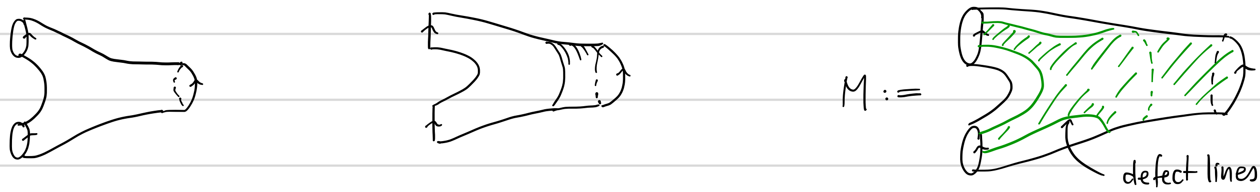
Clifford algebras and Defect TFT

History

- {closed 2D TFT} $\hookrightarrow$ { open-closed 2D TFTs } $\hookrightarrow$ { 2D defect TFTs }



= commutative Frobenius algebras = Calabi-Yau categories "=" pivotal 2-categories

- Carqueville-Runkel (arXiv: 0909.4381) initiated the study of the pivotal 2-category (actually bicategory) $\mathcal{LG}$ associated to the 2D defect TFT defined using $\mathcal{N}=2$ Landau-Ginzburg models.

- In order to compute $Z(M)$, need formulas for structure maps in $\mathcal{LG}$

  CR did this for $W(x)=x^d$, compared to A-type $\mathcal{N}=2$ super-Virasoro min. model.
  (composition in $\mathcal{LG} \longleftrightarrow$ fusion of defects in CFT)

- Carqueville-M (arXiv:1208.1481) (arXiv: 1303.1389) did all potentials W.

- Today: a bit of fun with Clifford algebras! (= case W quadratic)

Assume k char. 0 field.

$$\underset{\text{bicategory}}{\mathcal{C}liff_k} \xrightarrow[\text{fully faithful (strong) functor}]{F} \underset{\text{bicategory}}{\mathcal{LG}_k}$$

	$\mathcal{C}liff_k$	$\mathcal{LG}_k$
obj	Clifford algebras	potentials $W(x)$
1-mor	bimodules	matrix factorisations (MF)
2-mor	bimodule maps	htpy classes MF maps.

$\mathcal{C}liff_k$ pivotal
$\Rightarrow \exists$ a 2D defect TFT w/
phases = Clifford alg
defects = bimodules.

Def^n A 1-morphism $W(x) \longrightarrow V(y)$ is

$$\underbrace{\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}}_{\text{MF}}^2 = \begin{pmatrix} (W-V)\cdot I_r & 0 \\ 0 & (W-V)\cdot I_r \end{pmatrix}$$

block $r \times r$ matrices over $k[x,y]$.

Outline of talk

1. Brief introduction to Clifford algebras
2. Def^n of the functor F

1. Clifford algebras

$$\left[\sum_\mu \gamma^\mu \partial_\mu\right]^2 = -\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2$$

Recall: $g = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ Dirac matrices $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ $\quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \cdot I_4$

Quadratic space (V, B) = vector space V, $B : V \otimes_k V \longrightarrow k$ non-deg symmetric bilinear.

Def^n/Thm There is a universal pair (ι, A), A a k-algebra and $\iota : V \longrightarrow A$ linear satisfying $\{\iota(v), \iota(w)\} := \iota(v)\iota(w) + \iota(w)\iota(v) = 2B(v,w)\cdot 1_A$. It is the Clifford algebra $Cl(V,B) := A$.

