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Korea lectures 2017 – III

The aim of today's lecture is to compute (as far as we are able) the A_∞-minimal model of the DG-category $mf(R,W)$ where k is a $\mathbb{Q}$-algebra, $R = k[x_1,\ldots,x_n]$ and $W \in R$ is a potential.

Notation: $\mathcal{C} = mf(W)$, $I = (\partial_{x_1}W, \ldots, \partial_{x_n}W)$, $\hat{R}$ is I-adic completion, $Jac_W = R/I$

$$\sigma^* : Jac_W \otimes_k k[[t]] \xrightarrow{\cong} \hat{R}$$

derivatives in directions normal to $Crit(W) \subset \operatorname{spec} R$.

$$\nabla : \hat{R} \longrightarrow \hat{R} \otimes_{k[t]} \Omega^1_{k[t]/k}, \quad \nabla = \sum_i \frac{\partial}{\partial t_i} dt_i$$

Def^n An A_∞-category $\mathcal{A}$ over k consists of a class $ob(\mathcal{A})$ of objects, for each pair $a, b \in ob(\mathcal{A})$ a $\mathbb{Z}_2$-graded k-module $\mathcal{A}(a,b)$, and for $n \geq 1$ and a sequence $a_0, \ldots, a_n \in ob(\mathcal{A})$ a degree $2-n$ k-linear map

$$m_{a_n,\ldots,a_0} : \mathcal{A}(a_{n-1},a_n) \otimes_k \cdots \otimes_k \mathcal{A}(a_1,a_2) \otimes_k \mathcal{A}(a_0,a_1) \longrightarrow \mathcal{A}(a_0,a_n),$$

subject to the A_∞-constraints for $n \geq 1$ given by

$$\sum_{\substack{i \geq 0, j \geq 1 \\ 1 \leq i+j \leq n}} (-1)^{i+j+ij+n} m_{n-j+1}\big(x_n \otimes \cdots \otimes x_{i+j+1} \otimes m_j(x_{i+j} \otimes \cdots \otimes x_{i+1}) \otimes x_i \otimes \cdots \otimes x_0\big) = 0.$$

Remark (i) $m_{a_1,a_0} \circlearrowleft \mathcal{A}(a_0,a_1)$ $\quad m^2_{a_1,a_0} = 0$. Write $m_1 = \{m_{a_1,a_0}\}_{a_1,a_0}$.

(ii) $H^*_{m_1}(\mathcal{A})$ is a k-linear category.

(all our A_∞-categories are homologically unital but we will suppress this)

(iii) Any DG category is an A_∞-category.

Def^n Let $\mathcal{A}$ be an A_∞-category. Then a quasi-isomorphism of A_∞-categories $F: \mathcal{B} \longrightarrow \mathcal{A}$ is called a

(i) minimal model if $m_1^{\mathcal{B}} = 0$.

(ii) finite model if $\mathcal{B}(a,b)$ is a f.g. projective k-module for all $a, b \in ob(\mathcal{B})$, and each map

$$F_{a_1,a_0} : \mathcal{B}(a_1, a_0) \longrightarrow \mathcal{A}(a_1, a_0)$$

is a k-linear homotopy equivalence.

Motivation Why consider A_∞-categories? There are various reasons, but here are two:

① Given a DG-category $\mathcal{A}$ over a field k, it is common that each $\mathcal{A}(a,b)$ is an ∞-dim vector space but $H^*\mathcal{A}(a,b)$ is finite. The minimal model

$$\mathcal{B} = \left(H^*\mathcal{A}, \{m_k\}_{k\geq 2}\right) \xrightarrow[qis]{} (\mathcal{A}, m_1, m_2) = \mathcal{A}$$

therefore gives a finite-dimensional model of $\mathcal{A}$, which is "lossless" in the sense that $\mathcal{B}$, $\mathcal{A}$ are Morita equivalent. This is applied, for instance, to study deformations of objects of $\mathcal{A}$, or $\mathcal{A}$ itself.

