm{ob}(\mathcal{LG}) = \{ \text{pairs } (\mathbb{C}[\underline{x}], W) \mid W \text{ is a potential} \}.$$

$$\mathcal{LG}(W(\underline{x}), V(\underline{y})) = \mathrm{hmf}(\mathbb{C}[\underline{x}, \underline{y}], V - W)^{\omega}$$ ← idempotent completion

<u>Example</u> $x^n \xrightarrow{D_{\{2\}}(x,y)} y^n \xrightarrow{D_{\{3\}}(y,z)} z^n$ are 1-morphisms in $\mathcal{LG}$.

<u>Def^n</u> The composition of $W(x) \xrightarrow{D_1} V(y) \xrightarrow{D_2} U(z)$ is the representing object $D_2 \circ D_1$ in $\mathcal{LG}(W, U) = \mathrm{hmf}(\mathbb{C}[x,z], U - W)^{\omega}$ of

$$\left(X_2 \otimes_{\mathbb{C}[\underline{y}]} X_1 , \; D_2 \otimes 1 + 1 \otimes D_1 \right)$$ ← ∞-rank over $\mathbb{C}[\underline{x}, \underline{z}]$.

<u>Theorem</u> (Khovanov–Rozansky, Carqueville–M) A representing object exists.

<u>Example</u> $D_{\{3\}}(y,z) \circ D_{\{2\}}(x,y) \cong D_{\{5\}}(x,z)$

$z = \zeta^3 y \qquad y = \zeta^2 x \qquad z = \zeta^5 x$

This composition of 1-morphisms is (in quite a precise sense, since $D_{\{2\}}, D_{\{3\}}$ are "universal" objects in the matrix factorisation category associated to the graphs of f, g viewed as subschemes) a homological analogue of the function composition from earlier. We saw how a process of computation (consisting of substitution and elimination) lay behind the equality

$$T_g \circ T_f = T_h \qquad h(x) = \zeta^5 x \qquad (*)$$

and we now present a "homological" analogue for the isomorphism

$$D_{\{3\}} \circ D_{\{2\}} \cong D_{\{5\}}.$$

The appropriate kind of "projecting out" is <u>splitting an idempotent</u>

<u>implicit</u>	<u>explicit</u>
$X \circlearrowleft e$	$X \cong Y \oplus Z$
$e^2 = e$	$e \qquad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Actually we consider only a special kind of idempotent, splitting as

$$X \cong Y \oplus Y[1] \cong Y \otimes_k (k \oplus k\theta) \qquad |\theta| = 1$$

$$e \qquad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \theta^* \theta \qquad \theta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ \theta^* = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

3. An equivalence of bicategories

Def^n The Clifford algebra C_n is the $\mathbb{Z}_2$-graded $\mathbb{C}$-algebra generated by odd elements $\gamma_1, \dots, \gamma_n, \gamma_1^\dagger, \dots, \gamma_n^\dagger$ satisfying

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 0, \quad \gamma_i^\dagger\gamma_j^\dagger + \gamma_j^\dagger\gamma_i^\dagger = 0$$

$$\gamma_i\gamma_j^\dagger + \gamma_j^\dagger\gamma_i = \delta_{ij}.$$

Lemma There is an isomorphism of $\mathbb{Z}_2$-graded algebras

$$C_n \xrightarrow{\cong} \mathrm{End}_{\mathbb{C}}\big(\underbrace{\Lambda(\mathbb{C}\theta_1 \oplus \cdots \oplus \mathbb{C}\theta_n)}_{S_n}\big) \qquad |\theta_i| = 1$$

$$\gamma_i \longmapsto \theta_i \wedge (-)$$
$$\gamma_i^\dagger \longmapsto \theta_i^* \lrcorner (-).$$

C_n is Morita trivial
$\mathrm{Mod}(C_n) \cong \mathrm{Mod}(\mathbb{C})$.

Def^n Let $\mathcal{T}$ be an idempotent complete $\mathbb{Z}_2$-graded category (e.g. $\mathrm{hmf}(W)^\omega$). The Clifford thickening $\mathcal{T}^\bullet$ has as objects pairs (X, n) where $n \geqslant 0$ and X is a representation of C_n in $\mathcal{T}$.

$$\mathrm{Hom}_{\mathcal{T}^\bullet}\big((X,n),(Y,m)\big) = \mathrm{Hom}_{C_m}\big(S_m \underset{\mathbb{C}}{\otimes} S_n^* \underset{C_n}{\otimes} X, Y\big)$$

Note The inclusion $\mathcal{T} \longrightarrow \mathcal{T}^\bullet$ sending $X \longmapsto (X, 0)$ is an equivalence.

