FJRW theory

Cohomological field theory CohFT $(H, \{\Omega_{g,n}\}, \langle , \rangle, 1)$

$$\Omega_{g,n} : H^{\otimes n} \longrightarrow H^*(\overline{\mathcal{M}_{g,n}})$$

To compute/determine $\Omega_{g,n}$ one introduces $(M = \dim H)$

$$F_g(t) := \sum_{n=0}^{\infty} \sum_{\substack{a_1,\dots,a_n \in \mathbb{Z}_{\geq 0} \\ 0 \leq k_1,\dots,k_n \leq M}} \langle \tau_{a_1}(e_{k_1}) \cdots \tau_{a_n}(e_{k_n}) \rangle_{g,n} \frac{t_{a_1}^{k_1} \cdots t_{a_n}^{k_n}}{n!}$$

- $e_1, \dots, e_M$ is a basis of H s.t. $e_1 = 1$
- t_a^k formal variables $a \in \mathbb{Z}_{\geq 0}$, $k = 1, \dots, M$ correlation functions

$$\langle \tau_{a_1}(e_1) \cdots \tau_{a_n}(e_{k_n}) \rangle_{g,n} := \int_{\overline{\mathcal{M}_{g,n}}} \Omega_{g,n}(e_{k_1}, \dots, e_{k_n}) \cap \prod_{i=1}^{n} \psi_i^{a_i}$$

- $\psi_i = c_1(L_i) \in H^2(\overline{\mathcal{M}_{g,n}})$ $\quad L_i = T_{p_i}^* C$
  $(C, p_1, \dots, p_n) \in \overline{\mathcal{M}_{g,n}}$

- $Z(t, \hbar) := \exp\left(\sum_{g=0}^{\infty} \hbar^{2-2g} F_g(t) \right)$

Example The trivial CohFT $H = H^*(pt) = \mathbb{C}$, corresponds to "topological gravity" Kontsevich-Witten model. F_g satisfies a set of PDEs called dilaton eq (DE) string eq. (SE) topological recursion (TRR).

Thm (Kontsevich, conj. by Witten) $Z(t, \hbar)$ is a τ-function of the KdV hierarchy.

Calabi-Yau/non-semisimple

categorical level

open/closed CohFT

aka Costello?

$$\mathrm{Fuk}(X) \overset{\text{HMS}}{\underset{\cong}{\longleftrightarrow}} D^b(\mathrm{coh}\, X^\vee)$$

$$\mathrm{FS}_G(W) \overset{\cong}{\underset{\text{HMS}}{\longleftrightarrow}} \mathrm{HMF}(W^\vee, G^\vee)$$

$\mathrm{FS}_G(W) \overset{\cong}{\longleftrightarrow} \mathrm{Fuk}(X)$, $\mathrm{HMF}(W^\vee, G^\vee) \overset{\cong}{\longleftrightarrow} D^b(\mathrm{coh}\, X^\vee)$

not understood Fukaya-Seidel G-equivariant

HH

enumerative theory

closed CohFT

genus zero known

genus zero

$$GW(X) \overset{\text{mirror symmetry}\ \cong ?}{\underset{\text{conjecture}}{\longleftrightarrow}} \widehat{VHS}(X^\vee)$$

LG/CY $\cong$? conjecture

? $\cong$ conj.

$$\mathrm{FJRW}(W,G) \overset{\text{mirror sym}\ \cong ?}{\underset{\text{conjecture}}{\longleftrightarrow}} \widehat{\mathrm{SGT}}(W^\vee, G^\vee)$$

Si Li proves for $G=1$

Saito-Givental-Teleman ($G=1$)

space

$$X \overset{\text{mirror map}}{\underset{\text{SYZ, Batyrev–Borisov}}{\longleftrightarrow}} \hat{X}$$

