① 20/8/16

Generalised orbifolding of simple singularities

This talk was originally given at the Geometry at ANU conference in August 2016. The aim is to explain a remarkable theorem of the mathematical physicists Carqueville – Ros Camacho – Runkel, and my own contribution to some of the mathematical inputs to their theorem.

Schematically the result can be described as an unexpected series of equivalences of triangulated categories associated to isolated hypersurface singularities

$$\begin{array}{lcll} W \in \mathbb{C}[x_1,\dots,x_n] & \longmapsto & \text{triangulated category} & hmf(W) \\ \neq & & & \wr \\ V \in \mathbb{C}[y_1,\dots,y_m] & \longmapsto & \quad " \qquad\qquad " & hmf(V) \end{array}$$

different singularities

The equivalences are not $hmf(W) \simeq hmf(V)$ (this would be too strong to be interesting) but of the form

$$hmf(V) \cong \mathrm{Mod}_{hmf(W)}(A)$$

Frobenius algebra / monad

The interesting examples so far are pairs (V, W) of simple singularities (aka ADE singularities), and unimodular singularities, but there are probably many more.

Outline
1. Matrix factorisations i.e. $hmf(W)$
2. Frobenius algebras
3. Statement of theorem
4. Sketch of proof (uses work of mine with Carqueville).

↙ Eisenbud '80

<u>Def</u>

Def^N Let R be a commutative ring and $W \in R$. A <u>matrix factorisation</u> of W is a $\mathbb{Z}_2$-graded f.g. projective R-module $X = X^0 \oplus X^1$ together with an odd R-linear $d_X : X \longrightarrow X$ s.t. $d_X^2 = W \cdot 1_X$. A <u>morphism</u> $\varphi : (X, d_X) \to (Y, d_Y)$ is a degree zero map with $d_Y \varphi = \varphi d_X$.

Def^N $hmf(R,W) :=$ matrix factorisations of W with homotopy equivalence classes of morphisms (so $(R \oplus R, \begin{pmatrix} 0 & W \\ 1 & 0 \end{pmatrix}) \cong 0$).

Examples (1) $R = \mathbb{C}[x]$, $W = x^N$ $N \geq 2$, for $1 \leq i \leq N-1$

$$E_i := \left(R \oplus R, \begin{pmatrix} 0 & x^i \\ x^{N-i} & 0 \end{pmatrix} \right) \in hmf(\mathbb{C}[x], x^N)$$

(2) $R = \mathbb{C}[x,y]$, $W = y^N - x^N$ $N \geq 2$, $\eta = e^{2\pi i/N}$ so

$$y^N - x^N = \prod_{0 \leq i \leq N-1} (y - \eta^i x)$$

Given $S \subseteq \{0, \dots, N-1\}$ we have

$\boxed{\text{keep}}$

$$P_S := \left(R \oplus R, \begin{pmatrix} 0 & \prod_{i \in S} (y - \eta^i x) \\ \prod_{i \notin S} (y - \eta^i x) & 0 \end{pmatrix} \right) \in hmf(\mathbb{C}[x,y], y^N - x^N)$$

Theorem (Buchweitz, Orlov) There is an equivalence of triangulated categories

$$hmf(\mathbb{C}[x_1, \dots, x_n], W) \simeq D^b(\mathrm{coh}\, W^{-1}(0)) / \mathrm{Perf}(W^{-1}(0))$$

Proposition If $W \in \mathbb{C}[x_1,\dots,x_n]$ has isolated critical points then

Keep

$$\mathcal{E}_W = \operatorname{hmf}(\mathbb{C}[x]\otimes\mathbb{C}[x], W\otimes 1 - 1\otimes W)^{\omega}$$

Karoubi closure / idempotent closure

is naturally a $\otimes$-triangulated category $(\mathcal{E}_W, *)$, and $\operatorname{hmf}(\mathbb{C}[x], W)^{\omega}$ is an $\mathcal{E}_W$-module, i.e. there is an action $\mathcal{E}_W \times \operatorname{hmf}(W)^{\omega} \longrightarrow \operatorname{hmf}(W)^{\omega}$, where the tensor product $*$ is for $X, Y \in \mathcal{E}_W$

$$Y * X := \text{finite rank representative of } (Y \underset{\mathbb{C}[x]}{\otimes} X, \underbrace{d_Y \otimes 1 + 1 \otimes d_X}) \text{ in } \mathcal{E}_W.$$

squares to $W\otimes 1\otimes 1 - 1\otimes W\otimes 1 + 1\otimes W\otimes 1 - 1\otimes 1\otimes W$

and the action of X on $E \in \operatorname{hmf}(\mathbb{C}[x], W)$ is

$$Y * E := \text{finite rank representative of } (Y \underset{\mathbb{C}[x]}{\otimes} E, \underbrace{d_Y \otimes 1 + 1 \otimes d_E}) \text{ in } \operatorname{hmf}(W).$$

squares to $W \otimes 1$

Note $Y * (-) : \operatorname{hmf}(W) \longrightarrow \operatorname{hmf}(W)$ is a triangulated functor.

