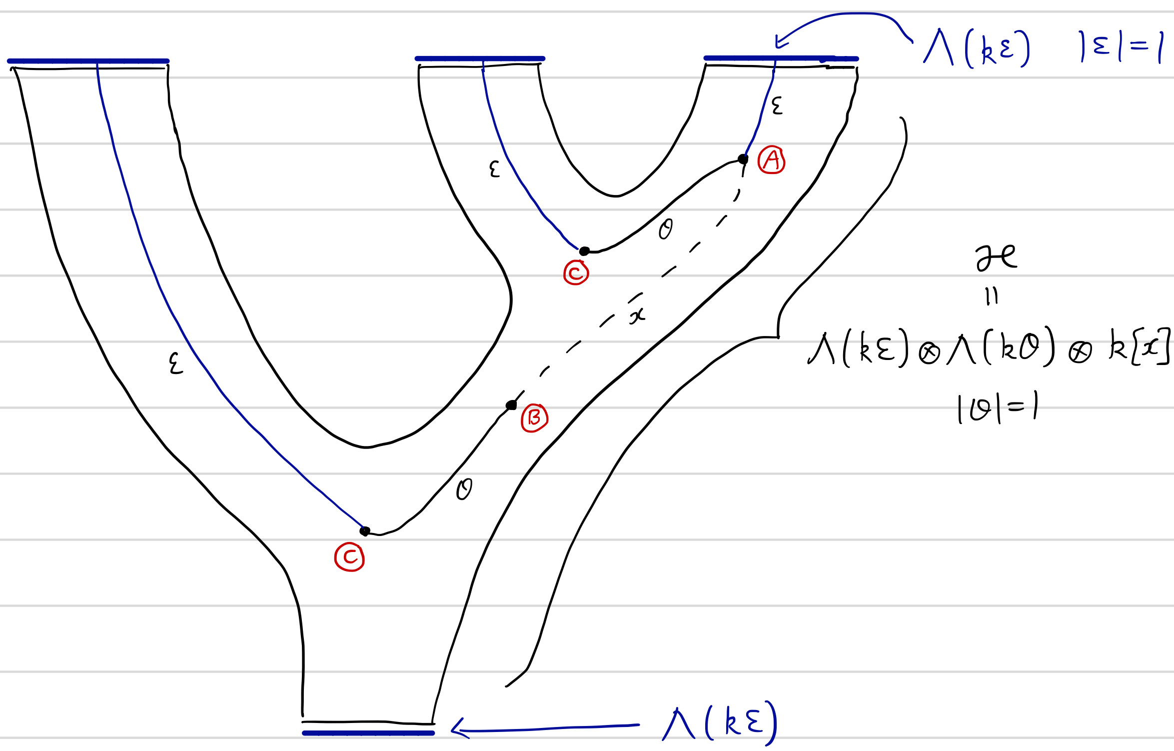
Korea lectures 2017 – I

The aim of these lectures is to present the Feynman rules governing the A_∞-minimal model of the DG-category of matrix factorisations of a potential $W \in k[x_1, \dots, x_n]$. Let us begin with a rough sketch of this destination, before we get into technical matters necessary to construct the minimal model.

$$\text{Potential } W \rightsquigarrow \text{DG-category } mf(W) \begin{cases} \overset{\text{lossy}}{\rightsquigarrow} \text{triangulated cat. } hmf(W) = H^0 mf(W) \\ \underset{\text{lossless}}{\rightsquigarrow} A_\infty\text{-category finite} / k \end{cases}$$

(topological string theory, LG/CFT)

Example For $W = x^3 \in k[x]$, k a char. 0 field, a relevant Feynman diagram is



$$\mathcal{H} = \Lambda(k\varepsilon) \otimes \Lambda(k\theta) \otimes k[x]$$
$$|\theta| = 1$$

As usual, the Feynman diagram depicts a certain pattern of contractions between creation and annihilation operators acting on the $\mathbb{Z}_2$-graded vector space $\mathcal{H}$. For this W there are exactly three kinds of interaction vertices, labelled A, B, C. This diagram computes the only nontrivial contribution to the A_∞-structure,

$$m_3 : \Lambda(k\varepsilon)^{\otimes 3} \longrightarrow \Lambda(k\varepsilon), \quad m_3(\varepsilon^{\otimes 3}) = 1.$$

