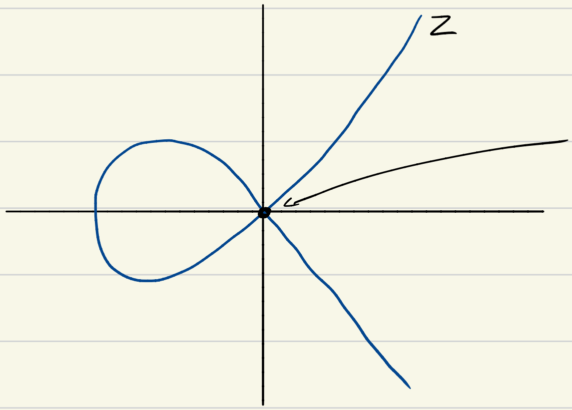
Introduction to matrix factorisations Part I

① What is a singularity? a point P of a curve $f(x,y)=0$ is nonsingular if $\nabla f(P) \neq 0$, so that locally near P, $Z = \{(x,y) \mid f(x,y)=0\}$ is a submanifold. Otherwise P is a singularity of Z

$$f(x,y) = y^2 - x^2(x+1)$$



singularity $P = \underline{0}$

(an A_1-singularity or node)

- Associated to the germ (Z,P) is the local coordinate ring

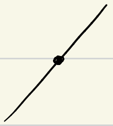
$$\mathbb{C}[[x,y]]/f(x,y) \cong \mathbb{C}[[u,v]]/(uv) =: R \quad \text{(Hartshorne Ex. 5.6.3)}$$

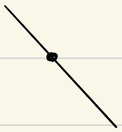
- Let mod(R) denote finitely generated R-modules. We can study the singularity (Z,P) by understanding

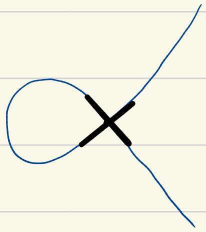
$$\text{mod}(R) \subseteq \mathbb{D}^b(\text{mod}\, R)$$

↑ morphisms are Ext classes

Example $R/u \cong \mathbb{C}[[v]]$, $R/v \cong \mathbb{C}[[u]]$







- Actually the "interesting" homological information about R-modules is concentrated in the <u>infinite tail of projective resolutions</u>.

<u>Example</u> $M = R/(u,v) \cong \mathbb{C}$ has (minimal) free resolution F

$$\cdots \longrightarrow \underbrace{R^{\oplus 2} \xrightarrow{\begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}} R^{\oplus 2} \xrightarrow{\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}} R^{\oplus 2} \xrightarrow{\begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}} \underset{-1}{R^{\oplus 2}} \xrightarrow{(u\ v)} \underset{0}{R}}_{F} \longrightarrow M \longrightarrow 0$$

To check this note that $fu + gv = 0$ in $R = \mathbb{C}[[u,v]]/(uv)$ implies $fu + gv \in (uv)$ in $\mathbb{C}[[u,v]]$ so $f \in (v)$, $g \in (u)$. For exactness at the other positions we need $fv \in (uv) \Rightarrow f \in (u)$ and $gu \in (uv) \Rightarrow g \in (v)$.

- The syzygies in the infinite periodic part are $R/v \oplus R/u$. These are <u>maximal Cohen–Macaulay R-modules</u> (MCM) which means $\operatorname{depth}(M) = \dim(R)$. Recall over a Noetherian local ring $(R, \mathfrak{m})$

$$\begin{aligned} \operatorname{depth}(M) &= \text{common length of maximal regular sequence in } M \\ &= \inf\{ i \mid \operatorname{Ext}^i(R/\mathfrak{m}, M) \neq 0 \} \\ &\leq \dim(M) \leq \dim(R) \end{aligned}$$

Krull dimension of $R/\operatorname{Ann}(M)$

We can compute in our example (i.e. $M = \mathbb{C}$, $R = \mathbb{C}[[u,v]]/uv$)

$$\operatorname{Ext}^i(R/\mathfrak{m}, M) = H^i \operatorname{Hom}_R(F, M) = H^i\left(\mathbb{C} \xrightarrow{0} \mathbb{C}^{\oplus 2} \xrightarrow{0} \mathbb{C}^{\oplus 2} \xrightarrow{0} \cdots\right)$$

Hence $\operatorname{depth}(M) = 0 = \dim(M) < \dim(R) = 1$ so M is <u>not</u> MCM. However

$$\operatorname{Ext}^i(R/m, R/u) = H^i \operatorname{Hom}_R(F, R/u)$$

$$= H^i\left(R/u \xrightarrow{\begin{pmatrix}0\\v\end{pmatrix}} (R/u)^{\oplus 2} \xrightarrow{\begin{pmatrix}v&0\\0&0\end{pmatrix}} (R/u)^{\oplus 2} \xrightarrow{\begin{pmatrix}0&0\\0&v\end{pmatrix}} (R/u)^{\oplus 2} \to \cdots \right)$$

$$\cong H^i\left(\mathbb{C}[|v|] \xrightarrow{\begin{pmatrix}0\\v\end{pmatrix}} \mathbb{C}[|v|]^{\oplus 2} \xrightarrow{\begin{pmatrix}v&0\\0&0\end{pmatrix}} \mathbb{C}[|v|]^{\oplus 2} \xrightarrow{\begin{pmatrix}0&0\\0&v\end{pmatrix}} \cdots \right)$$

$$= \underset{i=0}{0} \qquad \underset{i=1}{\mathbb{C}[|v|]/(v)} \qquad \underbrace{\qquad\qquad}_{\text{irrelevant}}$$

Hence $\operatorname{depth}_R(R/u) = 1 = \dim(R)$ so R/u is MCM, and similarly R/v.

There is a general phenomenon at work here:

<u>Lemma</u> If $R = \mathbb{C}[|x_1, \ldots, x_n|]/f$ is a hypersurface ring, and N is a f.g. R-module with free resolution

$$\begin{tikzcd}
\cdots \arrow[rr] \arrow[dr] & & F_2 \arrow[rr] \arrow[dr] & & F_1 \arrow[rr] \arrow[dr] & & F_0 \arrow[r] & N \arrow[r] & 0 \\
& S_2 \arrow[ur] & & S_1 \arrow[ur] & & S_0 \arrow[ur] & & &
\end{tikzcd}$$

then for $j \geq \dim(R) - 1$, the module S_j is MCM.

