Monoidal Categories:

Outline:

1. Monoidal Cat
2. Monoids in Monoidal Cat
3. Δ^+ & Monoidal Functors
4. Symmetric Monoidal Cat
5. Symmetric Monoidal Functors & Φ
6. Frobenius Objects.

Important Example:

Consider $\underline{Vec_{k}}$

Then the tensor product

$$\otimes_{k} : \underline{Vec_{k}} \times \underline{Vec_{k}} \longrightarrow \underline{Vec_{k}}$$

and $k \in \underline{Vec_{k}}$ define a monoidal category.

Note: Algebraic monoid is a set with an associative product and a unit.

Def^n: (Monoidal Category)

A (strict) monoidal category is a category V with two functors

$\mu : V \times V \longrightarrow V$ (eg $\otimes_{\mathbb{k}} : Vec_{\mathbb{k}} \times Vec_{\mathbb{k}} \longrightarrow Vec_{\mathbb{k}}$)

$\eta : \mathbb{1} \longrightarrow V$ (eg: $\eta(\mathbb{1}) = \mathbb{k}$)

category with one object and one morphism

such that the following diagrams commute

$$\begin{tikzcd}
 & V \times V \times V \arrow[dl, "\mu \times id_V"'] \arrow[dr, "id_V \times \mu"] & \\
V \times V \arrow[dr, "\mu"'] & & V \times V \arrow[dl, "\mu"] \\
 & V &
\end{tikzcd}$$

eg

$$\begin{tikzcd}
 & (A, B, C) \arrow[dl] \arrow[dr] & \\
(A \otimes_{\mathbb{k}} B, C) \arrow[d] & & (A, B \otimes_{\mathbb{k}} C) \arrow[d] \\
(A \otimes_{\mathbb{k}} B) \otimes_{\mathbb{k}} C \arrow[rr, phantom, "\cong"] & & A \otimes_{\mathbb{k}} (B \otimes_{\mathbb{k}} C)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{1} \times V \arrow[r, "\eta \times id_V"] \arrow[dr] & V \times V \arrow[d, "\mu"] \\
 & V
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathbb{1}, A) \arrow[r] \arrow[d] & (\mathbb{k}, A) \arrow[d] \\
A \arrow[r, phantom, "\cong"] & \mathbb{k} \otimes_{\mathbb{k}} A
\end{tikzcd}$$

$$\begin{tikzcd}
V \times V \arrow[d, "\mu"'] & V \times \mathbb{1} \arrow[l, "id_V \times \eta"'] \arrow[dl] \\
V &
\end{tikzcd}
\qquad
\begin{tikzcd}
(A, \mathbb{k}) \arrow[d] & (A, \mathbb{1}) \arrow[l] \arrow[dl] \\
A \otimes_{\mathbb{k}} \mathbb{k} \cong A &
\end{tikzcd}$$

For strict monoidal categories these are equalities

Functoriality of μ:

For a monoid category (V, μ, η) and $A, B \in Ob(V)$ we denote

$$\mu(A,B) = A \otimes B \quad (\text{or } A \otimes B,\ A \sqcup B)$$

Functoriallity of μ means that for

$$A \xrightarrow{f} A' \xrightarrow{f'} A'' \qquad B \xrightarrow{g} B' \xrightarrow{g'} B''$$

we have

$$\begin{tikzcd} A \otimes B \arrow[r, "f \otimes g"] \arrow[rr, bend right, "(f' \circ f) \square (g' \circ g)"'] & A' \otimes B' \arrow[r, "f' \otimes g'"] & A'' \otimes B'' \end{tikzcd}$$

And $id_{A \otimes B} = id_A \otimes id_B$

Examples: 1) $(Set, \times, \{*\})$ $\left(\begin{array}{l} (A \times B) \times C \cong A \times (B \times C) \\ \{*\} \times A \cong A \cong A \times \{*\} \end{array} \right)$

2) $(Set, \sqcup, \emptyset)$ $\left(\begin{array}{l} (A \sqcup B) \sqcup C \cong A \sqcup (B \sqcup C) \\ \emptyset \sqcup A \cong A \cong A \sqcup \emptyset \end{array} \right)$

3) $(Vec_{\mathbb{k}}, \otimes, \mathbb{k})$

4) $(2\mathrm{Cob}, \sqcup, \emptyset)$ — will see in detail next week

$\mathrm{Ob}(2\mathrm{Cob})$ = closed oriented 1-mflds

$$= \{\underbrace{S^1 \sqcup \cdots \sqcup S^1}_{n \text{ times}} : n \in \mathbb{Z}_{\geq 0}\}$$