② In examples, the calculation of $m_k^{\mathcal{B}}$'s tends to distill the "fundamental homological invariants" of objects of $\mathcal{A}$, i.e. the Atiyah class in the case of matrix factorisations.

Recall from last lecture:

Lemma (Completion comparison) The DG-functor $\mathcal{C} \longrightarrow \mathcal{C} \otimes_R \hat{R}$ is a k-linear homotopy equivalence (i.e. all induced $\mathcal{C}(X,Y) \longrightarrow \mathcal{C}(X,Y) \otimes_R \hat{R}$ are h.e.)

Def^n We define the DG-category

$$\mathcal{C}_\theta = \Lambda(k\theta_1 \oplus \cdots \oplus k\theta_n) \otimes_k \mathcal{C} \otimes_R \hat{R},$$

where $\Lambda(\oplus_i k\theta_i)$ and $\hat{R}$ are viewed as DG-algebras with zero differential.

Theorem (M) There is an SDR/k for any $X, Y \in \mathcal{C}$

$$H_\infty \circlearrowright \mathcal{C}_\theta(X,Y) \underset{\sigma_\infty}{\overset{\pi}{\rightleftarrows}} \mathcal{C}(X,Y) \otimes_R \mathrm{Jac}_W$$

and hence a finite model of the DG-category $\mathcal{C}_\theta$

$$\Big(\mathcal{C} \otimes_R \mathrm{Jac}_W, \underbrace{\{m_k\}_{k \geq 1}}_{\text{induced higher products}} \Big) \xrightarrow[\simeq]{qis} \mathcal{C}_\theta.$$

Next we sketch the proof, highlighting the role of Atiyah classes, before explaining how this may be used to give a minimal model of $\mathcal{C} = mf(W)$.

Remark As mentioned in Lecture 2, both Seidel and Efimov have used the SDR from Lecture 1 to compute minimal models of $\mathrm{End}_R(k^{stab})$. However this SDR does not apply to all of $\mathcal{C}$, so cannot be used to compute minimal models of e.g. $\mathrm{End}_R(X)$.

Def^n Set $\hat{\mathcal{C}}(X,Y) := \mathcal{C}(X,Y) \otimes_R \hat{R}$. There is a k-linear flat connection

$$\nabla : \hat{\mathcal{C}}(X,Y) \longrightarrow \hat{\mathcal{C}}(X,Y) \otimes_{k[t]} \Omega^1_{k[t]/k}$$

and the relative Atiyah class of the pair X, Y is the morphism of $k[t]$-complexes

$$[\nabla, d_{\hat{\mathcal{C}}(X,Y)}] : \hat{\mathcal{C}}(X,Y) \longrightarrow \hat{\mathcal{C}}(X,Y) \otimes_{k[t]} \Omega^1_{k[t]/k}.$$

Sketch of proof Fix $X, Y \in \mathcal{C} = mf(R,W)$. Then we have an SDR over k between the top and bottom rows

$$\begin{array}{lcl}
 & H_\infty \circlearrowright & \\
\boxed{\mathcal{C}_\theta} & \left(\Lambda(\oplus_i k\theta_i) \otimes_k \mathcal{C}(X,Y) \otimes_R \hat{R},\ d_{\mathcal{C}(X,Y)} \right) & \delta = \sum_i \lambda_i \theta_i^* \\
\cong & \exp(-\delta) \downarrow \quad \uparrow \exp(\delta) & \lambda_i : \mathcal{C}(X,Y) \to \mathcal{C}(X,Y) \\
 & & [\lambda_i, d_{\mathcal{C}(X,Y)}] = \partial_{x_i} W \cdot 1 \\
 & \left(\Lambda(\oplus_i k\theta_i) \otimes_k \mathcal{C}(X,Y) \otimes_R \hat{R},\ d_{\mathcal{C}(X,Y)} + d_K \right) & d_K = \sum_i \partial_{x_i} W\, \theta_i^* \\
\text{SDR}/k & \pi \downarrow \quad \uparrow \sigma_\infty & \\
\boxed{\mathcal{C} \otimes_R \mathrm{Jac}_W} & \left(\mathcal{C}(X,Y) \otimes_R \mathrm{Jac}_W,\ d_{\mathcal{C}(X,Y)} \otimes 1 \right) &
\end{array}$$

by homological perturbation starting from the Koszul SDR and adding $d_{\mathcal{C}(X,Y)}$ as the perturbation term

e.g. $\{\exp(\delta)\sigma_\infty\} \circ \{\pi \exp(-\delta)\} = 1 - [d_{\mathcal{C}(X,Y)}, H_\infty]$.