<u>Proof</u> Since R is Cohen-Macaulay $\operatorname{depth}(R) = \dim(R)$ so

$$\operatorname{Ext}^{<\dim(R)}(R/m, R) = 0.$$

The depths of the S_j <u>increase</u>, since for $i+1 < \dim(R)$

$$\operatorname{Ext}^i(R/m, S_j) \longrightarrow \operatorname{Ext}^{i+1}(R/m, S_{j+1}) \longrightarrow \operatorname{Ext}^{i+1}(R/m, F_{j+1})$$

is exact, so $\operatorname{depth}(S_j) = k < \dim(R) \Rightarrow \operatorname{Ext}^{<k}(R/m, S_j) = 0$

$$\Rightarrow \operatorname{Ext}^{\leq k}(R/m, S_{j+1}) = 0$$

$$\Rightarrow \operatorname{depth}(S_{j+1}) > k$$

Hence as long as $\operatorname{depth}(S_{j-1}) < \dim(R)$

$$\underset{\geq 0}{\operatorname{depth}(N)} < \underset{\geq 1}{\operatorname{depth}(S_0)} < \underset{\geq 2}{\operatorname{depth}(S_1)} < \cdots < \underset{\geq j+1}{\operatorname{depth}(S_j)}$$

and so clearly S_j is MCM for $j \geq \dim(R) - 1$. ☐

<u>Remark</u> This means that we have an exact sequence of complexes $(j \geq \dim(R) - 1)$

$$\begin{tikzcd}
0 \arrow[r] & F_{\leq j+1} \arrow[r] & F \arrow[r] & F_{\geq j} \arrow[r] & 0 \\
& \text{quasi-isomorphic to } S_j[j] \arrow[u] & \text{quasi-iso to } N \arrow[u] & \text{bounded cpx of free modules} \arrow[u] &
\end{tikzcd}$$

hence a triangle $S_j[j] \longrightarrow N \longrightarrow F_{\geq j} \xrightarrow{+}$ in $D^b(\operatorname{mod} R)$

and hence with $\operatorname{Perf}(R) = \{$bd. cpxs of f.g. proj. R-modules$\}$

$$\boxed{N \cong S_j[j] \text{ in } D^b(\operatorname{mod} R)/\operatorname{Perf}(R)}$$

Example $R = \mathbb{C}[[u,v]]/(uv)$

$$\mathbb{C} \cong S_1[1] = R/u[1] \oplus R/v[1] \quad \text{in} \quad \mathbb{D}^b(\text{mod}R)/\text{Perf}(R)$$

But we can do better, observe that

$$\begin{tikzcd} \cdots \arrow[r, "\bar{B}"] & R^{\oplus 2} \arrow[r, "\bar{A} = \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}"] & R^{\oplus 2} \arrow[r, "\bar{B} = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}"] & R^{\oplus 2} \arrow[r, "\bar{A}"] & \cdots \end{tikzcd}$$

is an infinite acyclic (zero cohomology) complex, with all syzygies $R/u \oplus R/v$. Hence by the same argument as above $(R/u \oplus R/v)[1] \cong R/u \oplus R/v$ in the Verdier quotient, hence

$$\mathbb{C} \cong R/u \oplus R/v \quad \text{in} \quad \mathbb{D}^b(\text{mod}R)/\text{Perf}(R)$$

Note these R-modules are not isomorphic in $\text{mod}(R)$!

Exercise $R/u \cong R/v[1]$ in $\mathbb{D}^b(\text{mod}R)/\text{Perf}(R)$.

Def^n (Buchweitz, Orlov) The singularity category of R is $\mathbb{D}^b_{sg}(R) := \mathbb{D}^b(\text{mod}R)/\text{Perf}(R)$.

Def^n The stable category of MCM R-modules $\underline{\text{MCM}}(R)$ has MCM R-modules as objects and morphisms denoted

$$\underline{\text{Hom}}_R(M,N) = \text{Hom}_R(M,N) \Big/ \left\{ \begin{array}{c} \varphi \text{ factoring as } M \longrightarrow P \longrightarrow N \\ \text{with } P \text{ projective} \end{array} \right\}$$

__Theorem__ (Buchweitz, Orlov) Every object of $\mathbb{D}^b_{sg}(R)$, for a hypersurface singularity $R = \mathbb{C}[[\underline{x}]]/f$, is isomorphic to an MCM module and the canonical functor

$$\mathrm{MCM}(R) \hookrightarrow \mathrm{mod}(R) \hookrightarrow \mathbb{D}^b(\mathrm{mod}\,R) \twoheadrightarrow \mathbb{D}^b_{sg}(R)$$

factors via an equivalence of triangulated categories

$$\underline{\mathrm{MCM}}(R) \xrightarrow[\Phi]{\cong} \mathbb{D}^b_{sg}(R).$$

__Example__ $R = \mathbb{C}[[u,v]]/(uv)$ we have computed $\Phi^{-1}(\mathbb{C}) \cong R/u \oplus R/v$. Actually we have an equivalence

$$\mathrm{Vect}^{\mathbb{Z}_2}_{\mathbb{C}} \cong \underline{\mathrm{MCM}}(R) \cong \mathbb{D}^b_{sg}(R)$$

$\mathbb{C}$	R/u	R/u
$\mathbb{C}[1]$	R/v	R/v
$\mathbb{C} \oplus \mathbb{C}[1]$	$R/u \oplus R/v$	$\mathbb{C}$

} Exercise: why is this asymmetry between u,v not a problem?

__Theorem__ (Eisenbud) Over a hypersurface ring $R = \mathbb{C}[[\underline{x}]]/f$ the minimal free resolution of every f.g. R-module N is __eventually 2-periodic__, that is, of the form

$$\cdots \underbrace{R^{\oplus d} \xrightarrow{\bar{A}} R^{\oplus d} \xrightarrow{\bar{B}} R^{\oplus d}}_{\text{repeats}} \longrightarrow \cdots \longrightarrow N \longrightarrow 0$$

where $A, B \in M_d(\mathbb{C}[[\underline{x}]])$ are polynomial matrices satisfying $AB = f \cdot \mathrm{Id}$, $BA = f \cdot \mathrm{Id}$. We call (A,B) a __matrix factorisation__ of f.