(m_3 zero on other input, e.g. $\varepsilon \otimes \varepsilon \otimes 1$)

Thus the tuple $\mathcal{B} = (\Lambda(k\varepsilon), m_2, m_3)$ is an A_∞-algebra (m_2 being the usual product in the exterior algebra) and $\mathcal{B}$ is the <u>minimal model</u> of the endomorphism DGA of the standard generator of the DG-category $mf(W)$. In particular, $\mathrm{Perf}_\infty \mathcal{B} \cong hmf(k[x], x^3)^\omega$.

<u>Outline of Lectures</u>

1. Connections and contracting homotopies.
2. The DG-category $mf(W)$ and generators.
3. The A_∞-minimal model.

$$\begin{tikzcd} \mathrm{DG} : mf(W) \ \textcircled{2} \arrow[d, "\text{minimal model}"', "\textcircled{1}"] \\ A_\infty : H^* mf(W) \ \textcircled{3} \end{tikzcd}$$

Lecture I

The technical core of the A_∞-calculations will be certain <u>connections</u> produced from quasi-regular sequences, which we now review. Let R be a commutative k-algebra, for some base ring k.

<u>Def</u>n A sequence $a_1, \dots, a_n \in R$ is <u>quasi-regular</u> if, writing $I = (a_1, \dots, a_n)$, the following morphism of R/I-algebras

$$\phi : R/I[t_1, \dots, t_n] \longrightarrow gr_I R = R/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots$$
$$\phi(t_i) = \bar{a}_i \in I/I^2$$

is an isomorphism. In particular $I/I^2 \cong \bigoplus_{i=1}^{n} R/I \cdot \bar{a}_i$.

<u>Remark</u> Set $Y = \mathrm{Spec}(R/I) \hookrightarrow \mathrm{Spec}(R) = X$, then $C_Y X := \mathrm{Spec}(\mathrm{gr}_I R)$ is a scheme over Y called the <u>normal cone</u>. When I is generated by a quasi-regular sequence, the normal sheaf (to Y in X) is a bundle

$$(I/I^2)^* := \mathrm{Hom}_{R/I}(I/I^2, R/I)$$

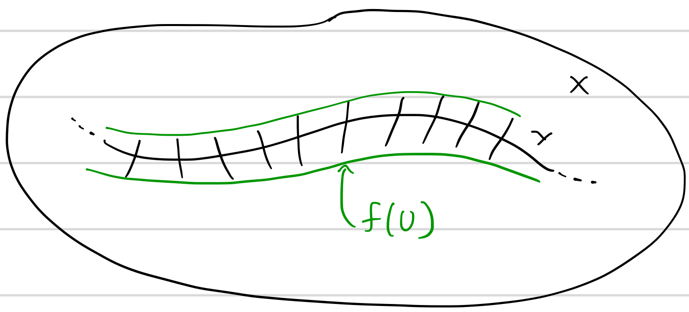
is free of rank n on the basis $\{a_i^*\}_i$, and the (total space of the) normal bundle is the relative Spec

$$\begin{aligned}
\underline{\mathrm{Spec}}_{\mathcal{O}_Y}(\mathrm{Sym}(\mathcal{N}^*_{Y/X})) &= \mathrm{Spec}(\mathrm{Sym}_{R/I}(I/I^2)) \\
&\cong \mathrm{Spec}(R/I[t_1, \dots, t_n]) \qquad t_i = \bar{a}_i \\
&\overset{\text{reg.}}{\cong} \mathrm{Spec}(\mathrm{gr}_I R) \\
&= C_Y X,
\end{aligned}$$

i.e. the normal cone is the normal bundle.