The Clifford representation on $\mathcal{C} \otimes_R \mathrm{Jac}_W$ induced by the θ_i, θ_i^* acting on $\mathcal{C}_\theta$ was worked out explicitly in [M] in terms of Atiyah classes.

Here the nontrivial components are (roughly speaking) both polynomials in Atiyah classes,

$$H_\infty \text{ "=" } \sum_{m\geq 0}(-1)^m[\nabla, d_{\mathcal{E}}]^m\nabla \qquad \delta_\infty \text{ "=" } \sum_{m\geq 0}(-1)^m[\nabla, d_{\mathcal{E}}]^m\delta$$

$$\text{DG} \xrightarrow{\text{finite/minimal model}} A_\infty$$

$$\begin{tikzcd}[column sep=huge]
\mathrm{End}_R(k(p)^{stab}) \arrow[r, "\text{complete minimal model}", "\simeq \text{ qis } A_\infty"'] \arrow[d, "\text{can}"] & (H^*\mathrm{End}_R(k(p)^{stab}), \{m_k\}_{k\geq 2}) \arrow[d, "\text{can}"] \\
\mathscr{C} \arrow[r, dashed, "\text{unknown / no SDR}"] \arrow[d, "\simeq \text{h.e. DG}"] & (H^*\mathscr{C}, \{m_k\}_{k\geq 2}) \arrow[d, "\text{injective (!)}"'] \\
\mathscr{C}\otimes_R \hat{R} \arrow[d, "\text{can}"] & (H^*(\mathscr{C}\otimes_R \mathrm{Jac}_W), \{m_k\}_{k\geq 2}) \arrow[u, bend right, "\text{explicit projector}"'] \\
\mathscr{C}_0 \arrow[r, "\simeq \text{h.e. } A_\infty", "\text{complete finite model}"'] & (\mathscr{C}\otimes_R \mathrm{Jac}_W, \{m_k\}_{\underline{\underline{k\geq 1}}}) \arrow[u, "\text{h.e.} \simeq"', "\text{minimal model}"]
\end{tikzcd}$$

$\mathscr{C}\otimes_R \mathrm{Jac}_W$: f.g. proj / k note $m_1 \neq 0$ in general

"complete" = we will give the Feynman rules in terms of W

<u>Summary</u> We extend $\mathscr{C}$ to $\mathscr{C}_0$, take a minimal model, and project back:

$$(H^*(\mathscr{C}\otimes_R \mathrm{Jac}_W), \{m_k\}_{k\geq 2}) \xrightarrow[\simeq]{\text{qis } A_\infty} \Lambda(k\theta_1\oplus\cdots\oplus k\theta_n)\otimes_k \underbrace{(H^*\mathscr{C}, \{m_k\}_{k\geq 2})}_{\bigcap_i \mathrm{Ker}(\theta_i^*)}$$

$\circlearrowleft \theta_i, \theta_i^*$ (induced action) $\qquad \circlearrowleft \theta_i, \theta_i^*$

Feynman rules $\left(\mathrm{End}_R(k^{stab}) \text{ case, } W \text{ has no quadratic terms} \right)$