Example $f = uv$ (i) $A = (u)$, $B = (v)$ (call this X)

(ii) $A = (v)$, $A = (u)$ (call this Y)

(iii) $A = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$, $B = \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}$ (call this Z)

Def^n The homotopy category of matrix factorisations $hmf(\mathbb{C}[\![x]\!], f)$ has

- objects are MFs (A, B) (square matrices of the same size)
- morphisms $(A,B) \xrightarrow{(\varphi,\psi)} (A', B')$ are commutative diagrams (writing $S = \mathbb{C}[\![x]\!]$, all maps S-linear)

$$\begin{tikzcd}
S^{\oplus d} \arrow[r, "A"] \arrow[d, "\varphi"'] & S^{\oplus d} \arrow[r, "B"] \arrow[d, "\psi"] & S^{\oplus d} \arrow[d, "\varphi"] \\
S^{\oplus d} \arrow[r, "A'"'] & S^{\oplus d} \arrow[r, "B'"'] & S^{\oplus d}
\end{tikzcd}$$

modulo the homotopy relation $(\varphi, \psi) \sim (\alpha, \beta)$ if there exist g, h such that $A'g + hB = \psi - \beta$, $B'h + gA = \varphi - \alpha$.

- triangulated structure with shift $(A,B)[1] := (-B, -A)$, so $[2] = Id$. and $(A,B) \oplus (A', B') = (A \oplus A', B \oplus B')$.

Example In $hmf(\mathbb{C}[\![u,v]\!], uv)$, $Y \cong X[1]$ and $Z \cong X \oplus X[1]$.

__Theorem__ For any hypersurface ring there are equivalences of triangulated categories

$$\mathrm{hmf}(\mathbb{C}[[\underline{x}]], f) \xrightarrow{\Lambda} \underline{\mathrm{MCM}}(R) \xrightarrow{\Phi} \mathbb{D}^b_{sg}(R)$$

shift here is "take syzygy"

where $\Lambda(A,B) = \operatorname{coker} A$.

__Proof sketch__ Observe that $A : S^{\oplus d} \longrightarrow S^{\oplus d}$ has cokernel N, and given $x \in N$ with $x = \bar{y}$, $y \in S^{\oplus d}$

$$fx = \overline{fy} = \overline{ABy} = 0$$

Hence N is an $R = \mathbb{C}[[\underline{x}]]/f$-module. To see N is MCM we prove that the infinite complex

$$\begin{tikzcd}
\cdots \arrow[rr, "\bar{B}"] & & R^{\oplus d} \arrow[rr, "\bar{A}"] & & R^{\oplus d} \arrow[rr, "\bar{B}"] \arrow[dr] & & R^{\oplus d} \arrow[rr, "\bar{A}"] & & \cdots \\
& & & & & N \arrow[ur] & & &
\end{tikzcd}$$

is acyclic, with syzygy N. Suppose $\bar{A}x = 0$, and $x = \bar{y}$. Then $\overline{Ay} = 0$ that is, $Ay = (a_1 f, \dots, a_n f)^T$ some $a_i \in S$. But then writing $a = (a_1, \dots, a_n)$

$$\begin{aligned} Ay = fa &\Rightarrow Ay = ABa \\ &\Rightarrow y = Ba \\ &\Rightarrow x = \bar{B}\bar{a}. \end{aligned}$$

(A is injective, as $f : S^{\oplus d} \to S^{\oplus d}$ is and $f = BA$)

By the earlier depth arguments we may conclude N is MCM. Fully-faithfulness requires a bit more "Ext work". □

Alternative def^N A matrix factorisation of $f \in \mathbb{C}[\![x]\!]$ is a $\mathbb{Z}_2$-graded f.g. free $S = \mathbb{C}[\![x]\!]$-module $X = X_0 \oplus X_1$ with an odd S-linear map $d_X : X \longrightarrow X$ such that $d_X^2 = f \cdot 1_X$.

$$d_X = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} : X_0 \oplus X_1 \longrightarrow X_0 \oplus X_1 .$$

$$\therefore \quad d_X^2 = \begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix}$$

In conclusion, for our original example $R = \mathbb{C}[\![u,v]\!]/(uv)$

$$\mathrm{Vect}_{\mathbb{C}}^{\mathbb{Z}_2} \cong \mathrm{hmf}(\mathbb{C}[\![u,v]\!], uv) \cong \underline{\mathrm{MCM}}(R) \cong \mathbb{D}^b_{sg}(R)$$

$\mathbb{C}$	$\begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}$	R/u	R/u
$\mathbb{C}[1]$	$\begin{pmatrix} 0 & v \\ u & 0 \end{pmatrix}$	R/v	R/v
$\mathbb{C} \oplus \mathbb{C}[1]$	$\begin{pmatrix} 0 & 0 & u & 0 \\ 0 & 0 & 0 & v \\ v & 0 & 0 & 0 \\ 0 & u & 0 & 0 \end{pmatrix}$	$R/u \oplus R/v$	$\mathbb{C}$

Theorem (Knörrer periodicity) For any $f \in \mathbb{C}[\![x_1, \dots, x_n]\!]$ with an isolated singularity

$$\mathrm{hmf}(\mathbb{C}[\![x]\!], f) \cong \mathrm{hmf}(\mathbb{C}[\![x,u,v]\!], f + uv)$$

Example $\mathrm{hmf}(\mathbb{C}[\![u,v]\!], uv) \cong \mathrm{hmf}(\mathbb{C}, 0) = \mathrm{Vect}_{\mathbb{C}}^{\mathbb{Z}_2}$.

Def^n We say $f \in \mathbb{C}[\![x_1, \dots, x_n]\!]$ has an isolated singularity if

$$\dim_{\mathbb{C}} \left(\mathbb{C}[\![x]\!] / (\partial_{x_1} f, \dots, \partial_{x_n} f) \right) < \infty .$$

Theorem Let $f \in \mathbb{C}[\![\underline{x}]\!]$ have an isolated singularity. Then with $\mathcal{T} = \mathrm{hmf}(\mathbb{C}[\![\underline{x}]\!], f)$,

- $\mathcal{T}$ has finite-dimensional Hom-spaces
- $\mathcal{T}$ is idempotent complete
- $\mathcal{T}$ is Krull-Remak-Schmidt, i.e.
  - every object is a direct sum of indecomposables
  - if $\bigoplus_{i=1}^{n} N_i \cong \bigoplus_{j=1}^{m} M_j$ with N_i, M_j all indecomposable then $m=n$ and after renumbering $N_i \cong M_i$ for all i.

