Korea lectures 2017 – II

The focus of this lecture is the DG-category of matrix factorisations of a potential $W \in k[x_1, \dots, x_n]$, k any commutative ring. We begin by giving the formal definition of potential. We set $R = k[x_1, \dots, x_n]$ throughout. Recall the overall plan:

$$\text{Potential } W \longmapsto \overset{\text{today}}{\longrightarrow} \text{DG-category } mf(W) \longmapsto \overset{\text{tomorrow}}{\longrightarrow} A_\infty\text{-category}$$

Def^n An element $W \in k[x_1, \dots, x_n]$ is a potential if (see [CM])

(i) The sequence $\partial_{x_1} W, \dots, \partial_{x_n} W$ is quasi-regular, and

(ii) With $I = (\partial_{x_1} W, \dots, \partial_{x_n} W)$, R/I is a f.g. free k-module.

(iii) The Koszul complex of $\partial_{x_1} W, \dots, \partial_{x_n} W$ is exact except in degree zero.

$$\boxed{\text{Throughout } W \text{ is a potential}, \quad a_i = \partial_{x_i} W \quad \text{and} \quad \mathrm{Jac}_W := R/I \quad 1 \le i \le n.}$$

The hypotheses (i), (ii) are, by Lecture 1, what we need to ensure that

- There is an isomorphism of $k[[\underline{t}]] = k[[t_1, \dots, t_n]]$-modules

$$\sigma^* : \mathrm{Jac}_W \otimes_k k[[\underline{t}]] \xrightarrow{\cong} \hat{R} = \varprojlim_i R/I^i$$

recall t_i acts as a_i here

- There is a k-linear flat connection

$$\nabla : \hat{R} \longrightarrow \hat{R} \otimes_{k[\underline{t}]} \Omega^1_{k[\underline{t}]/k}, \quad \nabla = \sum_i \frac{\partial}{\partial t_i} dt_i$$

which gives derivatives in the normal direction to the critical locus.

<u>Convention</u> From now on we do not distinguish $a_i \in R$ and the formal variable t_i.

<u>Example</u> (i) $k = \mathbb{C}$ and the critical points of W are isolated.

(ii) Consider $k = \mathbb{C}[u]$, $W(x,y,u) = x^2 + y^3 - 3u^2y + 2u^3 \in k[x,y]$ is the semi-universal deformation of the cusp, restricted to the discriminant. Observe $\partial_x W = 2x$, $\partial_y W = 3y^2 - 3u^2$ so

$$k[x,y]/(\partial_x W, \partial_y W) = \mathbb{C}[u,x,y]/(x, y^2-u^2) \cong \mathbb{C}[u] \oplus \mathbb{C}[u]y$$

so W is a potential over k.

<u>Def^n</u> A ($\mathbb{Z}_2$-graded) <u>DG-category</u> over k is a category $\mathcal{C}$ enriched over the monoidal category of $\mathbb{Z}_2$-graded complexes of k-modules, i.e. we have cochain maps

<u>composition</u> : $\mathcal{C}(b,c) \otimes_k \mathcal{C}(a,b) \longrightarrow \mathcal{C}(a,c)$

<u>unit</u> : $k \longrightarrow \mathcal{C}(a,a)$.

If $\mathcal{C}$ is a DG-category, the <u>cohomology category</u> of $\mathcal{C}$, denoted $H^0(\mathcal{C})$, is a k-linear category with $ob(H^0\mathcal{C}) = ob(\mathcal{C})$ and

$$(H^0\mathcal{C})(a,b) := H^0\mathcal{C}(a,b).$$

here R may be any commutative ring and $W \in R$ any element

<u>Def^n</u> The DG-category $\mathcal{C} = mf(R, W)$ has as objects finite rank matrix factorisations ($X = X^0 \oplus X^1$ free, $d_X : X \to X$ odd, $d_X^2 = W \cdot 1_X$) and

$$\mathcal{C}(X,Y) := \left(\mathrm{Hom}_R(X,Y),\ d_{\mathrm{Hom}}(\alpha) = d_Y \alpha - (-1)^{|\alpha|} \alpha d_X \right)$$

<u>Def</u>

<u>Def^n</u> $hmf(R,W) := H^0 mf(R,W)$ is the <u>homotopy category of matrix factorisations</u>.

