The category of simply-typed lambda terms (cuts)

What we want to do is define a functor $FV : \mathcal{L} \longrightarrow \mathcal{V}ar$ where $\mathcal{V}ar$ has a single object object $\Box$ and $\mathcal{V}ar(\Box,\Box) = \{\text{finite subsets of } \mathcal{Y}\}$ with composition as union. But even once we use FV_β so that is well-defined on arrows, it cannot be a functor because of the previous example. However (9.3) and (9.4) tell us we have a lax functor between 2-categories.

Def^n A 2-category $\mathscr{C}$ is the data of

(1) a class of objects $ob(\mathscr{C})$ written $a, b, c, \ldots$

(2) for each pair $a, b \in ob(\mathscr{C})$ a small category $\mathscr{C}(a,b)$ whose objects are called 1-morphisms and denoted $X, Y, \ldots : a \longrightarrow b$ and whose morphisms are called 2-morphisms, denoted $\alpha, \beta, \ldots$ e.g. $X \Rightarrow Y$.

(3) for each triple $a, b, c \in ob(\mathscr{C})$ a functor (composition)

$$\mathscr{C}(b,c) \times \mathscr{C}(a,b) \xrightarrow{c_{abc}} \mathscr{C}(a,c)$$

(4) for each $a \in ob(\mathscr{C})$ a unit $\Delta_a : a \longrightarrow a$.

satisfying the following conditions

(5) for any a, b, c, d the diagram

$$\begin{tikzcd}
\mathscr{C}(c,d) \times \mathscr{C}(b,c) \times \mathscr{C}(a,b) \arrow[r, "c \times 1"] \arrow[d, "1 \times c"'] & \mathscr{C}(b,d) \times \mathscr{C}(a,b) \arrow[d, "c"] \\
\mathscr{C}(c,d) \times \mathscr{C}(a,c) \arrow[r, "c"'] & \mathscr{C}(a,d)
\end{tikzcd}$$

commutes.

(6) For any a,b the diagrams ($\mathbb{1} = \{\cdot\}$ is the category with one object & one morphism)

$$\begin{tikzcd}
\mathbb{1} \times \mathscr{C}(a,b) \arrow[d, "\{\Delta_b\}\times 1"] & \mathscr{C}(a,b) \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d, equal] & \mathscr{C}(a,b) \times \mathbb{1} \arrow[d, "1 \times \{\Delta_a\}"] \\
\mathscr{C}(b,b) \times \mathscr{C}(a,b) \arrow[r, "c"'] & \mathscr{C}(a,b) & \mathscr{C}(a,b) \times \mathscr{C}(a,a) \arrow[l, "c"]
\end{tikzcd}$$

commute.

<u>Example</u> (1) If $\mathcal{J}$ is a monoidal category then there is a 2-category $\mathscr{C}$ with one object $\square$ and $\mathscr{C}(\square,\square) = \mathcal{J}$, composition being $\otimes$.

(2) If $(L, \leq)$ is a bounded lattice then there is a monoidal category $\mathcal{J} = \mathcal{J}(L, \leq)$ with $ob(\mathcal{J}) = L$ and $\otimes = \vee$ the join, with unit $\emptyset$, and morphisms given by $\leq$. Hence there is a 2-category $\mathscr{C}_L$ as above, with $\mathscr{C}_L(\square,\square) = \mathcal{J}$.

(3) Any category may be viewed as a 2-category with only identity 2-morphisms.

<u>Defn</u> We denote by $\mathcal{V}ar$ the 2-category associated to the lattice of finite subsets of $\mathcal{Y}$, so

- a single object $\square$
- 1-morphisms $\square \longrightarrow \square$ are finite sets of variables $Q \subseteq \mathcal{Y}$ (not ordered).
- 2-morphisms are inclusions $Q \subseteq Q'$
- composition $\square \xrightarrow{Q} \square \xrightarrow{P} \square$ is union $P \cup Q$.

<u>Def^n^</u> Let $\mathcal{A},\mathcal{B}$ be 2-categories. A <u>colax functor</u> $F:\mathcal{A}\longrightarrow\mathcal{B}$ is the data of

(1) a function $ob(\mathcal{A})\longrightarrow ob(\mathcal{B})$, denoted $a\longmapsto F(a)$

(2) for each pair $a,b\in ob(\mathcal{A})$ a functor $F_{ab}:\mathcal{A}(a,b)\longrightarrow\mathcal{B}(Fa,Fb)$,

(3) for each $a\in ob(\mathcal{A})$ a 2-morphism $F(\Delta_a)\to\Delta_{Fa}$

(4) for each composable pair $a\xrightarrow{X}b\xrightarrow{Y}c$ in $\mathcal{A}$ a 2-morphism

$$\mathcal{Y}_{Y,X}: F(Y\circ X)\longrightarrow F(Y)\circ F(X) \qquad (3.1)$$

natural in both variables,

all subject to some axioms which we omit here.