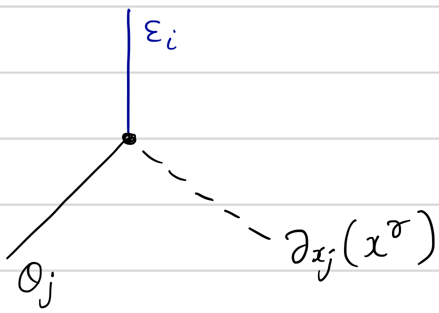
The minimal model is $\big(\underbrace{\Lambda(\oplus_{i=1}^{n} k\varepsilon_i)}_{\text{abbrev. } \Lambda(k\varepsilon)}, \{m_k\}_{k \geq 2} \big)$ where

$$m_k : \Lambda(k\varepsilon)^{\otimes k} \longrightarrow \Lambda(k\varepsilon)$$

is computed by a sum over connected planar rooted trees with k-inputs and internal vertices of degree 3, with trees decorated by Feynman diagrams built from the following local vertices, associated to a choice of factorisation

$$W = \sum_i x_i W^i \qquad W^i = \sum_{\gamma \in \mathbb{N}^n} W^i_\gamma x^\gamma$$

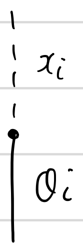
A-type (on inputs and internal edges, any number)



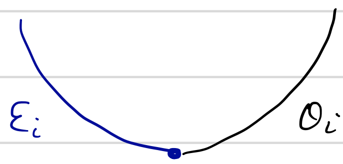
coefficient W^i_γ (possibly zero)

⌜Although it is obscured, this interaction vertex "is" the Atiyah class of $\mathrm{End}(k^{stab})$⌟

B-type (exactly one on each int. edge)

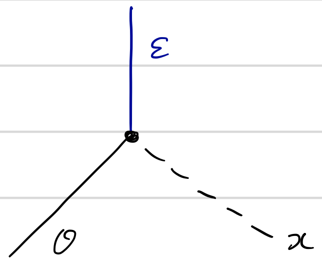


C-type (only at internal vertices, one "leg" in each incoming branch)