$\mathrm{Hom}(\underbrace{S^1\sqcup \cdots \sqcup S^1}_{m \text{ times}}, \underbrace{S^1 \sqcup \cdots \sqcup S^1}_{n \text{ times}})$ = closed oriented 2-mflds M with

$$\partial M = (\underbrace{S^1 \sqcup \cdots \sqcup S^1}_{m \text{ times}}) \sqcup (\underbrace{S^1 \sqcup \cdots \sqcup S^1}_{n \text{ times}})$$

<u>Note</u>: there is a sense of "in" and "out" boundaries which formalised by fixing embeddings $f_{in}: (S^1 \sqcup \cdots \sqcup S^1)^* \to \partial M$ and

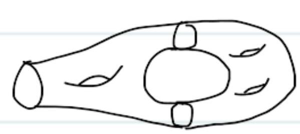
$f_{out}: (S^1 \sqcup \cdots \sqcup S^1) \to \partial M$.

* reversed orientation

eg of composition

$S^1, S^1 \sqcup S^1, \emptyset \in \mathrm{Ob}(2\mathrm{Cob})$ ie 1-mflds

$f: S^1 \to S^1 \sqcup S^1 =$ , $g: S^1 \sqcup S^1 \to \emptyset =$ 

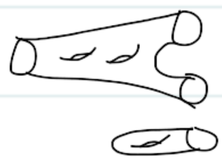
$g \circ f =$  $=$ 

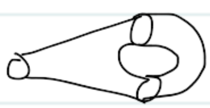
Here functoriality of $\amalg$ follows from the following pictures.

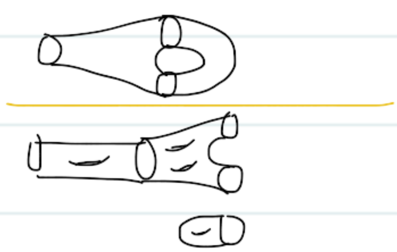
If $f: S^1 \longrightarrow S^1 \amalg S^1 =$ 

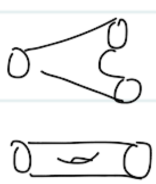
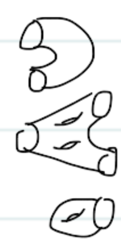
$g: S^1 \longrightarrow S^1 =$ 

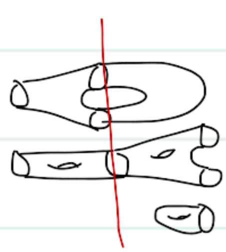
$h: S^1 \amalg S^1 \longrightarrow \emptyset =$ 

$k: S^1 \longrightarrow S^1 \amalg S^1 \amalg S^1 =$ 

Then $h \circ f =$  $k \circ g =$ 

So $h \circ f \amalg k \circ g =$ 

Now $f \amalg g =$  $h \amalg k =$ 

$(f \amalg g) \circ (h \amalg k) =$ 

4) Categories with $\begin{cases} \text{products and a terminal object} \\ \text{coproducts and an initial object} \end{cases}$

5) $\Delta^+ = \Delta \cup \{\emptyset\}$

simplex category $\to \Delta$

$ob(\Delta^+) = \{\langle n \rangle\}_{n \in \mathbb{Z}_{\geq 0}}$ for $n > 0$ $\langle n \rangle = [n-1]$

where $\langle n \rangle = (0, 1, \cdots, n-1)$ ordered set.

$Hom(\langle n \rangle, \langle m \rangle)$ = order preserving maps

Let $\otimes : \Delta^+ \times \Delta^+ \to \Delta^+$ s.t that

$$\langle n \rangle \otimes \langle m \rangle = \langle n+m \rangle$$

For $f : \langle n_1 \rangle \longrightarrow \langle m_1 \rangle$ order preserving
$g : \langle n_2 \rangle \longrightarrow \langle m_2 \rangle$

$f \otimes g : \langle n_1 + n_2 \rangle \to \langle m_1 + m_2 \rangle$ s.t

$$f \otimes g(x) = \begin{cases} f(x) & \text{if } 0 \leq x \leq n_1 - 1 \\ m_1 + g(x - n_1) & \text{if } n_1 \leq x \leq n_1 + n_2 \end{cases}$$

Then $(\Delta^+, \otimes, \langle 0\rangle)$ is a monoidal category.

proof: The main thing to check is functoriality of $\otimes$.