Example $\mathrm{hmf}(\mathbb{C}[\![x,y,z]\!], x^{n+1}+y^2+z^2) \cong \mathrm{hmf}(\mathbb{C}[\![x,u,v]\!], x^{n+1}+uv)$

A_n surface singularity

$$\overset{\text{Knörrer}}{\cong} \mathrm{hmf}(\mathbb{C}[\![x]\!], x^{n+1})$$

$$\overset{\text{Buchweitz-Orlov}}{\cong} \underline{\mathrm{MCM}}\left(\mathbb{C}[\![x]\!]/x^{n+1}\right)$$

But $R = \mathbb{C}[\![\underline{x}]\!]/x^{n+1}$ has $\dim(R) = 0$, so every f.g. R-module M is MCM. By the fundamental theorem for modules over a PID, we have in $\mathrm{mod}(R)$

$$M \cong R^{\oplus a} \oplus (R/x)^{\oplus a_1} \oplus \cdots \oplus (R/x^n)^{\oplus a_n}$$

Hence in $\underline{\mathrm{MCM}}(R)$, $M \cong \bigoplus_{i=1}^{n} (R/x)^{\oplus a_i}$, so the indecomposables are R/x^i for $1 \le i \le n$. Observe that we have an exact sequence over R

$$\begin{tikzcd}
\cdots \arrow[r, "x^{n+1-i}"] & R \arrow[rr, "x^i"] \arrow[dr] & & R \arrow[rr, "x^{n+1-i}"] \arrow[dr] & & R \arrow[r, "x^i"] & \cdots \\
& & R/x^{n+1-i} \arrow[ur] & & R/x^i \arrow[ur] & &
\end{tikzcd}$$

Hence a triangle $R/x^{n+1-i} \longrightarrow R \longrightarrow R/x^{i} \longrightarrow R/x^{n+1-i}[1]$ in $\mathbb{D}^b(\mathrm{mod}R)$, hence

$$R/x^{i} \cong R/x^{n+1-i}[1] \text{ in } \mathbb{D}^b_{sg}(R)$$

We can now complete the earlier table for $f = x^{n+1}$, $R = \mathbb{C}[[x]]/x^{n+1}$

$$\begin{array}{cccc}
\mathrm{hmf}(\mathbb{C}[[x]], x^{n+1}) \cong & \underline{\mathrm{MCM}}(R) \cong & \mathbb{D}^b_{sg}(R) & \\
Y_i := \begin{pmatrix} 0 & x^i \\ x^{n+1-i} & 0 \end{pmatrix} & R/x^i & R/x^i & 1 \leq i \leq n
\end{array}$$

<u>Example</u> Let us compute Homs in the different categories, $S = \mathbb{C}[[x]]$

$$\mathrm{Hom}_{hmf}(Y_i, Y_j) = \left\{ \begin{tikzcd} S \arrow[r, "x^i"] \arrow[d, "\varphi"] & S \arrow[r, "x^{n+1-i}"] \arrow[d, "\psi"] & S \arrow[d, "\varphi"] \\ S \arrow[r, "x^j"'] & S \arrow[r, "x^{n+1-j}"'] & S \end{tikzcd} \right\} / \text{htpy}$$

$i \leq j$

$$= \left\{ \varphi, \psi \in \mathbb{C}[[x]] \;\middle|\; \begin{array}{l} x^{n+1-i}\varphi = x^{n+1-j}\psi \\ x^j \varphi = x^i \psi \end{array} \right\} / \text{htpy}$$

$$= \left\{ \varphi, \psi \in \mathbb{C}[[x]] \mid \psi = x^{j-i}\varphi \right\} / \text{htpy}$$

where $(\varphi, x^{j-i}\varphi) \sim (\varphi', x^{j-i}\varphi')$ iff. there exist $g, h \in S$ such that $x^{j-i}(\varphi - \varphi') = x^j g + h x^{n+1-i}$, $\varphi - \varphi' = g x^i + x^{n+1-j} h$. Now $j \leq n$ so that $j - i < n+1-i$ and these conditions are equivalent to $\varphi - \varphi' = x^i g + h x^{n+1-j}$ some g, h. Hence to

$$\varphi - \varphi' \in (x^i, x^{n+1-j}) = (x^{\min(i, n+1-j)})$$

$$\therefore \mathrm{Hom}(Y_i, Y_j) \cong \mathbb{C}[[x]]/(x^{\min(i, n+1-j)})$$

[Introduction to matrix factorisations Part I]

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25/3/2020
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Mechanics of this talk

- Please mute your microphone & hold spacebar to speak
- Feel free to annotate the screen during a question (View Options > Annotate)
- I will write on partially complete slides (Paul calls this a "fill in the blanks" talk), but complete notes are available on my webpage, I recommend having them open in another window.
- Zoom talks ⊄ span(blackboard talk, slide talk). Help!

Ⅰ today

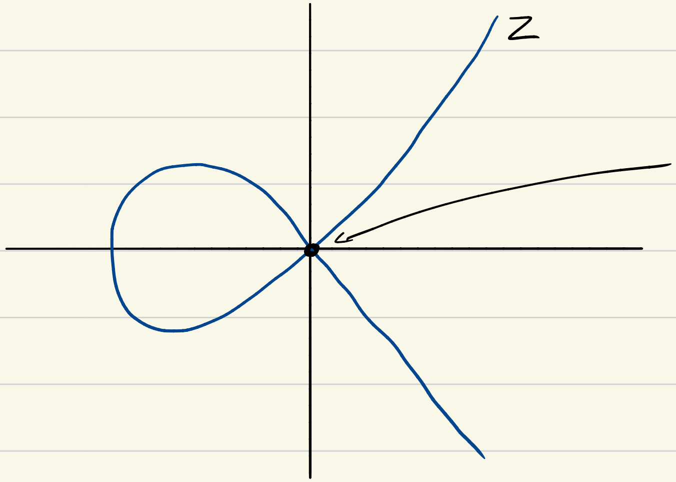
- What is a singularity?
- Maximal Cohen-Macaulay modules & $\underline{\mathrm{MCM}}(R)$
- Singularity category $\mathbb{D}^b_{sg}(R) = \mathbb{D}^b(\mathrm{mod} R)/\mathrm{Perf}(R)$
- Eisenbud's matrix factorisations & $\mathrm{hmf}(\mathbb{C}[[\underline{x}]], f)$
- <u>Theorem</u> $\mathrm{hmf}(\mathbb{C}[[\underline{x}]], f) \simeq \underline{\mathrm{MCM}}(R) \simeq \mathbb{D}^b_{sg}(R)$
- Knörrer periodicity
- Classification of matrix factorisations of A_n singularities