The method of <u>deformation to the normal cone</u> (used in e.g. intersection theory) relates the closed immersion $Y \hookrightarrow X$ by a flat deformation to the inclusion $Y \hookrightarrow C_Y X$. By the above, in the quasi-regular case, this latter immersion is just the zero section of the normal bundle. This is the algebraist's analogue of the tubular neighborhood theorem for smooth manifolds: if $Y \overset{i}{\subset} X$ is a submanifold a partial tubular neighborhood is a neighborhood U of the zero section of $N_Y X \to Y$ and an embedding $f: U \to X$ s.t. $f|_Y = i$ and $f(U)$ is open in X:

(smooth manifolds)



$$\begin{tikzcd}
N_Y X \arrow[d, shift left] & & \\
Y \arrow[u, shift left] \arrow[r] \arrow[rr, hook, bend right, "i"'] & U \arrow[ul, hook] \arrow[r, "f"] & X
\end{tikzcd}$$

Returning to the algebraic case, we do not expect Y to have isomorphic <u>Zariski</u> open neighborhoods in $C_Y X$ and X, but we could ask for the <u>formal</u> neighborhoods to be isomorphic, as in the diagram

(schemes)

$$\begin{tikzcd}
C_Y X \arrow[d, shift right] & (C_Y X)^{\wedge}_Y \arrow[l] \arrow[dr, phantom, "\cong \; (?)"] & \\
& & X^{\wedge}_Y \arrow[d, hook] \\
Y \arrow[u, shift right, hook] \arrow[uur] \arrow[urr] \arrow[rr, hook] & & X
\end{tikzcd} \qquad (*)$$

Now the completion of $C_Y X$ along Y just means passing from $R/I[t_1,\dots,t_n]$ to $R/I[[t_1,\dots,t_n]]$, and completing X along Y means taking the I-adic completion $\hat{R}$. So we are asking for a ring isomorphism (in fact a $k[[t]]$-algebra isomorphism)

$$R/I[[t_1,\dots,t_n]] \cong \hat{R}.$$

<u>Example</u> Let k be a field, $R = k[x]$ and $I = (x^d)$ for $d > 1$. By dimension count

$$\left(k[x]/(x^d) \right)[t] \xrightarrow{\cong} \bigoplus_{i \geq 0} (x^{di})/(x^{di+d})$$

so $a = x^d$ is certainly quasi-regular. The I-adic topology is the same as the (x)-adic topology, so

$$R/I[[t]] = k[x]/(x^d) \otimes_k k[[t]], \qquad \hat{R} = k[[x]].$$

These are certainly not isomorphic (one is reduced, the other is not).

The attempt $(*)$ at an algebraists "formal" tubular neighborhood is too naive, but there is a useful substitute:

<u>Lemma</u> (Lipman) Suppose $a_1, \dots, a_n \in R$ quasi-regular, that R/I is a finitely presented k-module and that the quotient $\pi : R \longrightarrow R/I$ has a k-linear section $\delta : R/I \longrightarrow R$. Then there is an induced isomorphism of $k[[t_1, \dots, t_n]]$-modules

$$\delta^* : R/I \otimes_k k[[\underline{t}]] \longrightarrow \hat{R}.$$

← "formal tubular neighborhood"

<u>Proof</u> First of all, there is a commutative diagram

$$\begin{tikzcd} R \arrow[r, "\pi"] \arrow[d] & R/I \arrow[d, "\cong"] \\ \hat{R} \arrow[r, "\pi"'] & \hat{R}/I\hat{R} \end{tikzcd}$$

(note $\hat{I} \subset \hat{R}$ is $I\hat{R}$)

so we do not distinguish R/I, $\hat{R}/I\hat{R}$. Since $\hat{R}$ is a $k[[\underline{t}]]$-algebra (t_i acting as a_i) any section δ induces a $k[[\underline{t}]]$-linear map δ^* defined by

$$\delta^*(\bar{r} \otimes f(\underline{t})) = \delta(\bar{r}) \cdot f(\underline{t}).$$

To show that δ^* is an isomorphism we must first show that every $r \in \hat{R}$ has a <u>unique</u> expression of the form

$$r = \sum_{M \in \mathbb{N}^n} \delta(r_M) t^M \qquad (\dagger)$$

for elements $r_M \in R/I$.