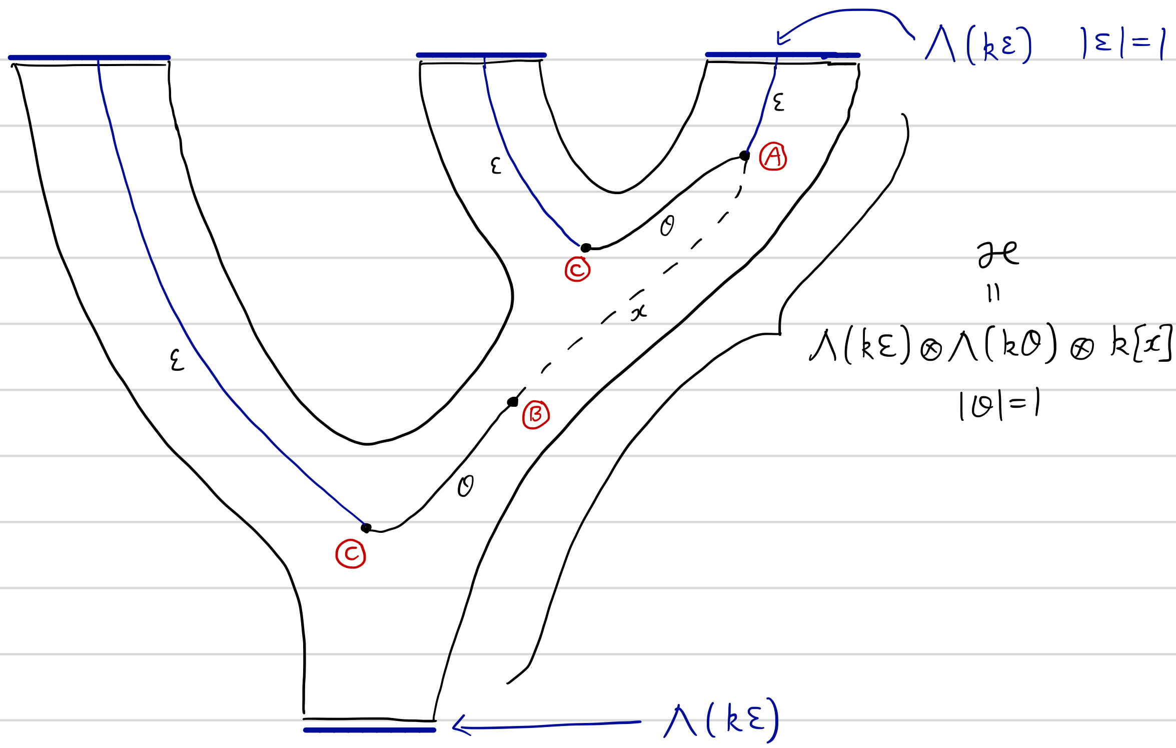


⌜here we are omitting various signs⌟

Example For $W = x^3 \in k[x]$, k a char. 0 field, we have $W = x \cdot x^2$, so there is only one kind of A-type



An example of a Feynman diagram constructed from these local interactions is



Scalar factors The operator $[\nabla, d_\theta]^{-1}$ contributes scalar factors to all of these diagrams, of the form

$$\sum_{\sigma \in S_n} \frac{1}{(a + a_{\sigma(1)})(a + a_{\sigma(1)} + a_{\sigma(2)}) \cdots (a + a_{\sigma(1)} + \cdots + a_{\sigma(n)})}$$

[a = # of "virtual" particles at a chosen edge, i.e. x's and θ's]

but while important, we will not describe these factors in these lectures.