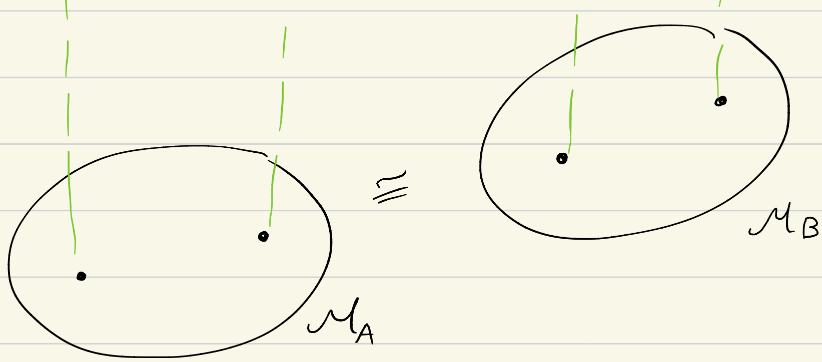
LG/CY

analytic continuation

W polynomial, G group

$$(W,G) \overset{\text{mirror map}}{\underset{\text{Berglund–Hübsch–Krawitz}}{\longleftrightarrow}} (W^\vee, G^\vee)$$

parameter space



Landau-Ginzburg orbifolds (Intriligator-Vafa)

- W: quasi-homogeneous polynomial in N variables.
- $\exists q_1, \dots, q_N \in \mathbb{Q} \cap [0,1)$ s.t. $W(\lambda^{q_1} x_1, \dots, \lambda^{q_N} x_n) = W(x_1, \dots, x_N)$ $\forall x \in \mathbb{C}^N$ $\forall \lambda \in \mathbb{C}$
- critical locus of W is compact, today $= \{\underline{0}\}$.

$$Aut(W) = \{ \mathrm{diag}(e^{2\pi i \alpha_1}, \dots, e^{2\pi i \alpha_N}) \mid W(e^{2\pi i \alpha_1} x_1, \dots, e^{2\pi i \alpha_N} x_N) = W(x_1, \dots, x_N) \ \forall x \}$$

$J \in Aut(W)$ $J = (e^{2\pi i \alpha_1}, \dots, e^{2\pi i \alpha_N})$

$\langle J \rangle \subseteq G \subseteq Aut(W)$.

subgroups of $Aut(W)$ are the most well-studied in the context of mirror symmetry, but the picture should hold more generally

$$H = \bigoplus_{\gamma \in G} (Jac(W_\gamma) \otimes dx_\gamma)^G = \bigoplus_{\sigma \in G} H_\sigma$$

$$W_\gamma = W|_{(\mathbb{C}^N)^\gamma} \qquad dx_\gamma = dx_1 \wedge \dots \wedge dx_N |_{(\mathbb{C}^N)^\gamma}$$

$$H_\gamma \overset{\text{FJR}}{=} H^N((\mathbb{C}^N)^\gamma; Re W_\gamma^\infty; \mathbb{C})^G$$

$$\overset{\text{PV}}{\cong} HH_*(HMF_G(W_\gamma))^G$$

Pairing: residue pairing $\langle , \rangle_\sigma$ on $Jac(W_\sigma)$

$\varepsilon : H_\gamma \xrightarrow{\cong} H_{\gamma^{-1}}$

$\rightsquigarrow \langle , \rangle_\gamma$

$$\Omega_{g,n} : H^{\otimes n} \longrightarrow H^*(W_{g,n}(W,G)) \xrightarrow{st_*} H^*(\overline{\mathcal{M}}_{g,n}) \qquad \begin{array}{l} g \geq 0, n \geq 0 \\ 2g-2+n \geq 0 \end{array}$$

$$\bar{\gamma} = (\gamma(1), \ldots, \gamma(n)) \in G^n$$

$$W_{g,n}(W,G)(\bar{\gamma}) = \left\{ (\mathcal{C}, z_1, \ldots, z_n ; \mathcal{L}_1, \ldots, \mathcal{L}_N ; \varphi_1, \ldots, \varphi \quad) \right\} / \sim$$

where $(\mathcal{C}, z_1, \ldots, z_N)$ is a d-stable orbicurve, with markings $z_1, \ldots, z_n$, i.e. markings and nodes are allowed to be orbifold points, $\mathcal{C}$ acted upon by a finite abelian group of order d (degree of W), $\mathcal{L}_j \longrightarrow \mathcal{C}$ orbi-line bundle $j=1,\ldots,N$ at the marked point the monodromy of the bundle is $M_{z_i}(\mathcal{L}_j) = \gamma_j(i) \in G$, φ_k are isomorphisms