Let $f:\langle n_1\rangle \to \langle n_2\rangle$ $g:\langle n_2\rangle \to \langle n_3\rangle$
$h:\langle m_1\rangle \to \langle m_2\rangle$ $k:\langle m_2\rangle \to \langle m_3\rangle$

$$(g\circ f)\otimes(k\circ h):\langle n_1\rangle\otimes\langle m_1\rangle \to \langle n_3\rangle\otimes\langle m_3\rangle$$

s.t $$(g\circ f)\otimes(k\circ h)(x) = \begin{cases} (g\circ f)(x) & \text{if } 0\le x\le n_1-1 \\ n_3+(k\circ h)(x-n_1) & \text{if } n_1\le x\le n_1+m_1 \end{cases}$$

$$f\otimes h(x) = \begin{cases} f(x) & \text{if } 0\le x\le n_1-1 \\ n_2+h(x-n_1) & \text{if } n_1\le x\le n_1+m_1 \end{cases}, \quad g\otimes k(x) = \begin{cases} g(x) & \text{if } 0\le x\le n_2-1 \\ n_3+k(x-n_2) & \text{if } n_2\le x\le n_2+m_2 \end{cases}$$

$$(g\otimes k)\circ(f\otimes h)(x) = (g\otimes k)\begin{cases} f(x) & \text{if } 0\le x\le n_1-1 \\ n_2+h(x-n_1) & \text{if } n_1\le x\le n_1+m_1 \end{cases} = \begin{cases} g\circ f(x) & \text{if } 0\le x\le n_1-1 \\ n_3+g(n_2+h(x-n_1)-n_2) & \text{if } n_1\le x\le n_1+m_1 \end{cases}$$

as $f(x)\le n_2-1$
$n_2\le n_2+h(x-n_1)$

Defn: (Monoid (in a monoidal category))

Let $(V, \otimes, I)$ be a monoidal category.

A monoid (M, μ, η) in V is an object $M \in \mathrm{Ob}(V)$ and morphisms $\mu \in \mathrm{Hom}(M \otimes M, M)$, $\eta \in \mathrm{Hom}(I, M)$ s.t the following diagrams commute

(associativity)

$$\begin{tikzcd} & M \otimes M \otimes M \arrow[dl, "\mu \otimes id_M"'] \arrow[dr, "id_M \otimes \mu"] & \\ M \otimes M \arrow[dr, "\mu"] & & M \otimes M \arrow[dl, "\mu"'] \\ & M & \end{tikzcd}$$

(identity)

$$\begin{tikzcd} I \otimes M \arrow[r, "\eta \otimes id_M"] \arrow[dr] & M \otimes M \arrow[d, "\mu"] & M \otimes I \arrow[l, "id_M \otimes \eta"'] \arrow[dl] \\ & M & \end{tikzcd}$$

Def^n (Monoid Morphism)

For monoids $(M, \mu, \eta), (N, \nu, \beta)$ in V, $f \in \mathrm{Hom}(M, N)$ is a morphism of monoids if

$$\begin{tikzcd} M \otimes M \arrow[r, "f \otimes f"] \arrow[d, "\mu"'] & N \otimes N \arrow[d, "\nu"] \\ M \arrow[r, "f"] & N \end{tikzcd} \qquad \begin{tikzcd} M \arrow[rr, "f"] & & N \\ & I \arrow[ul, "\eta"] \arrow[ur, "\beta"'] & \end{tikzcd}$$

Ex: $\underline{\underline{\mathrm{Mon}}} \subseteq \underline{\underline{\mathrm{Set}}}$ monoids in sets

$\underline{\underline{\mathrm{Rng}}} \subseteq \underline{\underline{\mathrm{Ab}}}$ rings in abelian groups

Monoidal Functors:

For $(V, \otimes, I), (V', \otimes', I')$ monoidal categories a monoidal functor from V to V' is a functor $F: V \to V'$ s.t the following digrams commute.

$$\begin{tikzcd} V \times V \arrow[r, "F \times F"] \arrow[d, "\mu"'] & V' \times V' \arrow[d] \\ V \arrow[r, "F"] & V' \end{tikzcd} \qquad \begin{tikzcd} V \arrow[rr, "F"] & & V' \\ & \mathbb{1} \arrow[ul, "\eta"] \arrow[ur, "\eta'"'] & \end{tikzcd}$$

Example: $F: Vec_{\mathbb{R}} \to Vec_{\mathbb{C}}$

$$A \longmapsto \mathbb{C} \otimes_{\mathbb{R}} A$$

Monoidal as $\mathbb{C} \otimes_{\mathbb{R}} (A \otimes_{\mathbb{R}} B) \cong (\mathbb{C} \otimes_{\mathbb{R}} A) \otimes_{\mathbb{C}} (\mathbb{C} \otimes_{\mathbb{R}} B)$

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R} \cong \mathbb{C}$$

Example: Let $(V, \otimes, I)$ be a monoidal category.