<u>Fact</u> $Cl(V,B)$ is naturally $\mathbb{Z}_2$-graded, $\iota(V) \subseteq Cl(V,B)^1$, $\dim_k Cl(V,B) = 2^{\dim V}$

<u>Example</u> (a) $k=\mathbb{R}$, $V=\mathbb{R}v$, $B(v,v)=-1$. $\quad (B=-x^2)$

$$\iota(v)^2=-1 \implies Cl(V,B)=\mathbb{C}=\overset{0}{\mathbb{R}}\oplus\overset{1}{\mathbb{R}i}$$

(b) $k=\mathbb{R}$, $V=\mathbb{R}v_1\oplus\mathbb{R}v_2$ $\quad B(v_i,v_j)=-\delta_{ij}$, $\quad (B=-x_1^2-x_2^2)$

$$\iota(v_i)^2=-1,\ \{\iota(v_1),\iota(v_2)\}=0 \implies Cl(V,B)=\mathbb{H}$$

(c) $V=\bigoplus_{i=1}^n \mathbb{R}v_i$, $\quad B\in \mathrm{Sym}^2(V^*)\subseteq k[x_1,\dots,x_n]$ $\quad x_i=e_i^*$

$Cl(r,s) = Cl(\mathbb{R}^{r+s}, x_1^2+\cdots+x_r^2-x_{r+1}^2 - \cdots - x_{r+s}^2)$

<u>Sylvester</u> every Clifford alg$/\mathbb{R} \cong Cl(r,s)$ some r,s.

(d) $Cl(3,1) \circlearrowright \mathbb{C}^4$ $\quad \iota(v_{i+1})$ acts as Dirac γ^i

(e) $Cl(V,B)^{op} \cong Cl(V,-B)$

$Cl(V_1,B_1)\otimes Cl(V_2,B_2) \cong Cl(V_1\oplus V_2, B_1+B_2)$

<u>Def^n</u> $\mathcal{C}\mathit{liff}_{\mathbb{R}}$ = bicategory of Clifford algs$/\mathbb{R}$

f.g. free ($\mathbb{Z}_2$-graded)

$$\mathcal{C}\mathit{liff}_{\mathbb{R}}((V_1,B_1),(V_2,B_2)) := \{Cl(V_2,B_2)-Cl(V_1,B_1)-\text{bimodules}\} \ni E$$
$$\cong \{Cl(V_1,B_1)^{op}\otimes Cl(V_2,B_2)-\text{modules}\}$$
$$\cong \{Cl(V_1\oplus V_2, B_2-B_1)-\text{modules}\}$$

$$\begin{tikzcd}
(V_1,B_1) \arrow[r, bend left=50, "E"{name=U}] \arrow[r, bend right=50, "E'"'{name=D}] & (V_2,B_2)
\arrow[from=U, to=D, Rightarrow]
\end{tikzcd}$$

<u>Remark</u> $\mathcal{C}\mathit{liff}_{\mathbb{R}}(C(r,s), C(r',s')) = C(r'+s, s'+r)\text{-mod}$

② The functor $F : \mathcal{C}liff_{\mathbb{R}} \longrightarrow \mathcal{LG}_{\mathbb{R}}$

<u>obj</u> $F(\mathbb{R}v_1 \oplus \cdots \oplus \mathbb{R}v_n, B) := (k[x_1, \dots, x_n], B(x_i))$ $\quad x_i = v_i^*$

<u>1-mor</u>

$$\begin{tikzcd}
(V = \oplus_{i=1}^n \mathbb{R}v_i, B_1) \arrow[r, mapsto] \arrow[d] & (k[\underline{x}], B_1) \arrow[d] \\
(W = \oplus_{j=1}^m \mathbb{R}w_j, B_2) \arrow[r, mapsto] & (k[\underline{y}], B_2)
\end{tikzcd}$$