Example $\mathcal{E}_{x^N} = \operatorname{hmf}(\mathbb{C}[x,y], y^N - x^N)$ is monoidal, with $0 \le \lambda \le N-2$

$$P_{a:\lambda} := P_{\{a, a+1, \dots, a+\lambda\}} \in \mathcal{E}_{x^N}$$

⌜Example $P_{a:\lambda} * P_{b:0} = P_{a+b:\lambda}$⌟

$$P_{a:\lambda} * P_{b:\mu} = \bigoplus_{\substack{\nu = |\lambda-\mu| \\ \text{by steps of } 2}}^{\min(\lambda+\mu,\, 2N-4-\lambda-\mu)} P_{a+b-\frac{1}{2}(\mu+\lambda-\nu) : \nu}$$

(Brunner–Roggenkamp '07)

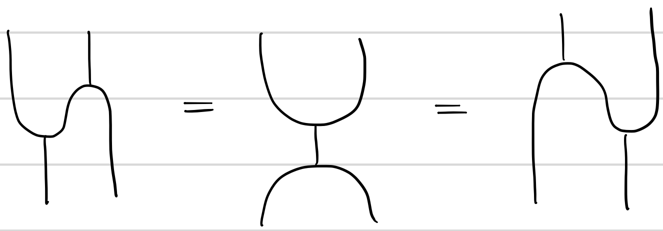
related to fusion / $\widehat{su(2)}_{N-2}$

2 Frobenius algebras

Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a monoidal category. A Frobenius algebra in $\mathcal{C}$ is an object $A \in \mathrm{ob}(\mathcal{C})$ equipped as an

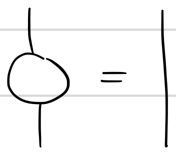
- associative, unital algebra, $\mu: A^{\otimes 2} \to A$, $\eta: A \longrightarrow \mathbb{1}$
- coassociative, counital coalgebra, $\Delta: A \to A^{\otimes 2}$, $\varepsilon: \mathbb{1} \longrightarrow A$

such that the Frobenius identity holds:



i.e. $$(1_A \otimes \mu) \circ (\Delta \otimes 1_A) = \Delta \circ \mu = (\mu \otimes 1_A) \circ (1_A \otimes \Delta)$$

A Frobenius algebra is separable if



, i.e. $\mu \circ \Delta = 1_A$

If there is an action $\mathcal{C} \times \mathcal{T} \xrightarrow{\otimes} \mathcal{T}$, $A \otimes - : \mathcal{T} \longrightarrow \mathcal{T}$ is a monad, and $\mathrm{Mod}_{\mathcal{T}}(A)$ denotes modules over this monad.

Theorem (Balmer) If $\mathcal{C}$ is a $\otimes$-triangulated category acting on a triangulated category $\mathcal{T}$, and A is a separable algebra in $\mathcal{C}$, then $\mathrm{Mod}_{\mathcal{T}}(A)$ is naturally triangulated (some caveats). $\boxed{\text{skip}}$

③ From now on $W \in \mathbb{C}[x_1, \dots, x_n]$, $V \in \mathbb{C}[y_1, \dots, y_m]$ have isolated crit. points.

Def^N V is a weak generalised orbifold (wGO) of W, denoted $W \longrightarrow_{\mathrm{wGO}} V$, if there is a separable Frobenius algebra $A \in \mathcal{E}_W$ and an equivalence of triangulated categories

$$\mathrm{Mod}_{\mathrm{hmf}(W)}(A) \simeq \mathrm{hmf}(V) \qquad \left(\text{note } \mathrm{hmf}(W) \circlearrowleft A * -\right)$$

Example (1) $W \longrightarrow_{\mathrm{wGO}} W$ $\quad A = \Delta_W \in \mathcal{E}_W$ which is the monoidal unit

(2) $W \longrightarrow_{\mathrm{wGO}} W + u^2 + v^2$ $\quad A = \Delta_W \otimes_{\mathbb{C}} \mathrm{Cliff}(u^2+v^2)$ $\quad$ skip
(a form of Knörrer periodicity)