$$\left.\begin{array}{l} X \circlearrowleft \gamma, \gamma^\dagger \text{ satisfying Clifford relations in } \mathcal{T} \\ \mathrm{Im}(\gamma\gamma^\dagger) \otimes (\mathbb{C} \oplus \mathbb{C}\theta) \circlearrowleft \theta, \theta^* \\ \mathrm{Im}(\gamma\gamma^\dagger) \end{array}\right\} \text{all isomorphic in } \mathcal{T}^\bullet$$

<u>Theorem</u> (M) The obvious inclusions define an equivalence of bicategories

$$\mathcal{Z} : \mathcal{LG} \longrightarrow \mathcal{LG}^{\bullet}$$

where $\mathcal{LG}^{\bullet}$ has the same objects as $\mathcal{LG}$, but

$$\mathcal{LG}^{\bullet}(W(x), V(y)) := \{ hmf(\mathbb{C}[\underline{x},\underline{y}], V-W)^{\omega} \}^{\bullet}$$

and the composition in $\mathcal{C}$ is given by <u>explicit formulas</u> (unlike $\mathcal{LG}$)

$$\mathcal{LG}^{\bullet}(V,U) \times \mathcal{LG}^{\bullet}(W,V) \longrightarrow \mathcal{LG}^{\bullet}(W,U)$$

$$(\underset{\text{MF of } U-V}{X_2 \circlearrowleft D_2}, \underset{\text{MF of } V-W}{X_1 \circlearrowleft D_1}) \longmapsto \underbrace{(X_2 | X_1 \circlearrowleft D_2 | D_1, \{\gamma_i, \gamma_i^{\dagger}\}_{i=1}^{n})}$$

we call this the <u>cut</u> of D_2, D_1
(a hat tip to logic, and cut-elimination)

where $n = |\underline{y}|$ and

$$X_2 | X_1 = X_2 \underset{\mathbb{C}[\underline{y}]}{\otimes} \mathrm{Jac}_V \underset{\mathbb{C}[\underline{y}]}{\otimes} X_1 \circlearrowleft D_2 | D_1 = D_2 \otimes 1 + 1 \otimes D_1$$

$$\circlearrowleft$$

$$At_i := [D_2 \otimes 1 + 1 \otimes D_1, \tfrac{\partial}{\partial t_i}] \qquad t_i = \tfrac{\partial}{\partial y_i} V \qquad \widehat{\mathbb{C}[\underline{y}]} \xrightarrow{\nabla} \widehat{\mathbb{C}[\underline{y}]} \underset{\mathbb{C}[\underline{t}]}{\otimes} \Omega^1_{\mathbb{C}[\underline{t}]/\mathbb{C}}$$

(Atiyah classes) $\qquad$ $(\underline{t})$-adic completion

$$\gamma_i = At_i \qquad \gamma_i^{\dagger} = -1 \otimes \tfrac{\partial}{\partial y_i}(D_1) - \tfrac{1}{2} \sum_{q=1}^{n} \tfrac{\partial}{\partial y_q} \tfrac{\partial}{\partial y_i}(V) At_q$$

These operators satisfy Clifford relations up to homotopy.

To complete the story let us elaborate the "computational" view this equivalence provides on the composition of 1-morphisms in $\mathcal{LG}$

$$\begin{tikzcd}
(D_1, D_2) \arrow[rrr, mapsto] & & & D_2 \,|\, D_1 \ni \gamma_i, \gamma_i^+ \arrow[d, dash, "\cong \text{ in } \mathcal{J}^\bullet \ \mathrm{I}\ \text{“rearrange”}"] \\
\rotatebox{90}{$\in$}\ \mathcal{LG}^\bullet(V,U) \times \mathcal{LG}^\bullet(W,V) \arrow[rr] & & \mathcal{LG}^\bullet(W,U) & \\
\mathcal{LG}(V,U) \times \mathcal{LG}(W,V) \arrow[u, "\cong"] \arrow[rr, dashed] & & \mathcal{LG}(W,U) \arrow[u, "\cong"'] & \\
(D_1, D_2) \arrow[uuu] & & & (D_2 \circ D_1) \otimes \bigwedge(\mathbb{C}\theta_1 \oplus \cdots \oplus \mathbb{C}\theta_n) \ni \theta_i, \theta_i^* \arrow[d, "\mathrm{II}\ \text{“projection”}"] \\
& & & D_2 \circ D_1
\end{tikzcd}$$

Implemented in software, we calculate $D_2 \,|\, D_1$ and the γ_i, γ_i^+ as matrices and then split idempotents $\gamma_1\gamma_1^+, \ldots, \gamma_n\gamma_n^+$ to compute $D_2 \circ D_1$ as an explicit matrix of polynomials.

<u>Application</u> : this calculation of $D_2 \circ D_1$ via $D_2 \,|\, D_1$ is the basis of one approach to calculating an A_∞-minimal model of the DG-enhancement of $\mathrm{hmf}(\mathbb{C}[\underline{x}], W)$.