Feynman rules (General case, $\mathrm{End}_R(X)$)

For simplicity we will assume X is of Koszul type

$$X=\{\underline{f},\underline{g}\}=\left(\Lambda(k\xi_1\oplus\cdots\oplus k\xi_e),\ \sum_i f_i\xi_i^* + \sum_i g_i\xi_i\right).$$

The B,C-type Feynman rules are as before. To describe the A-type interactions we choose a k-basis $\mathrm{Jac}_W \cong \bigoplus_{i=1}^{\mu} k\cdot z_i$. We need the tensor T discussed in Lecture 1, namely

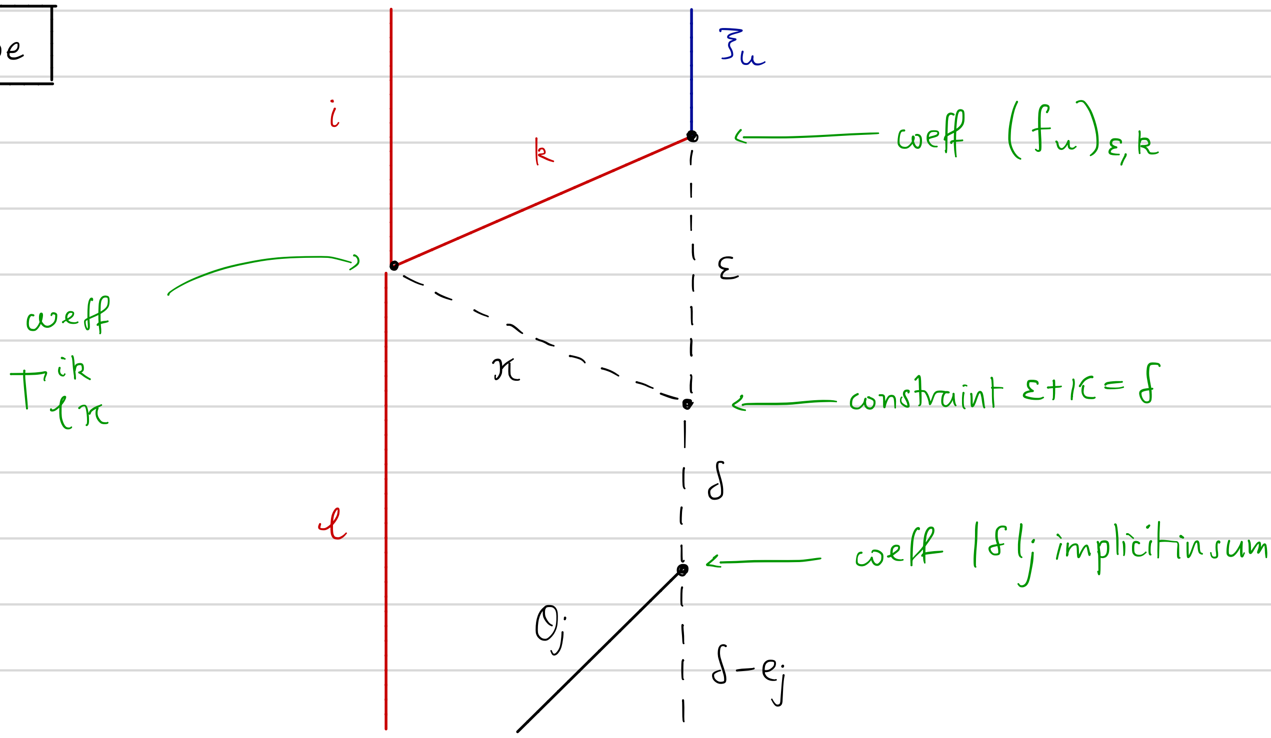
$$T=\left(T^{ij}_{l\beta}\right)\in \mathrm{Jac}_W^*\otimes_k \mathrm{Jac}_W^*\otimes_k \mathrm{Jac}_W\otimes_k k[[\underline{t}]] \qquad (\beta\in\mathbb{N}^n)$$