$Cl(V \oplus W, B_2 - B_1)\text{-mod} \longrightarrow E \longmapsto \tilde{E}^2 = \begin{pmatrix} B_2 - B_1 & 0 \\ 0 & B_2 - B_1 \end{pmatrix}$ $\qquad y_j = w_j^*$

matrix factorisation $\tilde{E} := \sum_{i=1}^n x_i \, \iota(v_i) + \sum_{j=1}^m y_j \, \iota(w_j) \circlearrowright E \otimes_{\mathbb{R}} k[\underline{x}, \underline{y}]$

has block shape $\begin{pmatrix} 0 & /// \\ /// & 0 \end{pmatrix} \circlearrowright E$

$$\tilde{E}^2 = -\sum_{i_1 \le i_2} x_{i_1} x_{i_2} B_1(v_{i_1}, v_{i_2}) + \sum_{j_1 \le j_2} y_{j_1} y_{j_2} B_2(w_{j_1}, w_{j_2})$$

$$= B_2(\underline{y}) - B_1(\underline{x})$$

Note $\partial_{x_i}(\tilde{E}) = \iota(v_i)$

<u>Theorem</u> (Buchweitz-Eisenbud-Herzog) F is fully faithful, i.e.

$$Cl(V, B)\text{-mod} \xrightarrow{\simeq} hmf(\operatorname{Sym} V^*, B)^{\omega}$$

$$\begin{array}{ccc}
\therefore \quad \mathcal{C}liff_{\mathbb{R}}((V_1, B_1), (V_2, B_2)) & & \mathcal{LG}_{\mathbb{R}}((\operatorname{Sym} V_1^*, B_1), (\operatorname{Sym} V_2^*, B_2)) \\
\| & & \| \\
Cl(V_1 \oplus V_2, B_2 - B_1)\text{-mod} & \xrightarrow[\simeq]{} & hmf(\operatorname{Sym} V_1^* \otimes \operatorname{Sym} V_2^*, B_2 - B_1)
\end{array}$$

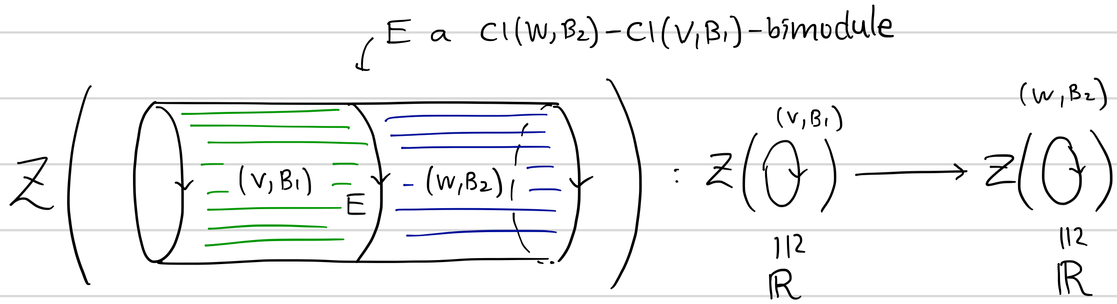
$\Longrightarrow$ F is fully faithful

well, conjecturally

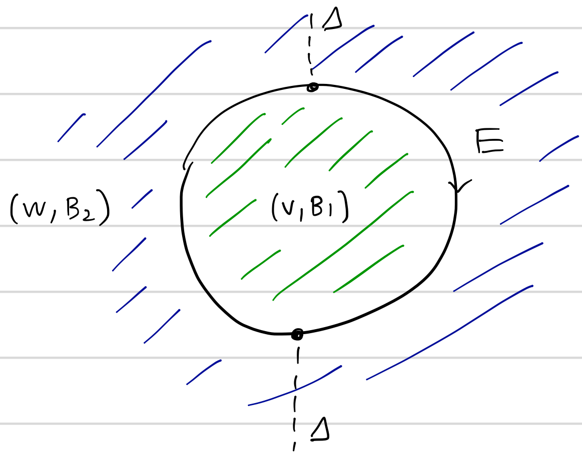
Corollary $\mathcal{C}liff_{\mathbb{R}}$ is a pivotal bicategory (∴ defines 2D defect TFT) since $\mathcal{L}\mathcal{G}$ is.