The following technical lemma will be important for the third lecture:

<u>Lemma</u> (Completion comparison) There is a DG-functor

$$F : mf(R,W) \longrightarrow mf(\hat{R}, W)$$

which has the property that for all X, Y the map

$$F_{XY} : \mathrm{Hom}_R(X,Y) \longrightarrow \mathrm{Hom}_{\hat{R}}(\hat{X}, \hat{Y})$$

is a k-linear homotopy equivalence (we say F is <u>h-fully faithful</u>).

<u>Proof</u> By (iii) the hypotheses of [DM, §7] are satisfied (one does not need noetherianness, see e.g. [CM2, Rem. C2], i.e. there is a deformation retract of the Koszul cpx of $\underline{t}$ over R to R/I (although it is not described by a connection). Thus the claim follows from [DM, Rem 7.7] (since we do not assume R is noetherian we do not know $R \to \hat{R}$ is flat, but actually to run the argument there we only need to know the image of $\underline{t}$ in $\hat{R}$ is quasi-regular, which it is – in fact it is <u>regular</u>). $\square$

<u>Remark</u> It is much easier to see F_{XY} is always a quasi-isomorphism, but unless k is a field this is too weak for our purposes.

The plan is now to give some examples of objects in $mf(W)$ and a calculation of $H^*\mathcal{C}(X,Y)$.

<u>Def^n</u> Given sequences $f_1,\dots,f_e$ and $g_1,\dots,g_e$ in R with $\sum_{i=1}^e f_i g_i = W$ the associated <u>Koszul factorisation</u> is

$$\{\underline{f},\underline{g}\} := \Big(\textstyle\bigwedge(k\xi_1\oplus\cdots\oplus k\xi_e)\otimes_k R\,,\ \underbrace{\textstyle\sum_{i=1}^e f_i\xi_i^*}_{\text{(contraction)}} + \underbrace{\textstyle\sum_{i=1}^e g_i\xi_i}_{\text{(wedge)}}\Big)$$

where $|\xi_i|=1$. Observe that (all commutators are graded, $[a,b]=ab-(-1)^{|a||b|}ba$)

$$\Big(\sum_i f_i\xi_i^* + \sum_i g_i\xi_i\Big)^2 = \sum_i f_i g_i[\xi_i^*,\xi_i] = \sum_i f_i g_i = W.$$

<u>Example</u> Suppose k is a field and $P=(P_1,\dots,P_n)\in Z(W)$, so we may write for some g_i, $W=\sum_{i=1}^n(x_i-P_i)g_i$. The associated Koszul MF is denoted

$$k(P)^{stab} := \{\underline{x}-P,\underline{g}\}.$$

<u>Lemma</u> There is an isomorphism of matrix factorisations of $-W$

$$\{\underline{g},-\underline{f}\}\xrightarrow{\ \cong\ }\{\underline{f},\underline{g}\}^\vee \qquad (\mathrm{I})$$

$$\xi_{i_1}\cdots\xi_{i_p}\longmapsto(-1)^{\binom{p}{2}}(\xi_{i_1}\cdots\xi_{i_p})^* \qquad (i_1<\cdots<i_p)$$

and an isomorphism of matrix factorisations of W

$$\{\underline{g},-\underline{f}\}\xrightarrow{\ \cong\ }\{\underline{f},-\underline{g}\}[e]. \qquad (\mathrm{II})$$

In particular we have R-linear isomorphisms of complexes

$$
\begin{aligned}
\mathrm{Hom}_R(k(P)^{stab}, X) &= \mathrm{Hom}_R(\{\underline{x}-P, \underline{g}\}, X) \\
&\cong \{\underline{x}-P, \underline{g}\}^{\vee} \otimes_R X \\
&\overset{(\mathrm{I})}{\cong} \{\underline{g}, -(\underline{x}-P)\} \otimes_R X \qquad (\mathrm{III}) \\
&\overset{(\mathrm{II})}{\cong} \Big(\{\underbrace{\underline{x}-P}_{\text{Koszul differential}}, \underbrace{-\underline{g}}_{\text{Koszul}}\} \otimes_R \underbrace{X}_{d_X} \Big)[n]
\end{aligned}
$$

The homological perturbation lemma, applied to the strong deformation retract on Koszul complexes from last lecture, provides an efficient method for computing the cohomology of the complex $\mathrm{Hom}_R(k(P)^{stab}, X)$. We will discuss this in some detail, as the same basic tools go into constructing the A_∞-structure on $\mathcal{C} = mf(W)$.