Ⅱ

- Graded and equivariant matrix factorisations
- McKay correspondence (classification for ADE singularities)

What is a singularity? a point P of a curve $f(x,y)=0$ is nonsingular if $\nabla f(P) \neq 0$, so that locally near P, $Z = \{(x,y) \mid f(x,y)=0\}$ is a submanifold. Otherwise P is a singularity of Z

$$f(x,y) = y^2 - x^2(x+1)$$



singularity $P = 0$

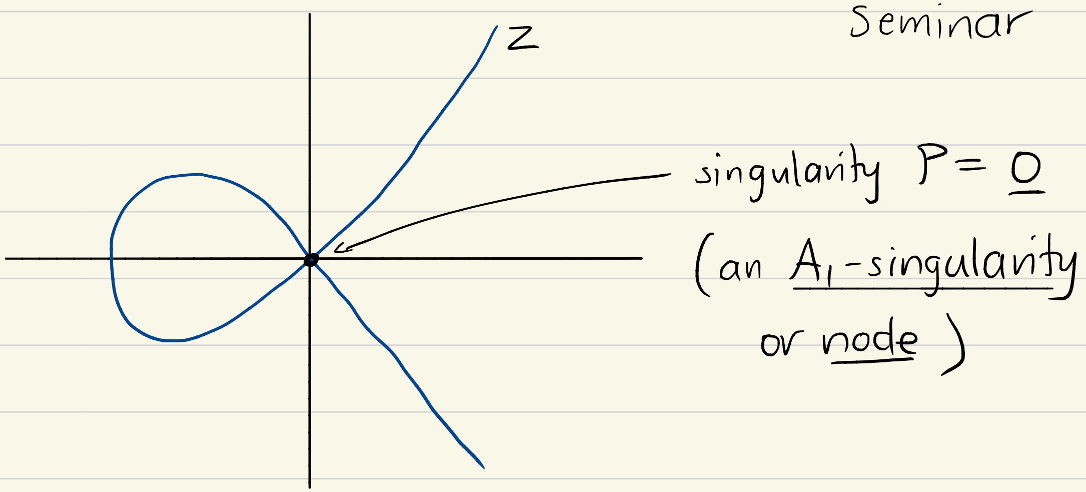
(an A_1-singularity or node)

- Associated to the germ (Z, P) is the local coordinate ring (Hartshorne Ex. 5.6.3)

$$\mathbb{C}[[x,y]]/f(x,y) \cong \mathbb{C}[[u,v]]/(uv) \cong \mathbb{C}[[u,v]]/(u^2+v^2)$$

this means "I am from a previous page"

$$f(x,y) = y^2 - x^2(x+1)$$

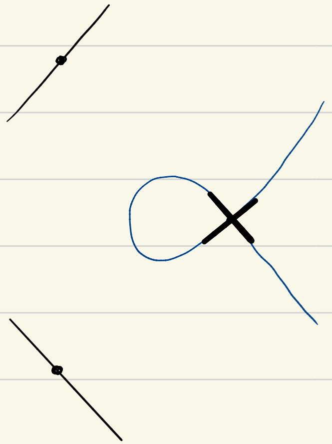


- Associated to the germ (Z, P) is the local coordinate ring $R = \mathbb{C}[u,v]/(uv)$

- Let $\mathrm{mod}(R)$ denote finitely generated R-modules. We can study the singularity via

$$\mathrm{mod}(R) \subseteq \mathbb{D}^b(\mathrm{mod}\,R)$$

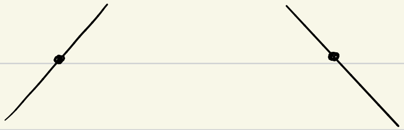
morphisms are Exts

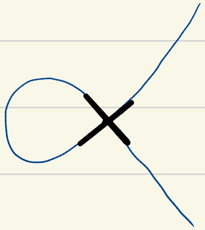


Example $R/u \cong \mathbb{C}[v]$, $R/v \cong \mathbb{C}[u]$

- Associated to the germ (Z, P) is the local coordinate ring $R = \mathbb{C}[[u,v]]/(uv)$

<u>Example</u> $R/u \cong \mathbb{C}[[v]]$, $R/v \cong \mathbb{C}[[u]]$





$\mathbb{D}^b(\operatorname{mod} R)$

- Actually the "interesting" homological information about R-modules is concentrated in the <u>infinite tail of projective resolutions</u>.

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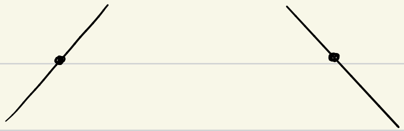
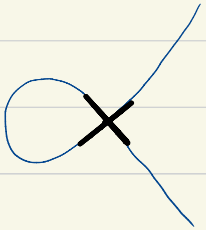
\cdots \arrow[rr, "\begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}"] & & R^{\oplus 2} \arrow[rr, "\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}"] \arrow[dr, two heads] & & R^{\oplus 2} \arrow[rr, "\begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}"] \arrow[dr, two heads] & & R^{\oplus 2} \arrow[rr, "(u\ v)"] \arrow[dr, two heads] & & R \arrow[r] & M \arrow[r] & 0 \\
& & & R/u \oplus R/v \arrow[ur, hook] & & R/u \oplus R/v \arrow[ur, hook] & & T \arrow[ur, hook] & & &
\end{tikzcd}$$

<u>Note</u> The syzygies in this resolution are $R/v \oplus R/u$.