Let (M, μ, η) be a monoid in V.

We have a monoidal functor given by

$$F : \Delta^+ \longrightarrow V \quad \text{s.t} \quad F(\langle n \rangle) = M^{\otimes n}$$

and for $f : \langle n \rangle \to \langle m \rangle$ we can write f as a combination of $z : \langle 0 \rangle \to \langle 1 \rangle$, $m : \langle 2 \rangle \to \langle 1 \rangle$ and $+$.

Therefore defining $F(z) = \eta$ and $F(m) = \mu$ determines $F(f)$ and therefore a functor F.

Likewise given monoidal $F : \Delta^+ \to V$ $(F(\langle 1 \rangle), F(z), F(m))$ is a monoid in V.

Def^n: (Category of Monoids in V)

Let $(V, \otimes, I)$ be a monoidal category.

Let $ob(Mon(V))$ be the monoids in V.

For $M, N \in ob(Mon(V))$ let $Hom_{Mon(V)}(M,N)$ be morphisms of monoids.

Mon(V) is a category.

Theorem 1: $MonCat(\Delta^+, V) \underset{\text{cat}}{\overset{\text{equiv. of}}{\longleftrightarrow}} Mon(V)$

(monoidal functors from Δ^+ to V)

Remark: Δ^+ is therefore the "free monoidal cat on a single object".

Remark: We have avoided a some technicalities by avoiding the issues around "strictness".
This can all be dealt with but we can igror these issues for the most part.

See 3.2. 17—19 of Kock for a few details.

Def$^{\underline{n}}$: (Symmetric Monoidal Category)

A (strict) monoidal category $(V, \otimes, I)$ with maps $\tau_{X,Y} : X \otimes Y \to Y \otimes X$ for each $X, Y \in ob(V)$ is called a <u>symmetric monoidal category</u> if

i) the maps $\tau_{X,Y}$ are natural i.e.

$$\begin{tikzcd} X \otimes Y \arrow[r, "\tau_{X,Y}"] \arrow[d, "f \otimes g"] & Y \otimes X \arrow[d, "g \otimes f"] \\ X' \otimes Y' \arrow[r, "\tau_{X',Y'}"] & Y' \otimes X' \end{tikzcd}$$

commutes

ii) For $X, Y, Z \in ob(V)$

$$\begin{tikzcd} X \otimes Y \otimes Z \arrow[rr, "\tau_{X, Y \otimes Z}"] \arrow[dr, "\tau_{X,Y} \square id_Z"'] & & Y \otimes Z \otimes X \\ & Y \otimes X \otimes Z \arrow[ur, "id_Y \square \tau_{X,Z}"'] & \end{tikzcd} \qquad \begin{tikzcd} X \otimes Y \otimes Z \arrow[rr, "\tau_{X \otimes Y, Z}"] \arrow[dr, "id_X \square \tau_{Y,Z}"'] & & Z \otimes X \otimes Y \\ & X \otimes Z \otimes Y \arrow[ur, "\tau_{X,Z} \square id_Y"'] & \end{tikzcd}$$

iii) $\tau_{Y,X} \circ \tau_{X,Y} = id_{X \otimes Y}$

Example: 1) $A\otimes B \xrightarrow{\tau_{A,B}} B\otimes A$ the linear map such that $\tau_{A,B}(a\otimes b) = b\otimes a$

2) Previous examples such as set with $\times, \sqcup$

3) $grVec_{\mathbb{k}}$ = graded vector spaces

for $V = \bigoplus_{n\in\mathbb{Z}} V_n$ and $W = \bigoplus_{n\in\mathbb{Z}} W_n$

$(V\otimes W)_n = \bigoplus_{p+q=n} V_p \otimes W_q$

$I = \bigoplus_{n\in\mathbb{Z}} I_n$ with $I_n = \begin{cases} \mathbb{k} & \text{if } n=0 \\ 0 & \text{o.w} \end{cases}$

Can define two twist maps s.t for $v\in V_p$, $w\in W_q$

$\tau_{V,W}(v,w) = v\otimes w$

$\tau'_{V,W}(v,w) = (-1)^{pq} v\otimes w$ ← Koszul's sign change.