<u>NOTE</u>: There is no noetherian hypothesis here

<u>Existence</u> for $r \in R$ since $\pi(r - \mathfrak{z}(r)) = 0$ we have

$$r - \mathfrak{z}(r) \in I\hat{R} \quad \Rightarrow \quad r - \mathfrak{z}(r) = \sum_{i=1}^{n} a_i f_i \qquad \text{some } a_i \in \hat{R}$$

But $a_i - \mathfrak{z}(a_i) \in I\hat{R} \Rightarrow a_i - \mathfrak{z}(a_i) = \sum_{j=1}^{n} a_{ij} f_j$

$$\therefore \quad r = \mathfrak{z}(r) + \sum_{i=1}^{n} \{ \mathfrak{z}(a_i) + \textstyle\sum_j a_{ij} f_j \} f_i$$

$$= \mathfrak{z}(r) + \sum_{i=1}^{n} \mathfrak{z}(a_i) f_i + \sum_{i,j} a_{ij} f_i f_j$$

continuing in this way produces a series converging to r.

<u>Uniqueness</u> follows from quasi-regularity. Suppose to the contrary that

$$\sum_M \mathfrak{z}(r_M) t^M = 0 \quad \text{in } \hat{R}$$

with not all r_M zero in R/I. Let $m := \min\{ |M| \mid r_M \neq 0 \text{ in } R/I \}$. Since the Cauchy sequence $\{ \sum_{|M| \leq d} \mathfrak{z}(r_M) t^M \}_{d \in \mathbb{N}}$ converges to 0, we can find D s.t. for all $d \geqslant D$,

$$\sum_{|M| \leq d} \mathfrak{z}(r_M) t^M \in (t_1, \dots, t_n)^{m+1} \quad \text{in } R.$$

$$\parallel$$

$$\sum_{|M|=m} \mathfrak{z}(r_M) t^M + \sum_{m < |M| \leq d} \mathfrak{z}(r_M) t^M$$

This implies $\sum_{|M|=m} \mathfrak{z}(r_M) t^M \in I^{m+1}$, so $\sum_{|M|=m} r_M t^M = 0$ in I^m/I^{m+1} and by def^{n} of quasi-regularity this forces $r_M \in I$ so $r_M = 0$ in R/I for all M with $|M| = m$. But this is a contradiction, so we arrive at the desired uniqueness of the representation $(\dagger)$.

This shows there is an isomorphism of k-modules

$$\begin{aligned} \hat{R} &\xrightarrow{\cong} \prod_{M\in\mathbb{N}^n} R/I \\ r &\longmapsto (r_M)_M \end{aligned}$$

But then since R/I is f.p. over k, z^* is the isomorphism

$$R/I \otimes_k k[[\underline{t}]] \cong R/I \otimes_k \prod_M k \cong \prod_M (R/I \otimes_k k) \cong \prod_M R/I \cong \hat{R}\,. \quad \square$$

<u>Corollary</u> If R/I is a f.g. projective k-module there is a k-linear connection

$$\nabla : \hat{R} \longrightarrow \hat{R} \otimes_{k[\underline{t}]} \Omega^1_{k[\underline{t}]/k}$$

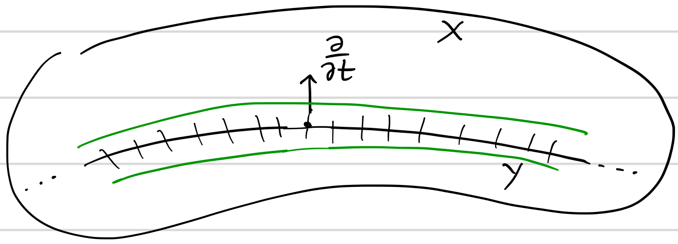
$$\nabla(r f(\underline{t})) = \nabla(r) f(\underline{t}) + r \otimes df.$$

Relative to a fixed connection we introduce $\frac{\partial}{\partial t_i} \in \mathrm{End}_k(\hat{R})$ via $\nabla = \sum_j \frac{\partial}{\partial t_j} dt_j$.

<u>Proof</u> $\hat{R}$ is a direct summand of a finite direct sum of copies of $k[[\underline{t}]]$, which has a connection. Choosing a section z we obtain a connection ∇_z,

$$\nabla_z(r) = \sum_{M\in\mathbb{N}^n} \sum_{j=1}^{n} M_j\, z(r_M)\, t^{M-e_j} \otimes dt_j\,. \quad \square$$

The "formal" tubular neighborhood is not an actual tubular neighbourhood, so the intuition can be misleading, but one can think of the connection ∇ as differentiation in the directions normal to $Y \subset X$, i.e.