- Associated to the germ (Z, P) is the local coordinate ring $R = \mathbb{C}[[u,v]]/(uv)$

Example $R/u \cong \mathbb{C}[[v]]$, $R/v \cong \mathbb{C}[[u]]$





$M = R/(u,v) \cong \mathbb{C}$

$$\cdots \longrightarrow \underbrace{R^{\oplus 2} \xrightarrow{\begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}} R^{\oplus 2} \xrightarrow{\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}} R^{\oplus 2} \xrightarrow{\begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}} \underset{-1}{R^{\oplus 2}} \xrightarrow{(u \ v)} \underset{0}{R}}_{F} \longrightarrow M \longrightarrow 0$$

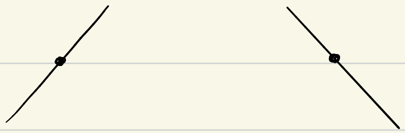
$$\boxed{y^n - x^u}$$

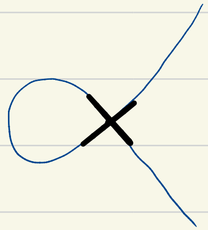
Recall If $(R, \mathfrak{m})$ is a Noetherian local ring, M f.g. R-module

$$\begin{aligned} \operatorname{depth}(M) &= \text{common length of maximal regular seq. in } M \\ &= \inf\{ i \mid \operatorname{Ext}^i(R/\mathfrak{m}, M) \neq 0 \} \\ &\leq \dim(M) \leq \dim(R) \end{aligned}$$

Krull dim. of $R/\operatorname{Ann}(M)$

$R = \mathbb{C}[u,v]/(uv)$, $\quad M = R/(u,v) \cong \mathbb{C}$ $\qquad R/u \cong \mathbb{C}[v]$, $R/v \cong \mathbb{C}[u]$





$$\underbrace{\cdots \longrightarrow R^{\oplus 2} \xrightarrow{\begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}} R^{\oplus 2} \xrightarrow{\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}} R^{\oplus 2} \xrightarrow{\begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}} \underset{-1}{R^{\oplus 2}} \xrightarrow{(u\ v)} \underset{0}{R}}_{F} \longrightarrow M \longrightarrow 0$$

- A f.g. R-module M is MCM if $\operatorname{depth}(M) = \dim(R)$

$\operatorname{depth}(M) = \inf\{ i \mid \operatorname{Ext}^i(R/m, M) \neq 0 \}$

$$\operatorname{Ext}^i(R/m, M) = H^i \operatorname{Hom}_R(F, M) = H^i\left(\mathbb{C} \xrightarrow{0} \mathbb{C}^{\oplus 2} \xrightarrow{0} \mathbb{C}^{\oplus 2} \xrightarrow{0} \cdots\right)$$

$\therefore \operatorname{depth}(M) = 0 = \dim(M) < \dim(R) = 1.$

<u>Defn</u> A f.g. R-module is <u>maximal Cohen-Macaulay</u> (MCM) if $\operatorname{depth}(M) = \dim(R)$.

$\therefore$ M is <u>not</u> MCM. But R/u, R/v are MCM.

- A f.g. R-module M is MCM if $\operatorname{depth}(M) = \dim(R)$ $\operatorname{depth}(M) = \inf\{ i \mid \operatorname{Ext}^i(R/m, M) \neq 0 \}$

Lemma If $R = \mathbb{C}[[x_1, \dots, x_n]]/f$ is a hypersurface ring, and N is a f.g. R-module with free resolution

$$\begin{tikzcd}
\cdots \arrow[rr] \arrow[dr, two heads] & & F_2 \arrow[rr] \arrow[dr, two heads] & & F_1 \arrow[rr] \arrow[dr, two heads] & & F_0 \arrow[r] & N \arrow[r] & 0 \\
& S_2 \arrow[ur, hook] & & S_1 \arrow[ur, hook] & & S_0 \arrow[ur, hook] & & &
\end{tikzcd}$$

then for $j \geqslant \dim(R) - 1$, the module S_j is MCM.

Proof Since R is Cohen-Macaulay ring $\operatorname{depth}(R) = \dim(R)$ $\operatorname{Ext}^{<\dim(R)}(R/m, R) = 0$.

Claim The depths of the S_j increase, since $i+1 < \dim(R)$ we have an exact sequence (from $0 \to S_{j+1} \to F_{j+1} \to S_j \to 0$)

$$\operatorname{Ext}^i(R/m, S_j) \xrightarrow{\cong} \operatorname{Ext}^{i+1}(R/m, S_{j+1}) \longrightarrow \cancelto{0}{\operatorname{Ext}^{i+1}(R/m, F_{j+1})}$$

so $\operatorname{depth}(S_j) = k < \dim(R) \Rightarrow \operatorname{Ext}^{<k}(R/m, S_j) = 0$

$\Rightarrow \operatorname{Ext}^{\leq k}(R/m, S_{j+1}) = 0$ $\operatorname{depth}(S_j) = \dim(R)$

$\Rightarrow \operatorname{depth}(S_{j+1}) > k$ $\therefore$ for $j \gg 0$ $\therefore$ S_j is MCM. $\square$

- A f.g. R-module M is MCM if $\operatorname{depth}(M) = \dim(R)$ $\quad$ $\operatorname{depth}(M) = \inf\{ i \mid \operatorname{Ext}^i(R/m, M) \neq 0 \}$

Lemma If $R = \mathbb{C}[x_1, \dots, x_n]/f$ is a hypersurface ring, and N is a f.g. R-module with free resolution

$$\begin{tikzcd}
\cdots \arrow[rr] \arrow[dr] & & F_2 \arrow[rr] \arrow[dr] & & F_1 \arrow[rr] \arrow[dr] & & F_0 \arrow[r] & N \arrow[r] & 0 \\
& S_2 \arrow[ur] & & S_1 \arrow[ur] & & S_0 \arrow[ur] & & &
\end{tikzcd}$$

then for $j \geq \dim(R) - 1$, the module S_j is MCM.