Theorem (Carqueville-Ros Camacho-Runkel '13) With the notation

$$\begin{aligned}
V^{(\mathrm{A}_{d-1})} &= x_1^d + x_2^2 & c &= 3 - 3 \cdot \tfrac{2}{d} & &(d \geqslant 2)\\
V^{(\mathrm{D}_{d+1})} &= x_1^d + x_1 x_2^2 & c &= 3 - 3 \cdot \tfrac{2}{2d} & &(d \geqslant 3)\\
V^{(\mathrm{E}_6)} &= x_1^3 + x_2^4 & c &= 3 - 3 \cdot \tfrac{2}{12} & &\\
V^{(\mathrm{E}_7)} &= x_1^3 + x_1 x_2^3 & c &= 3 - 3 \cdot \tfrac{2}{18} & &\\
V^{(\mathrm{E}_8)} &= x_1^3 + x_2^5 & c &= 3 - 3 \cdot \tfrac{2}{30} & &
\end{aligned}$$

we have

$$\begin{aligned}
(d \text{ even}) \quad V^{(A_{d-1})} &\longrightarrow_{\mathrm{wGO}} V^{(D_{d/2+1})}\\
V^{(A_{11})} &\longrightarrow_{\mathrm{wGO}} V^{(E_6)}\\
V^{(A_{17})} &\longrightarrow_{\mathrm{wGO}} V^{(E_7)}\\
V^{(A_{29})} &\longrightarrow_{\mathrm{wGO}} V^{(E_8)}
\end{aligned}$$

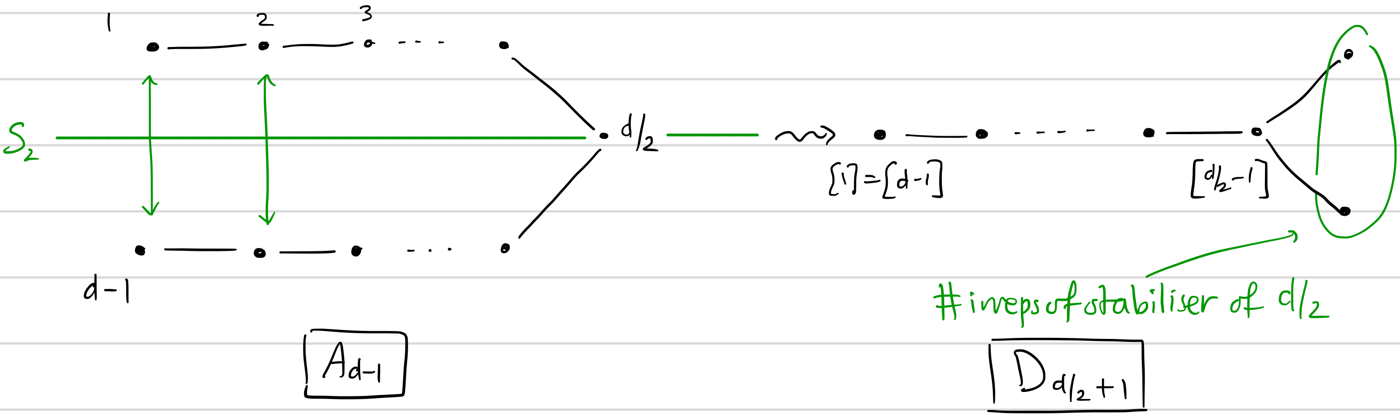
i.e. "ADE from A"

(true for ADE in any dimension not just curves)

<u>Note</u> By a theorem of Kajiura-Saito-Takahashi for V an ADE singularity

$$\mathrm{hmf}^{gr}(\mathbb{C}[\underline{x},t], V+t^2) \simeq D^b(\mathrm{rep}.\,\mathbb{C}\vec{Q})$$

where Q is the corresponding Dynkin quiver. The first of the above wCO relations was known (Reiten – Riedtmann '85) and corresponds to a "folding"



<u>But</u> the $A \longrightarrow E$ relations don't seem to arise some group actions on quivers in this straightforward way.