<u>Lemma</u> (Dyckerhoff) Let k be a field and $P \in Z(W)$. Then as k-modules there is an isomorphism for any (even infinitely generated) matrix factorisation X of W,

$$H^* \mathrm{Hom}_R(k(P)^{stab}, X) \cong H^*(X \otimes_R k(P))[n]$$

where $k(P) = R/(x_1 - P_1, \ldots, x_n - P_n)$.

<u>Proof</u> See [D]. We give here a different proof using perturbation. Wlog take $P = \underline{0}$.
By Lecture 1 there is a strong deformation retract (SDR) over k

$$H \circlearrowright \Big(\Lambda(\oplus_i k\xi_i) \otimes_k R, \underbrace{\textstyle\sum_i x_i \xi_i^*}_{d_K}\Big) \underset{\sigma}{\overset{\pi}{\rightleftarrows}} (k, 0).$$

$H = [\nabla, d_K]^{-1}\nabla, \quad \nabla = \sum_i \frac{\partial}{\partial x_i} \xi_i$

(note there is no need for completions here!)

Tensoring with X gives another SDR over k

$$H = H \otimes 1_X \circlearrowright \left(\Lambda(\oplus_i k\xi_i) \otimes_k X, \textstyle\sum_i x_i \xi_i^* \right) \underset{\sigma}{\overset{\pi}{\rightleftarrows}} (X \otimes_R k, 0).$$

The "perturbation" $\delta = \sum_i (-g_i)\xi_i + d_X$ satisfies

$$(H\delta)^m = 0 \quad \text{for } m \gg 0 \qquad (\text{i.e. } \delta \text{ is "small"})$$

and so by the homological perturbation lemma there is an SDR $\quad \sum_m (-1)^m \pi \delta (H\delta)^m \sigma$

$$H_\infty \circlearrowright \left(\Lambda(\oplus_i k\xi_i) \otimes_k X, \textstyle\sum_i x_i \xi_i^* + \sum_i (-g_i)\xi_i + d_X \right) \underset{\sigma_\infty}{\overset{\pi}{\rightleftarrows}} (X \otimes_R k, d_X \otimes 1)$$

But by (III) the LHS here is $H^* \mathrm{Hom}_R(k(P)^{stab}, X)[n]$. $\square$

<u>Corollary</u> (Schoutens, later Orlov, Dyckerhoff, Keller-M-Vanden Bergh) If k is a field and $W \in k[[\underline{x}]]$ has an isolated singularity, then $k^{stab} = k(0)^{stab}$ is a split generator of the triangulated category

$$\mathcal{T} = \mathrm{hmf}(k[[\underline{x}]], W).$$

That is, every object in $\mathcal{T}$ can be built from k using shifts, cones, sums and direct summands.

<u>Proof</u> Consider $\mathcal{T} \subseteq \mathrm{HMF}(W)$, the infinite rank MFs of W. This is a triangulated category with arbitrary coproducts and it is clear $\mathcal{T} \subseteq \mathrm{HMF}(W)^c$, the compact objects. We claim k^{stab} compactly generates $\mathrm{HMF}(W)$, from which it follows by general theory (see Neeman's book) that $\langle k^{stab} \rangle = \mathrm{HMF}(W)^c$ and hence $\langle k^{stab} \rangle = \mathcal{T}$, where $\langle - \rangle$ denotes the smallest thick triangulated subcategory containing a given set.

Suppose $\mathcal{T}(k^{stab}, X[i]) = 0$ for $i \in \mathbb{Z}_2$ and $X \in HMF(W)$. Then, by the Proposition

$$0 = H^* \mathrm{Hom}_{k[[x]]}(k^{stab}, X) \cong H^*(X \otimes k)[n].$$

But X is homotopy equivalent over $k[[x]]$ to a matrix factorisation X_{min} whose differential (as a matrix) contains no units (intuitively this is Gaussian elimination, but since X is infinite this needs to be made precise using a Zorn's lemma argument). It follows that

$$H^*(X \otimes k) \cong H^*(X_{min} \otimes k) \cong X_{min} \otimes k$$

so we conclude $X \cong 0$ in $\mathcal{T}$. By standard results this shows k^{stab} compactly generates. □