Remark For $j \geq \dim(R) - 1$ we have an exact sequence of complexes

$$0 \longrightarrow F_{\leq j+1} \longrightarrow F \longrightarrow F_{\geq j} \longrightarrow 0$$

(brutal truncation)

projective res. of S_j
quasi-isomorphic to $S_j[j]$

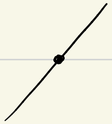
quasi-iso to N

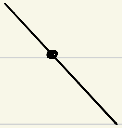
bounded
complex of free modules

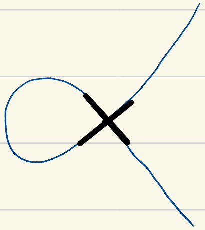
$\Longrightarrow$ is a triangle $\quad S_j[j] \longrightarrow N \longrightarrow F_{\geq j} \xrightarrow{+}$ in $\mathbb{D}^b(\operatorname{mod} R)$

$\Longrightarrow$ in $\mathbb{D}^b(\operatorname{mod} R)/\operatorname{Perf}(R)$ we have $N \cong S_j[j]$

$R = \mathbb{C}[u,v]/(uv)$, $M = R/(u,v) \cong \mathbb{C}$ $R/u \cong \mathbb{C}[v]$, $R/v \cong \mathbb{C}[u]$







$$\begin{tikzcd}
\cdots \arrow[r] & R^{\oplus 2} \arrow[r, "\begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}"] & R^{\oplus 2} \arrow[r, "\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}"] & R^{\oplus 2} \arrow[r, "\begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}"] \arrow[dr, two heads] & & R^{\oplus 2} \arrow[r, "(u\ v)"] \arrow[dr, two heads] & & R \arrow[r] & M \arrow[r] & 0 \\
& & & & S_1 \arrow[ur] \arrow[d, phantom, "\cong" rotate=90] & & S_0 \arrow[ur] & & & \\
& & & & R/u \oplus R/v & & & & &
\end{tikzcd}$$

<u>Example</u> $R = \mathbb{C}[u,v]/(uv)$

$$\mathbb{C} \cong S_1[1] = R/u[1] \oplus R/v[1] \cong R/u \oplus R/v. \quad \text{in } D^b(\mathrm{mod} R)/\mathrm{Perf}(R)$$

But we can do better, observe that $\mathrm{mod}(R) \subseteq D^b(\mathrm{mod} R)$

$\bar{A} = \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}$ $\bar{B} = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$

$$\cdots \xrightarrow{\bar{B}} R^{\oplus 2} \xrightarrow{\bar{A}} R^{\oplus 2} \xrightarrow{\bar{B}} R^{\oplus 2} \xrightarrow{\bar{A}} \cdots$$

<u>Ex</u> $R/u \cong R/v[1]$ in $D^b(\mathrm{mod} R)/\mathrm{Perf}(R)$.

is an infinite acyclic complex, with syzygies $R/u \oplus R/v$. Hence by the same argument $(R/u \oplus R/v)[1] \cong R/u \oplus R/v$ in $D^b(\mathrm{mod} R)/\mathrm{Perf}(R)$

Def^n (Buchweitz, Orlov) The singularity category of R is $\mathbb{D}^b_{sg}(R) := \mathbb{D}^b(\mathrm{mod}R) / \mathrm{Perf}(R)$.

$$\mathrm{Perf}(R) = \{ \text{bounded cpxs of f.g. projective } R\text{-modules} \}$$

$$\mathbb{D}^b(\mathrm{mod}R) \xrightarrow{\pi} \mathbb{D}^b(\mathrm{mod}R) / \mathrm{Perf}(R)$$ sends $\mathrm{Perf}(R)$ to zero (universally)

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

" $\frac{g}{f}$ "

Def^n The stable category of MCM R-modules $\underline{\mathrm{MCM}}(R)$ has MCM R-modules as objects and morphisms denoted

$$\underline{\mathrm{Hom}}_R(M, N) = \mathrm{Hom}_R(M, N) \Big/ \left\{ \varphi : M \to N \text{ which factor as } M \to P \to N \text{ where } P \text{ is projective} \right\}$$

Theorem (Buchweitz, Orlov) Every object of $D^b_{sg}(R)$, for a hypersurface singularity $R = \mathbb{C}[[\underline{x}]]/f$ is isomorphic to an MCM module and and the canonical functor below factors via an equivalence

$$\begin{tikzcd}
MCM(R) \arrow[r, hook] \arrow[dr] & mod(R) \arrow[r, hook] & D^b(mod R) \arrow[r] & D^b(mod R)/Perf(R) \\
 & \underline{MCM}(R) \arrow[urr, "\Phi", "\cong"'] & &
\end{tikzcd}$$

Example $R = \mathbb{C}[[u,v]]/(uv)$ $\quad \Phi^{-1}(\mathbb{C}) \cong R/u \oplus R/v \quad (A_1\text{-sing})$

$$\mathrm{Vect}^{\mathbb{Z}_2}_{\mathbb{C}} \xrightarrow{\cong} \underline{MCM}(R) \xrightarrow{\cong} D^b_{sg}(R)$$

$\mathbb{C}$	R/u	R/u
$\mathbb{C}[1]$	R/v	R/v
$\mathbb{C} \oplus \mathbb{C}[1]$	$R/u \oplus R/v$	$\mathbb{C}$

$\underline{(M \otimes_R N)^{MCM}}$

Def^n (Buchweitz, Orlov) The singularity category of R is $\mathbb{D}^b_{sg}(R) := \mathbb{D}^b(\mathrm{mod}R)/\mathrm{Perf}(R)$.

Def^n The stable category of MCM R-modules $\underline{\mathrm{MCM}}(R)$ has MCM R-modules as objects

Theorem (Buchweitz, Orlov) $\underline{\mathrm{MCM}}(R) \xrightarrow[\Phi]{\cong} \mathbb{D}^b_{sg}(R)$

Example $R = \mathbb{C}[u,v]/(uv)$ we have computed $\Phi^{-1}(\mathbb{C}) \cong R/u \oplus R/v$.

Def^n (Buchweitz, Orlov) The singularity category of R is $\mathbb{D}^b_{sg}(R) := \mathbb{D}^b(\text{mod}R)/\text{Perf}(R)$.