$\boxed{\text{say, don't write}}$

4 Sketch of proof (assume m, n even)

Prop (Carqueville-M '12) Any $X \in hmf(\mathbb{C}[\underline{x},\underline{y}], V(\underline{y}) - W(\underline{x}))$ has a dual $X^\vee \in hmf(\mathbb{C}[\underline{x},\underline{y}], W(\underline{x}) - V(\underline{y}))$ which is an adjoint (on both sides) in an appropriate bicategory

$$W \underset{X^\vee}{\overset{X}{\rightleftarrows}} V \qquad \left(\text{i.e. } hmf(\mathbb{C}[\underline{x}], W) \underset{X^\vee * -}{\overset{X * -}{\rightleftarrows}} hmf(\mathbb{C}[\underline{y}], V)\right)$$

This leads us morphisms in $\mathcal{E}_W$, $\mathcal{E}_V$ resp.

$$\mathrm{qdiml}(X) := \Delta_W \xrightarrow{\text{unit}} X^\vee * X \xrightarrow{\text{counit}} \Delta_W$$
$$\mathrm{qdimr}(X) := \Delta_V \xrightarrow{\text{unit}} X * X^\vee \xrightarrow{\text{counit}} \Delta_V$$

Prop If both qdim's of X are scalar multiples of 1_Δ, then $A := X^\vee * X$ is a separable Frobenius alg. in $\mathcal{E}_W$ and $\mathrm{Mod}_{hmf(W)}(A) \cong hmf\, V$, that is, $W \longrightarrow_{wGO} V$. (a version of the Barr-Beck theorem)

Def^N If X as in the proposition exists we say W, V are orbifold equivalent $V \sim_{GO} W$. This is an equivalence rel^N stronger than $\longrightarrow_{wGO}$.

One actually proves the ADE singularities are orbifold equivalent, e.g. $V^{(A_{d-1})} \sim_{GO} V^{(D_{d/2+1})}$

Note For a grading $|x_i| \in \mathbb{Q}$ s.t. $|W| = 2$, the central charge is $c(W) = 3\sum_i (1 - |x_i|)$

Lemma $V \sim_{GO} W \Rightarrow m \equiv n \pmod 2$ and $c(W) = c(V)$.

$V^{(D_{d+1})} = x_1^d + x_1 x_2^2 \quad |x_1| = \frac{2}{d}, \; |x_2| = 1 - \frac{1}{d}, \; c = 3 - \frac{3}{d}$, the same as
$V^{(A_{2d-1})} = y_1^{2d} + y_2^2 \quad |y_1| = \frac{1}{d}, \; |y_2| = 1.$

<u>Theorem</u> (Carqueville-M '12) With $X : W \to V$ as above,

$$\mathrm{qdim}_e(X) = \pm \mathrm{Res}_{\mathbb{C}[x,y]/\mathbb{C}[x]} \left(\frac{\mathrm{str}(\partial_{x_1} d_X \cdots \partial_{x_n} d_X \partial_{y_1} d_X \cdots \partial_{y_m} d_X)\, dy_1 \cdots dy_m}{\partial_{y_1} V \cdots \partial_{y_m} V} \right) \cdot 1_\Delta$$

$$\mathrm{str}(M) = \textstyle\sum_i (-1)^{|e_i|} M_{ii}$$

the residue is a polynomial in $\mathbb{C}[x]$

and similarly for $\mathrm{qdim}_r(X)$.

Finally: Carqueville – Ros Camacho – Runkel prove their theorem by searching the space of matrices d_X over $\mathbb{C}[x,y]$ with (i) $d_X^2 = V - W$ and (ii) $\mathrm{qdim}_e(X) \in \mathbb{C}^*$, $\mathrm{qdim}_r(X) \in \mathbb{C}^*$ in a clever way, and finding an explicit d_X in each case.

(5) Notes

- In all cases the <u>Frobenius algebra</u> (of ADE pair) $A := X^\vee * X \in \mathcal{E}_W$ has as underlying object a direct sum of P_S matrix factorisations for some S (Carqueville)

- <u>Conjecture</u> Strangely dual unimodular exceptional singularities are orbifold equivalent (there are four nontrivial cases, as 6 out of 14 are self-dual).

<u>Known</u>:
- $Q_{10} : x^4 + y^3 + xz^2 \sim_{GO} E_{14} : x^4 z + y^3 + z^2$ (Ros-Camacho, Newton '15)

- Exceptional unimodular sing. of same weight $(a_1, a_2, a_3; h)$ are GO-equivalent (Ros-Camacho, Newton '16).

  $|x_i| = \frac{2a_i}{h}$ $\quad c_W = \frac{h+2}{h}$

Q1/ What is the geometric origin of orbifold equivalences?

Q2) Is there a better way of generating examples?

Appendix B ADE orbifolding defects X

- $V^{(A_{d-1})} \sim V^{(D_{d/2+1})}$ rank $X = 2$ (i.e. d_X^1, d_X^0 are 2×2 matrices)
- $V^{(A_{11})} \sim V^{(E_6)}$ rank $X = 2$
- $V^{(A_{17})} \sim V^{(E_7)}$ rank $X = 2$
- $V^{(A_{29})} \sim V^{(E_8)}$ rank $X = 4$

References

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