<u>Lemma</u> Viewing $\mathcal{L}$ as a 2-category with only identity 2-morphisms, there is a colax functor ($\mathcal{V}ar$ as defined on p. 12)

$$FV_\beta:\mathcal{L}\longrightarrow\mathcal{V}ar$$

<u>Proof</u> FV_β sends each object of $\mathcal{L}$ to $\square$, and each λ-term M to $FV_\beta(M)$, with $FV_\beta(*):=\phi$. The lemmas on p. 9 imply the "colaxity" of (3.1). $\square$

From now on we will silently view any category $\mathcal{J}$ as a 2-category with only identity 2-morphisms.

<u>Def^n</u> Let $\mathcal{A}$ be a category and $F: \mathcal{A} \longrightarrow \mathcal{V}ar$ a colax functor. Define for P a subset of $\mathcal{Y}$ a subcategory $\mathcal{A}_P \subseteq \mathcal{A}$ with the same objects as $\mathcal{A}$, but

$$\mathcal{A}_P(a,a') := \{ f \in \mathcal{A}(a,a') \mid F(f) \subseteq P \}.$$

<u>Lemma</u> $\mathcal{A}_P$ is a subcategory.

<u>Proof</u> Condition (3) of a colax functor implies $F(1_a) = \emptyset$ for all a, and (4) shows $\mathcal{A}_P$ is closed under composition, as

$$F(f_2 \circ f_1) \subseteq F(f_2) \cup F(f_1) \subseteq P. \quad \square$$

Our categorical description of (9.5.1) says that $\mathfrak{U}$ is something like a strong cartesian morphism, but weakened in the 2-categorical setting to an adjunction rather than an equivalence. To state this properly, note that for simple types β, ρ we have a functor

$$FV_\beta^{\beta,\rho} : \mathcal{L}(\beta,\rho) \longrightarrow \mathcal{V}ar(\square,\square), \quad M \longmapsto FV_\beta(M) \qquad (4.1)$$

and for $Q \subseteq \mathcal{Y}$ viewed as a functor $-\cup Q : \mathcal{V}ar(\square,\square) \to \mathcal{V}ar(\square,\square)$ we can form the <u>comma category</u> (a certain 2-limit) of the diagram made of $FV_\beta^{\beta,\rho}$ and $-\cup Q$.

$$\begin{tikzcd}
FV_\beta^{\beta,\rho}/Q \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\swarrow\mu"] & \mathcal{L}(\beta,\rho) \arrow[d, "FV_\beta^{\beta,\rho}"] \\
\mathcal{V}ar(\square,\square) \arrow[r, "-\cup Q"'] & \mathcal{V}ar(\square,\square)
\end{tikzcd} \qquad (4.2)$$

The comma category $FV_\beta^{\beta,\rho}/Q$ is universal filling in this diagram with a 2-cell (i.e. natural transformation) μ, and can be described concretely as:

Def^n $FV_\beta^{\mathfrak{z},\rho}/Q$ has

- objects: triples (M, P, γ) where $M \in \mathcal{X}(\mathfrak{z},\rho)$, $P \subseteq \mathcal{Y}$ is finite, and γ is a 2-morphism $FV_\beta(M) \longrightarrow P \cup Q$. This is a condition, not data, so objects are really (M, P) s.t. $FV_\beta(M) \subseteq P \cup Q$.

- morphisms: $(M,P) \longrightarrow (M', P')$ are pairs $M \to M'$ (forcing $M = M'$) and $P \subseteq P'$ such that the diagram

$$\begin{tikzcd} FV_\beta(M) \arrow[r, equal] \arrow[d] & FV_\beta(M') \arrow[d] \\ P \cup Q \arrow[r] & P' \cup Q \end{tikzcd}$$

commutes, which is again no condition, so morphisms are just $(M,P) \to (M, P')$ with $P \subseteq P'$.

- The functors $FV_\beta^{\mathfrak{z},\rho}/Q \xrightarrow{\pi_1} \mathcal{X}(\mathfrak{z},\rho)$ and $FV_\beta^{\mathfrak{z},\rho}/Q \xrightarrow{\pi_2} \mathcal{V}ar(\square,\square)$ $(M,P) \mapsto M$ and $(M,P) \mapsto P$ respectively, while the natural transformation $\mu : FV_\beta^{\mathfrak{z},\rho} \circ \pi_1 \Longrightarrow (- \cup Q) \circ \pi_2$ is determined.

Remark: Every morphism in $\mathcal{V}ar(\square,\square)$ (thus also in $FV_\beta^{\mathfrak{z},\rho}/Q$) is an epimorphism (and monomorphism).