$X = \operatorname{Spec} R$
$\uparrow$
$Y = \operatorname{Spec} R/I$

The plan for the remainder of the lecture is to use this connection to define residues and provide a natural homotopy equivalence between the Koszul complex of $\underline{a} = (a_1, \dots, a_n)$ and its cohomology R/I. But first, an example:

Example k a field, $R = k[x]$, $a = x^d$. Choose the k-linear section

$$\delta : R/I \longrightarrow R, \qquad \delta(x^i) = x^i \qquad 0 \le i \le d-1,$$

and let ∇ be the associated connection, with $\frac{\partial}{\partial t} : R \to R$. Then $(d > 2)$

$$\frac{\partial}{\partial t}\left(x^2 + x^{d+1}\right) = \frac{\partial}{\partial t}\left(\underbrace{[x^2]}_{R/I} \cdot \underbrace{1}_{t^0} + \underbrace{[x]}_{R/I} \cdot \underbrace{x^d}_{t^1} \right) = x.$$

Remark Since the t_i are the coordinates in $(I/I^2)^*$ arising from the a_i^*, we will often write $\frac{\partial}{\partial a_i}$ for $\frac{\partial}{\partial t_i}$ where it will not cause confusion.

Proposition Let $\underline{a} = (a_1, \dots, a_n)$ be a sequence in R. Then

(i) $\underline{a}$ regular $\Rightarrow$ $\underline{a}$ quasi-regular

(ii) If R is noetherian, $\underline{a}$ is quasi-regular $\Longleftrightarrow$ the Koszul complex on $\underline{a}$ is exact except in degree zero.

(iii) If $(R, \mathfrak{m}, k)$ is local and $\underline{a} \subseteq \mathfrak{m}$ then regular $\Longleftrightarrow$ quasi-regular.

<u>Def^n</u> Suppose $\underline{a} \subseteq R$ quasi-regular and R/I f.g. projective over k.
Given $r_0, r_1, \ldots, r_n \in R$ we define

$$\operatorname{Res}_{R/k}\left(\frac{r_0\, dr_1 \cdots dr_n}{a_1 \cdots a_n}\right) := tr^k_{R/I}\Big(\underset{\circledast}{r_0 [\nabla, r_1] \cdots [\nabla, r_n]}\Big) \in k$$

$$\begin{array}{ccccccccc}
\hat{R} & \longrightarrow & \hat{R}\otimes\Omega^1 & \longrightarrow & \hat{R}\otimes\Omega^2 & \longrightarrow \cdots & \longrightarrow & \hat{R}\otimes\Omega^n \cong & \hat{R} \\
\uparrow{\scriptstyle \delta} & & & & & & & & \downarrow{\scriptstyle \pi} \\
R/I & & - - - - - - & \overset{\circledast}{-} & - - - - - - & & & \dashrightarrow & R/I
\end{array}$$

<u>Remark</u> This is essentially the definition of residues given by Lipman, rephrased to use connections. He proves the basic properties (e.g. the transformation rule) for these residues.

<u>Example</u> $\operatorname{Res}_{k[x]/k}\left(\frac{x^i\, dx}{x^d}\right) = tr_{k[x]/x^d}\left(x^i\left[\frac{\partial}{\partial t}x - x\frac{\partial}{\partial t}\right]\right) = tr_{k[x]/x^d}\left(x^i \frac{\partial}{\partial t} x\right)$

$$= tr \begin{array}{c c} & \begin{array}{ccccc} 1 & x & x^2 & \cdots & x^{d-1} \end{array} \\ i & \left(\begin{array}{ccccc} \boxed{} & & & & \begin{matrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{matrix} \end{array}\right) \end{array} = \delta_{i,d-1}.$$

<u>Remark</u> If $r_0 \in I$ the residue is zero.