The road to A_∞

It follows that there is a quasi-isomorphism of DG-categories

$$\mathrm{mf}(k[[x]], W) \xrightarrow[\cong]{\text{Keller/Lefevre}} \mathrm{Perf}(\underbrace{\mathrm{End}_R(k^{stab})}_{\text{DGA}})$$

$$X \longmapsto \mathrm{Hom}_R(k^{stab}, X)$$

<u>Upshot</u> The DG-algebra $\mathrm{End}_R(k^{stab})$ contains all the information of $\mathrm{mf}(W)$.
<u>But</u> This DGA is ∞-dimensional over k. We want a <u>finite</u>, <u>lossless</u> model.

The standard approach is to take the A_∞-minimal model

$$\mathrm{End}_R(k^{stab}) \xrightarrow[\text{(using the Koszul contraction above)}]{\text{min. model}} \mathcal{A} = \underbrace{\left(H^* \mathrm{End}_R(k^{stab}), \{m_k\}_{k \geq 2} \right)}_{A_\infty\text{-algebra}}.$$

This is finite$/k$ and also lossless, as $\mathrm{Perf}\,\mathcal{A} \cong \mathrm{Perf}(\mathrm{End}(k^{stab}))$.

<u>Remark</u> $H^* \mathrm{Hom}_R(k^{stab}, k^{stab}) \cong H^*(k^{stab} \otimes_R k)[n] \cong \Lambda(k\xi_1 \oplus \cdots \oplus k\xi_n)[n]$.

This approach to constructing A_∞-minimal models of $\mathrm{End}_R(k^{stab})$ first appears in work of Seidel [Se] on mirror symmetry, and a continuation of this work by Efimov [E]. It was also considered in [D]. We will describe how to (in theory) compute these m_k's next lecture (explicit calculations for generic W can be hard).

However we aim to do substantially more. Computing the A_∞-minimal model of $\mathrm{End}_R(k^{stab})$ is relatively easy because k^{stab} is a "perturbation" of a Koszul complex. To compute the A_∞-minimal model of $\mathrm{End}_R(X)$, X a generic matrix factorisation, requires new methods, which we will discuss.

<u>Remark</u> One might expect computing the minimal model of $\mathrm{End}_R(k^{stab})$ would be leveraged into computing the minimal model of $\mathrm{End}_R(X)$, as k^{stab} is a generator. But this seems to require first explicitly presenting X as a (summand of a) twisted complex over k^{stab}, i.e. explicitly building X from k^{stab}. My impression is that this is not a practical approach to doing such calculations.

$$\begin{tikzcd}[column sep=large]
(\Delta ed) & \mathrm{Perf}(\mathcal{A}) & \mathrm{Perf}(\mathrm{End}(k^{stab})) \arrow[l, "\cong"'] & \mathrm{hmf}(W) \arrow[l, "\cong"'] \\
& (H^*\mathrm{Hom}_R(k^{stab}, X), \{m_k\}) \arrow[u, phantom, "\in" rotate=90] & H^*\mathrm{Hom}_R(k^{stab}, X) \arrow[l, mapsto] \arrow[u, phantom, "\in" rotate=90] & X \arrow[l, mapsto] \arrow[u, phantom, "\in" rotate=90]
\end{tikzcd}$$

↺ End here is a vector space

$$\begin{tikzcd}[column sep=large]
(DG) & \mathrm{Perf}_{dg}(\mathcal{A}) & \mathrm{Perf}_{dg}(\mathrm{End}(k^{stab})) \arrow[l, "\simeq"', "qis"] & \mathrm{mf}(W) \arrow[l, "\simeq"', "qis"] \\
& (H^k\mathrm{Hom}_R(k^{stab}, X), \{m_k\}) \arrow[u, phantom, "\in" rotate=90] & \mathrm{Hom}_R(k^{stab}, X) \arrow[l, mapsto] \arrow[u, phantom, "\in" rotate=90] & X \arrow[l, mapsto] \arrow[u, phantom, "\in" rotate=90] \arrow[dll, bend left, "(*)"]\\
(A_\infty) & \mathrm{Tw}(\mathcal{A}) & &
\end{tikzcd}$$

↺ End here is a DG-algebra

requires expressing X as a twisted cpx.

↺ End here is an A_∞-algebra, the minimal model of $\mathrm{End}(X)$

References

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