monodromy

$$\varphi_k : W_k(\mathcal{L}_1, \ldots, \mathcal{L}_N) \xrightarrow{\sim} \omega_{\mathcal{C},\log}$$

sheaf of differential forms
with poles of order ≤ 1 at marked points

where

$$W = \sum_k c_k \prod_j x_j^{M_{jk}} \qquad W_k(\mathcal{L}_1, \ldots, \mathcal{L}_N) = \bigotimes \mathcal{L}_j^{\otimes M_{jk}}$$

'07

<u>Theorem</u> (FJR) $W_{g,n}(W,G) \xrightarrow[st]{|G|^n:1} \overline{\mathcal{M}}_{g,n}$ is a finite cover, in particular $W_{g,n}(W,G)$ is compact (since $\langle J \rangle \subseteq G$). Branched over the boundary of $\overline{\mathcal{M}}_{g,n}$, and $W_{g,n}(W,G)$ is smooth.

conjectured/constructed by Witten '91 $\quad W = x^d$, $G = \mathbb{Z}/d\mathbb{Z}$

This is the analog of $\overline{\mathcal{M}_{g,n}}(X,\beta)$ in GW theory.

Still need a <u>virtual</u> fundamental class of $W_{g,n}(W,G)$.

$$[W_{g,n}(W,G)]^{vir} \in H^*(W_{g,n}(W,G)) \otimes \bigotimes_{i=1}^{n} (H_{\gamma(i)})^{\vee}$$

$s_j \in H^0(\mathcal{C}, \mathcal{L}_j)$ need to satisfy a PDE $\quad \bar{\partial} s_j + \overline{\partial_j W} = 0 \quad \forall j=1,\dots,N$

$$\begin{array}{l} \mathcal{M} = \{(s_1,\dots,s_N) \in H^0(\mathcal{C}, \oplus_j \mathcal{L}_j) \mid \bar{\partial} s_j + \overline{\partial_j W} = 0\}/\sim \\ \quad \big\downarrow st_{\mathcal{M}} \\ W_{g,n}(W,G) \end{array}$$

if $\gamma_j(i) \neq 1 \quad \forall i,j$ then $st_{\mathcal{M}} = id$.

Let $\alpha_1,\dots,\alpha_N \in H$

Poincaré dual

$$\Omega_{g,n}^{W,G}(\alpha_1,\dots,\alpha_n) := \frac{|G|^g}{\deg(st)} \, PD\, st_*\Big([W_{g,n}(W,G)]^{vir} \cap \prod_{i=1}^{n} \alpha_i\Big)$$

$$\cap$$

$$H^*(\overline{\mathcal{M}}_{g,n})$$

<u>Theorem</u> (FJR '07) $\quad (H, \Omega_{g,n}^{W,G}(\alpha_1,\dots,\alpha_n), \langle,\rangle, 1)$
defines a CohFT satisfying DE, SE, TRR.

<u>Theorem</u> (FJR '07) Let W be one of the ADE potentials. Then $Z(t,\hbar)$ is a τ-function of the Drinfeld-Sokolov hierarchy corresponding to ADE.

$\longleftarrow$ Kontsevich-Witten is $W = x^2$

$W = x^{n+1}$ conjectured by Witten in '91

If $\mathrm{diag}(g) \neq 1$ then $H_{\gamma(i)}$ is called <u>narrow</u>, otherwise broad.

in A-cases all sectors are narrow, which is why the A-case was understood earlier

PV:

$$\begin{tikzcd}
\mathrm{HMF}_G(W) \arrow[rr, "\text{using } \Psi_k \ \otimes PV_{\bar{\gamma}}", "K(\alpha)"'] \arrow[d, "ch"'] & & MF(W \times \mathbb{C}^{\bar{\gamma}}, 0) \arrow[d, "ch(\,)\cdot td"] \\
H^{\otimes n} \arrow[rr] & & H^*(\mathcal{W}_{g,n}(W,G))
\end{tikzcd}$$

commutes in narrow sectors in general
for all sectors in ADE.