$\Rightarrow$ "defines" $Z : \mathrm{Bord}^{2,1}_{\mathrm{def}} \longrightarrow \mathrm{Vect}_{\mathbb{R}}$ in terms of $\gamma^i = \iota(v_i) \circlearrowright E$

Example



is computed in $\mathcal{C}liff_{\mathbb{R}}$ via a string diagram



write: $\delta_\alpha = \iota(v_\alpha)$ $\circlearrowright E$
$\gamma^\mu = \iota(w_\mu)$

$$\Delta \xrightarrow{\mathrm{coev}} E \underset{Cl(V,B_1)}{\otimes} E^\vee \xrightarrow{\widetilde{\mathrm{ev}}} \Delta$$

$\Vert$

$Cl(W,B_2)$

$\Vert$ as vector sp.

$$\Lambda W \ni 1 \longmapsto \sum_{i,j} \{\gamma^1 \cdots \gamma^m\}_{ij}\, e_i \otimes e_j^* \longmapsto \sum_{i,j} (-1)^{s+(n+1)|e_j|} \cdot \frac{1}{2^{r+s}} \{\gamma^1 \cdots \gamma^m\}_{ij} \{\delta_1 \cdots \delta_n\}_{ji}$$

(assume $Cl(V,B_1) = C(r,s)$)

$$= \frac{(-1)^s}{2^{r+s}} \operatorname{str}_E\big(\underbrace{\gamma^1 \cdots \gamma^m \delta_1 \cdots \delta_n}_{\text{"volume form"}}\big) + \text{higher terms in } \omega\text{'s}$$

where

Formulas CM '08 —

$$\mathrm{coev}(\omega_1\cdots\omega_k) = \sum_{i,j}\{\gamma^{k+1}\cdots\gamma^m\}_{ij}\, e_i\otimes e_j^*$$

$$\widetilde{ev}(e_i\otimes e_j^*) = \sum_{l\geq 0}\sum_{a_1<\cdots<a_l} \omega_{a_1}\cdots\omega_{a_l}\frac{(-1)^{s+(n+1)|e_j|}}{2^{r+s}} \cdot\{\gamma^{a_l}\cdots\gamma^{a_1}\delta_1\cdots\delta_n\}_{ji}$$

<u>Example</u> $Cl(r,r) = \left(\oplus_{i=1}^r \mathbb{R}v_i^+ \oplus \oplus_{i=1}^r \mathbb{R}v_i^-,\ B(v_i^+,v_i^+)=2,\ B(v_i^-,v_i^-)=-2\right)$

We define a representation E of $Cl(r,r)$ as follows (viewed as $\mathbb{R}\xrightarrow{E} Cl(r,r)$)

$$\left.\begin{aligned} e_i &:= \tfrac{1}{2}(v_i^+ + v_i^-)\\ e_i^* &:= \tfrac{1}{2}(v_i^+ - v_i^-)\end{aligned}\right\}\ \text{check}\quad \begin{aligned} &B(e_i,e_i) = B(e_i^*,e_i^*) = 0\\ &B(e_i,e_i^*) = \tfrac{1}{4}\big(B(v_i^+,v_i^+) - B(v_i^-,v_i^-)\big)\\ &\qquad = \tfrac{1}{4}(2+2) = 1\end{aligned}$$

Define $P = \mathrm{span}\langle e_1,\dots,e_n\rangle$, $Q = \mathrm{span}\langle e_1^*,\dots,e_n^*\rangle$ so $V = P\oplus Q$, and set

$E = \Lambda P$ $\circlearrowleft$ e_i acts as $e_i\wedge(-)$, e_i^* acts as $e_i^*\lrcorner(-)$, defines an action of $Cl(r,r)$.

In this case the value of $Z(M)$, i.e. the supertrace of the volume form, is

$$\mathrm{str}_E(e_n^*\cdots e_1^* e_1\cdots e_n) = 1.$$