<u>Remark</u> Suppose $r_i = z(s_i)$ for $1 \le i \le n$ (this is typical). Then the residue is

$$\sum_{i_1,\dots,i_n} \mathrm{tr}^k_{R/I}\left(z(s_0) \circ \frac{\partial}{\partial t_{i_1}} \circ z(s_1) \quad \cdots \frac{\partial}{\partial t_{i_n}} \circ z(s_n) \right) \qquad (*)$$

If $z^*: R/I[[t]] \xrightarrow{\cong} \hat{R}$ were an isomorphism of rings, we would have (in $\hat{R}$) $z(s)z(s') = z(ss')$ and so the residue would have to be zero. The residue reflects the <u>failure</u> of z^* to be a ring isomorphism, or more precisely, it is an expression in the <u>t-derivatives</u> of the following tensor:

<u>Defn</u> With the same hypotheses as in the definition of residues, let T be the tensor

$$T \in (R/I)^* \otimes_k (R/I)^* \otimes_k (R/I) \otimes_k k[[t]]$$

given by the k-linear map

$$R/I \otimes_k R/I \xrightarrow{z \otimes z} \hat{R} \otimes_k \hat{R} \xrightarrow{\text{mult}} \hat{R} \xrightarrow{\cong} R/I \otimes_k k[[t]].$$

<u>Example</u> $R = k[x]$, $a = x^d$, $R/I = \bigoplus_{i=0}^{d-1} kx^i$, for $0 \le i,j \le d-1$ write

$$i + j = qd + r \qquad 0 \le r, q < d$$

$$z(x^i)z(x^j) = x^{i+j} = x^r \cdot (x^d)^q = z(x^r)t^q$$

Hence

$$T(x^i, x^j) = x^r \otimes t^q, \quad T^{ij}_{kl} = \delta_{k,r(i,j)}\,\delta_{l,q(i,j)}$$

(now assuming k is a $\mathbb{Q}$-algebra)

<u>Contractions on the Koszul complex</u> Let $\underline{a}$ be a quasi-regular sequence, and assume R/I is f.g. projective over k. Consider the quasi-isomorphism

$$K := \left(\Lambda(k\theta_1 \oplus \cdots \oplus k\theta_n) \otimes_k \hat{R},\ \ d_K = \sum_{i=1}^n a_i \theta_i^* \right)$$
$$\downarrow \pi$$
$$R/I = (R/I, 0).$$

free resolution of R/I as an R-module (even as a DG-R-algebra)

In fact this is a <u>homotopy equivalence</u> over k. Choose σ a k-linear section and ∇ the associated homotopy. We view $\nabla = \sum_i \frac{\partial}{\partial t_i}\theta_i$ as an odd k-linear operator on K. Then

$$[\nabla, d_K](r\theta_{i_1}\cdots\theta_{i_p}) = (\nabla d_K + d_K\nabla)(r\theta_{i_1}\cdots\theta_{i_p})$$

$$= \nabla\Big(\sum_{j=1}^p (-1)^{j+1} r a_{i_j} \theta_{i_1}\cdots\hat{\theta}_{i_j}\cdots\theta_{i_p} \Big) + d_K\Big(\sum_{k=1}^n \frac{\partial}{\partial t_k}(r)\theta_k\theta_{i_1}\cdots\theta_{i_p} \Big)$$

$$= \sum_{j,k} (-1)^{j+1} \frac{\partial}{\partial t_k}(r a_{i_j})\theta_k\theta_{i_1}\cdots\hat{\theta}_{i_j}\cdots\theta_{i_p}$$

$$+ \sum_{k=1}^n \frac{\partial}{\partial t_k}(r) a_k \theta_{i_1}\cdots\theta_{i_p}$$

$$+ \sum_{j,k} (-1)^j \frac{\partial}{\partial t_k}(r) a_{i_j} \theta_{i_1}\cdots\hat{\theta}_{i_j}\cdots\theta_{i_p}$$

by def^n of a connection $\frac{\partial}{\partial t_k}(r a_{i_j}) = a_{i_j}\frac{\partial}{\partial t_k}(r)$ unless $k = i_j$

$$= \sum_j \Big\{ \frac{\partial}{\partial t_{i_j}}(r a_{i_j}) - \frac{\partial}{\partial t_{i_j}}(r) a_{i_j} \Big\} \theta_{i_1}\cdots\theta_{i_p}$$

$$+ \sum_{k=1}^n \frac{\partial}{\partial t_k}(r) a_k \theta_{i_1}\cdots\theta_{i_p}$$

$$= \Big\{ \sum_j r + \sum_k \frac{\partial}{\partial t_k}(r) a_k \Big\} \theta_{i_1}\cdots\theta_{i_p} = \{ p + d_K\nabla \}(r)\theta_{i_1}\cdots\theta_{i_p}.$$