5) Δ^+ is not symmetric. (Idea: $\tau_{V,W}$ would have to be id. $id\circ f\otimes g \neq g\otimes f\circ id$!!! ← in general)

Def: (Symetric Monoidal Functor)

For symetric monoidal categories $(V, \otimes, I, \tau)$ and $(V', \otimes', I', \tau')$ a functor $F: V\to V'$ is a symetric monoidal functor if it is monoidal and for $X, Y \in ob(V)$

$$\tau_{X,Y} F = \tau'_{F(X),F(Y)}$$

eg Extension of scalars (as before).

Defn: (Commutative Monoid in Sym. Mon. Cat. V)

Let $(V, \otimes, I, \tau)$ sym. mon. cat.

If $(M, \mu, \eta) \in ob(\underline{\underline{Mon(V)}})$ and

$$\begin{tikzcd} M \otimes M \arrow[rr, "\tau"] \arrow[dr, "\mu"'] & & M \otimes M \arrow[dl, "\mu"] \\ & M & \end{tikzcd}$$

is commutative. Then we say M is a commutative monoid.

Defn: (Category of Commutative Monoids in V)

The class of commutative objects forms a category with morphisms given by monoid morphisms.

Denote this category by $\underline{\underline{cMon(V)}}$

Defn: Φ = cat. of finite sets with morphisms given by fn.

Remark: $(\Phi, \sqcup, \emptyset, \tilde{\tau})$ is a symmetric monoidal cat.

Theorem 2: $SymMonCat(\Phi, V) \simeq cMon(V)$

Aim: Define Frobenius objects and find the analogue of Δ^+ and Φ to make a thm 3.

Defn: (Frobenius Objects)

A <u>Frobenius Object</u> in a monoidal category $(V, \otimes, I)$ is an object $A \in Ob(V)$ with four maps

$$\eta : I \to A \qquad \mu : A \square A \to A \qquad \delta : A \to A \square A \qquad \varepsilon : A \to I$$

s.t. 1) $\mu \circ (id_A \square \eta) = \underset{\substack{\uparrow \\ \text{proj onto } A}}{id_A} = \mu(\eta \square id_A)$ (multiplication μ and unit η)

2) $(id_A \square \varepsilon) \circ \delta = id_A = (\varepsilon \square id_A) \circ \delta$ (comulti μ and counit ε)

3) $(id_A \square \mu) \circ (\delta \square id_A) = \delta \circ \mu = (\mu \square id_A) \circ (id_A \otimes \delta)$
(Frobenius relation)

We denote the collection of frobenious objects by Frob(V).

Ex: In $\underline{\mathrm{Vec}_{\mathbb{K}}}$ the frobenius objects are given by the frobenius algebras.

See 2.3.24 of Kock

Def: Frobenius homomorphisms are morphisms between frobenius object preserving the structures $\mu, \eta, \delta, \varepsilon$.

Remark: Frob(V) is a category with frobenius morphisms.

Def: (Commutative Frobenius Objects)

A commutative frobenius object in a symmetric monoidal category $(V, \Box, I, \tau)$ is commutative if

$$\mu \circ \tilde{\tau}_{x,x} = \mu$$

and cocommutative if

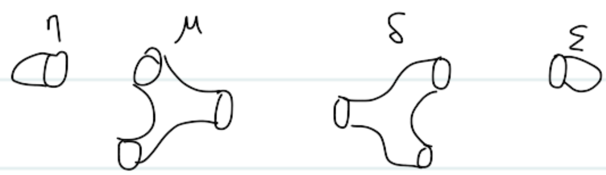
$$\tilde{\tau}_{x,x} \circ \delta = \delta$$

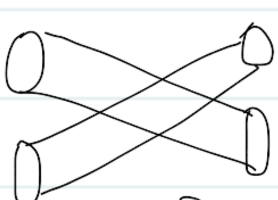
Lemma: A Frobenius object is commutative iff it is cocommutative.

Remark: The set of commutative frobenius objects $cFrob(V)$ forms a category with frobenius morphisms.

Next: 2Cob.

2Cob symmetric monoidal category





twists

every object is frobenius.