given by the k-linear map

$$\mathrm{Jac}_W\otimes_k\mathrm{Jac}_W \xrightarrow{\sigma\otimes\sigma} \hat{R}\otimes_k\hat{R}\xrightarrow{\text{mult}}\hat{R}\xrightarrow{\cong}\mathrm{Jac}_W\otimes_k k[[\underline{t}]].$$

Given $f\in\hat{R}$ recall that $f=\sum_M \sigma(f_M)t^M = \sum_{M,i} f_{M,i}\,\sigma(z_i)t^M$, $f_{M,i}\in k$.
With this notation there are two A-type interactions, of which we display only one:

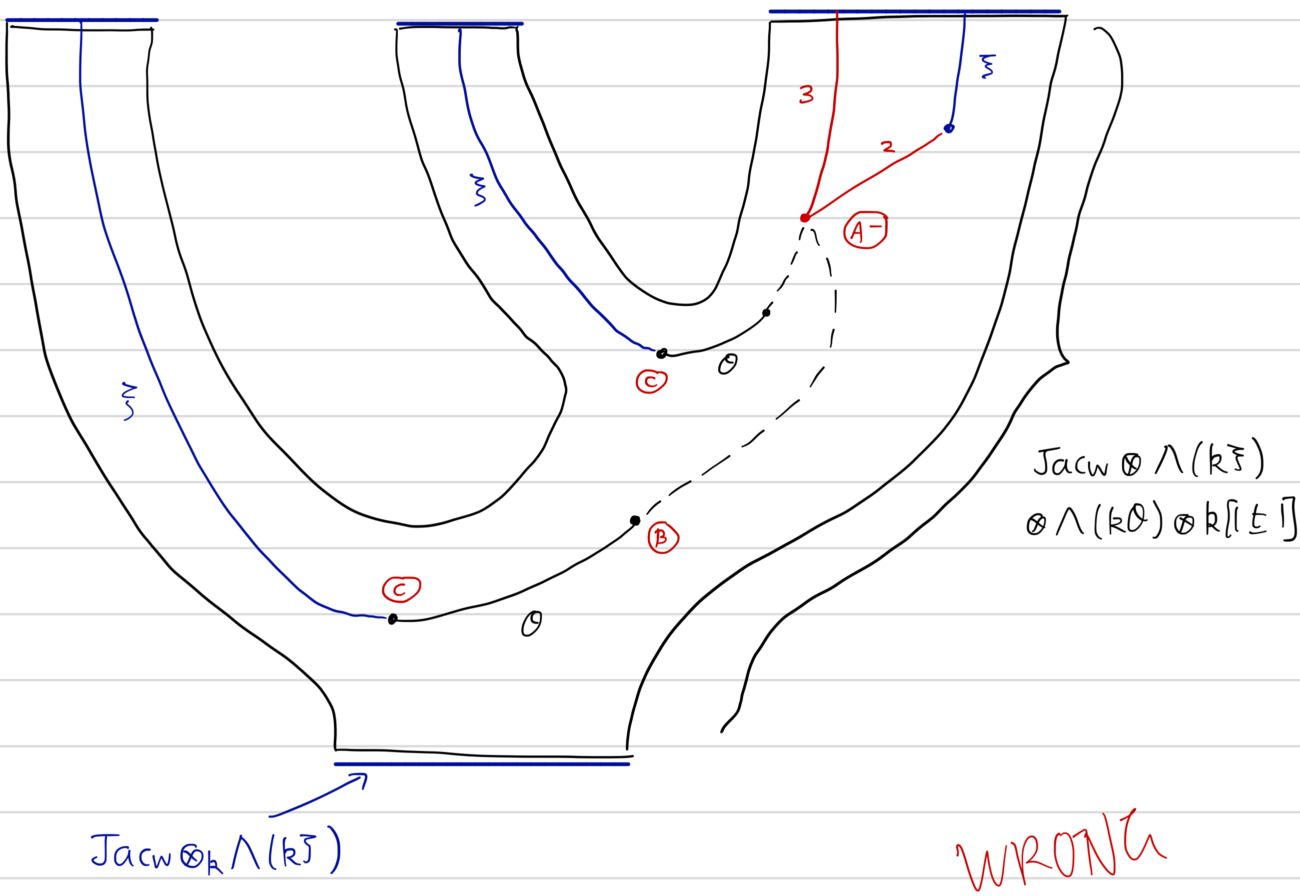
$\boxed{A^{-}\text{-type}}$



<u>Example</u> For $W = \frac{1}{5}x^5 \in k[x]$, k char. 0 field, and $t = x^4$,

$$X = \{x^2, x^3\} = \left(\Lambda(k\xi),\ x^2\xi^* + \frac{1}{5}x^3\xi\right)$$

$$\mathrm{Jac}_W = \bigoplus_{j=0}^{3} k z_j \qquad z_j = x^j \qquad (f = x^2,\ g = x^3)$$



WRONG

This diagram contibutes to

$$m_3(\xi \otimes \xi \otimes x^3\xi) = 1 + \cdots$$

Again, the A^- type interaction is a piece of the Atiyah class of $\mathrm{End}_R(X)$.

<u>Conclusion</u> We have discussed, for a potential $W \in k[x_1, \dots, x_n]$, to define connections, for $R = k[\underline{x}]$,

$$\nabla : \hat{R} \longrightarrow \hat{R} \otimes_{k[\underline{t}]} \Omega^1_{k[\underline{t}]/k}$$

which "differentiate" in the normal directions $\underline{t}$ to the critical locus. Using these connections we constructed strong deformation retracts, and used them to prove various facts about the DG-category of matrix factorisations, including a description of the Feynman rules in minimal models.

Perhaps the main conceptual insight to be gained from the construction of A_∞-minimal models that we have presented is that it "proves" the following empirical observation:

<u>Slogan</u> : Every homological invariant of matrix factorisations is a function of Atiyah classes.

We can justify this as follows. Let $Z(-)$ be a homological invariant:

$$\begin{tikzcd}
mf(W) \arrow[rr, "\overset{\text{qis } F}{\simeq}"] \arrow[dr, "Z"'] & \arrow[d, phantom, "\circlearrowleft"] & \left(H^0 mf(W), \{m_k\}_{k \geq 2} \right) \arrow[dl, "Z"] \\
& \mathcal{V} &
\end{tikzcd}$$

Then it should be invariant under DG quasi-isomorphism. But then $Z(X) \cong ZF(X)$ and the only way that X enters the A_∞-structure on $F(X)$ is via its Atiyah class, which determine the A-type interactions.

References

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