Observe that

$$d_K \nabla\left(\sum_M z(r_M) t^M \right) = \sum_M \sum_j M_j a_j z(r_M) t^{M-e_j}$$

$$= \sum_M \sum_j M_j z(r_M) t^M$$

$$= \sum_M |M| z(r_M) t^M$$

Lemma $\operatorname{Im}(d_K \nabla^\circ : R \longrightarrow R)$ is the ideal I.

Lemma Let $K_{\geq 1} \subseteq K$ denote the submodule spanned by $r \otimes \theta_{i_1} \cdots \theta_{i_p}$ with $p \geq 1$. Then $K_{\geq 1}$ is closed under $[\nabla, d_K]$ and the k-linear map $[\nabla, d_K] : K_{\geq 1} \longrightarrow K_{\geq 1}$ is invertible.

Proof $[\nabla, d_K]^{-1}(r\theta_{i_1} \cdots \theta_{i_p}) = \sum_M \frac{1}{p + |M|} z(r_M) t^M \theta_{i_1} \cdots \theta_{i_p}$. $\square$

Theorem Consider the diagram of k-linear maps

$$H \circlearrowleft (K, d_K) \underset{z}{\overset{\pi}{\rightleftarrows}} (R/I, 0)$$

where $H = [\nabla, d_K]^{-1} \circ \nabla$. Then we have

(a) π, z are cochain maps (of k-complexes)
(b) $\pi z = 1$
(c) $z\pi = 1_K - [d_K, H]$
(d) $H^2 = 0$, $Hz = 0$, $\pi H = 0$.

This kind of situation is called a <u>strong deformation retract</u>.

Proof We prove (c). Observe that, writing $J = [\nabla, d_K]$

$$d_K J = d_K[\nabla, d_K] = d_K(\nabla d_K + d_K \nabla)$$
$$= d_K \nabla d_K = [\nabla, d_K] d_K = J d_K \tag{$*$}$$

Write $K_p \subseteq K$ for the submodule spanned by $r \otimes \theta_{i_1} \cdots \theta_{i_p}$, $r \in \hat{R}$. Then on K_p for $p > 0$ we have by $(*)$ that $d_K = J d_K J^{-1}$ on K_p, and hence

$$[d_K, H] = d_K H + H d_K$$
$$= d_K J^{-1} \nabla + J^{-1} \nabla d_K$$
$$= J^{-1} \underline{J d_K J^{-1}} \nabla + J^{-1} \nabla d_K$$
$$= J^{-1} d_K \nabla + J^{-1} \nabla d_K$$
$$= J^{-1}(d_K \nabla + \nabla d_K) = J^{-1} J = 1$$

whereas on K_0 we have for $r \in \hat{R}$ that $r - \mathfrak{z}(r) \in I$ and for $x \in I$ we know there exists $y \in \hat{R}$ with $x = d_K \nabla(y)$. Then

$$(1 - d_K H)(x) = (1 - d_K H) d_K \nabla(y)$$
$$= d_K \nabla(y) - d_K J^{-1} \nabla d_K \nabla(y)$$
$$= d_K \nabla(y) - d_K J^{-1}(J - d_K \nabla) \nabla(y)$$
$$= 0.$$

But this shows that on K_0,

$$(1_K - [d_K, H])(r) = (1_K - d_K H)(r)$$
$$= (1_K - d_K H)(\mathfrak{z}(r))$$
$$= \mathfrak{z}(r). \quad \square$$

References

- J. Lipman, "Residues and traces of differential forms via Hochschild homology", Contemporary Math, Vol. 61, Amer. Math. Soc 1987.

  "the purpose of this paper is to provide an elementary development of the theory of residues."

- T. Dyckerhoff, D. Murfet, "Pushing forward matrix factorisations", Duke Math. J., 2013.