Lemma: The projective objects of $FV_\beta^{\mathfrak{z},\rho}/Q$ are precisely the pairs (M, P) with $P = FV_\beta(M) \setminus Q$.

Proof: An object (M,P) will be projective iff. P is minimal s.t. $FV_\beta(M) \subseteq P \cup Q$, which gives the claim. $\square$

<u>Lemma</u> For every morphism $u : \theta \to \rho$ with $FV_\beta(u) \subseteq Q$ there is a canonically induced functor
$$\Phi_u : \mathcal{L}(\zeta, \theta) \longrightarrow FV_\beta^{\zeta,\rho}/Q.$$

<u>Proof</u> Consider

$$\begin{tikzcd}
\mathcal{L}(\zeta,\theta) \arrow[dr, dashed, "\Phi_u"] \arrow[drr, bend left, "u \circ -"] \arrow[ddr, bend right, "FV_\beta^{\zeta,\theta}"'] & & \\
& FV_\beta^{\zeta,\rho}/Q \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow\mu"] & \mathcal{L}(\zeta,\rho) \arrow[d, "FV_\beta^{\zeta,\rho}"] \\
& \mathcal{V}ar(\square,\square) \arrow[r, "-\cup Q"'] & \mathcal{V}ar(\square,\square)
\end{tikzcd} \tag{6.1}$$

since there is a 2-cell $FV_\beta^{\zeta,\rho} \circ (u \circ -) \Rightarrow (- \cup Q) \circ FV_\beta^{\zeta,\theta}$ this is automatic. $\square$

Explicitly, Φ_u sends $N \in \mathcal{L}(\zeta,\theta)$ to the pair $(u \circ N, FV_\beta(N))$.

<u>Proposition</u> For each simple type ρ and finite set $Q \subseteq \mathcal{Y}$ there is an object $Q^*\rho$ in $\mathcal{L}$ and morphism $\mathfrak{U}^Q : Q^*\rho \to \rho$ with $FV_\beta(\mathfrak{U}^Q) = Q$ such that the canonical induced functor for every type ζ

$$\mathcal{L}(\zeta, Q^*\rho) \xrightarrow{\ \Phi\ } FV_\beta^{\zeta,\rho}/Q \tag{6.2}$$

restricts to a bijection natural in ζ,

$$\mathcal{L}(\zeta, Q^*\rho)_{Q^c} \xrightarrow{\ \cong\ } \mathrm{Proj}\left(FV_\beta^{\zeta,\rho}/Q \right) \tag{6.3}$$

where the LHS is the full subcategory of N with $FV_\beta(N) \cap Q = \emptyset$ and the RHS is the full subcategory of projective objects. In particular $(Q^*\rho, \mathfrak{U}^Q)$ is unique up to unique isomorphism.

<u>Remark</u> This says in particular that for given ρ, Q the functor (see p. 14 for $\mathcal{L}_{Q^c}$)

$$\begin{aligned} \mathcal{L}_{Q^c}^{op} &\longrightarrow \underline{Set} \\ \mathfrak{z} &\longmapsto \mathrm{Proj}\left(FV_\beta^{\mathfrak{z},\rho} / Q \right) \end{aligned}$$

projective objects of comma of

$$\begin{array}{ccc} & & \mathcal{L}(\mathfrak{z},\rho) \\ & \swarrow & \downarrow FV_\beta \\ \mathcal{V}ar(\square,\square) & \xrightarrow[-\cup Q]{} & \mathcal{V}ar(\square,\square) \end{array}$$

is representable, and the representing pair is $(Q^*\rho, \mathfrak{U}^Q)$.

<u>Proof</u> Let $\rho \in \Phi_\to$ and Q be a finite set of variables. If $Q = \phi$ define $Q^*\rho := \rho$. Otherwise choose an ordering $Q = \{q_1 : J_1, \dots, q_k : J_k\}$

$$Q^*\rho := J_1 \to J_2 \to \cdots \to J_k \to \rho. \qquad (7.1)$$

and define

$$\mathfrak{U}^Q = \lambda u^{Q^*\rho}. (\cdots((u\, q_1)\, q_2) \cdots q_k) : Q^*\rho \to \rho \qquad (7.2)$$

As discussed earlier $Q^*\rho$ is independent up to isomorphism of the chosen ordering, in a way which is clearly compatible with $\mathfrak{U}^Q$. In any case the uniqueness statement of the Proposition absolves us from caring about this. Clearly $FV_\beta(\mathfrak{U}^Q) = Q$ so we have $\Phi = \Phi_{\mathfrak{U}^Q}$ as in (16.2) defined by

$$\Phi(N) = (\mathfrak{U}^Q \circ N, FV_\beta(N))$$

// end. At this point we realised this was overly complicated ... (belatedly)