Def^n The stable category of MCM R-modules $\underline{\text{MCM}}(R)$ has MCM R-modules as objects

Theorem (Buchweitz, Orlov) $\underline{\text{MCM}}(R) \xrightarrow[\Phi]{\cong} \mathbb{D}^b_{sg}(R)$

Theorem (Eisenbud) Over a hypersurface ring $R = \mathbb{C}[[\underline{x}]]/f$ the minimal free resolution of every f.g. R-module N is eventually 2-periodic, that is, of the form

$$\cdots \underbrace{R^{\oplus d} \xrightarrow{\bar{A}} R^{\oplus d} \xrightarrow{\bar{B}} R^{\oplus d}}_{\text{repeats}} \longrightarrow \cdots \longrightarrow N \longrightarrow 0$$

where $A, B \in M_d(\mathbb{C}[[\underline{x}]])$ are power-series matrices satisfying $AB = f \cdot Id$, $BA = f \cdot Id$.
We call (A,B) a matrix factorisation of f.

Example $f = uv$

(i) $A = (u)$, $B = (v)$ (call this X)
(ii) $A = (v)$, $B = (u)$ (call this Y)
(iii) $A = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$, $B = \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}$ (call this Z)

$Z \cong X \oplus Y$
$X \cong Y[1]$

> If $A, B \in M_d(\mathbb{C}[[\underline{x}]])$ are polynomial matrices satisfying $AB = f \cdot \mathrm{Id}$, $BA = f \cdot \mathrm{Id}$, we call (A,B) a <u>matrix factorisation</u> of f.

<u>Def^n</u> The <u>homotopy category of matrix factorisations</u> $\mathrm{hmf}(\mathbb{C}[[\underline{x}]], f)$ has

- objects are MFs (A,B) (square matrices of the same size)
- morphisms $(A,B) \xrightarrow{(\varphi,\psi)} (A',B')$ are commutative diagrams $(S = \mathbb{C}[[\underline{x}]])$

$$\begin{tikzcd}
& S^{\oplus d} \arrow[r, "A"] \arrow[d, "\varphi"] \arrow[dl, dashed, "h"'] & S^{\oplus d} \arrow[r, "B"] \arrow[d, "\psi"] \arrow[dl, "g"'] & S^{\oplus d} \arrow[d, "\varphi"] \arrow[dl, "h"] & {} \arrow[dl, dashed, "g"] \\
{} \arrow[r, dashed, "B'"'] & S^{\oplus e} \arrow[r, "A'"'] & S^{\oplus e} \arrow[r, "B'"'] & S^{\oplus e} \arrow[r, dashed, "A'"'] & {}
\end{tikzcd}$$

modulo the homotopy relation $(\varphi, \psi) \sim (\alpha, \beta)$ if there exist g, h such that $A'g + hB = \psi - \beta$, $B'h + gA = \varphi - \alpha$.

- triangulated category with shift $(A,B)[1] = (-B,-A) \cong (B,A)$ $[2] = \mathrm{Id}$.
  $(A,B) \oplus (A',B') = (A \oplus A', B \oplus B')$

Example $f = uv$ (i) $A = (u)$, $B = (v)$ (call this X)

(ii) $A = (v)$, $A = (u)$ (call this Y)

(iii) $A = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$, $B = \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}$ (call this Z)

Example In $\mathrm{hmf}(\mathbb{C}[[u,v]], uv)$, Def^n (Buchweitz, Orlov) The singularity category of R is $\mathbb{D}^b_{sg}(R) := \mathbb{D}^b(\text{mod}R)/\text{Perf}(R)$.

Def^n The stable category of MCM R-modules $\underline{\text{MCM}}(R)$ has MCM R-modules as objects

Theorem (Buchweitz, Orlov) $\underline{\text{MCM}}(R) \xrightarrow[\Phi]{\cong} \mathbb{D}^b_{sg}(R)$

Theorem For any hypersurface ring there are equivalences of triangulated categories

$$R = \mathbb{C}[[\underline{x}]]/f$$

$$S = \mathbb{C}[[\underline{x}]]$$

$$\text{hmf}(\mathbb{C}[[\underline{x}]], f) \xrightarrow[\cong]{\Lambda} \underline{\text{MCM}}(R) \xrightarrow[\cong]{\Phi} \mathbb{D}^b_{sg}(R)$$

where $\Lambda(A, B) = \text{coker} A$.

Proof Observe that $A : S^{\oplus d} \longrightarrow S^{\oplus d}$ has cokernel N, given $x \in N$, say $x = \overline{y}$ $y \in S^{\oplus d}$

$$fx = \overline{fy} = \overline{ABy} = 0$$

$\therefore$ N is an R-module.

Theorem For any hypersurface ring there are equivalences of triangulated categories

$$\mathrm{hmf}(\mathbb{C}[[x]], f) \xrightarrow[\cong]{\Lambda} \underbrace{\underline{MCM}(R) \xrightarrow[\cong]{\Phi} \boxed{D^b_{sg}(R)}}_{\text{zero if } R \text{ is regular local ring. by ABS.}}$$

where $\Lambda(A,B) = \operatorname{coker} A$.

Theorem If X is a noetherian semi-sep scheme which is regular then $D^b_{sg}(X) = 0$.

Proof cont To see $N = \operatorname{coker}(A)$ is MCM

$$\begin{tikzcd}
\cdots \arrow[r] & R^{\oplus d} \arrow[r, "\bar{A}"] & R^{\oplus d} \arrow[r, "\bar{B}"] \arrow[dr] & & R^{\oplus d} \arrow[r, "\bar{A}"] & {} \\
& & & N \arrow[ur] & &
\end{tikzcd}$$

$A : S^{\oplus d} \to S^{\oplus d}$

is acyclic with syzygy N. Suppose $\bar{A}x = 0$, $x = \bar{y}$. Then $\overline{Ay} = 0$ that is $Ay = fa$ for $a \in S^{\oplus d}$

$f : S^{\oplus d} \to S^{\oplus d}$ is injective

$f = BA$

$$Ay = fa \Rightarrow Ay = ABa$$

$$\Rightarrow y = Ba \text{ since } A \text{ is injective}$$

$$\Rightarrow x = \bar{B}\bar{a}$$

By the earlier depth arguments we may conclude N is MCM.

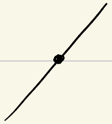
$\square$.

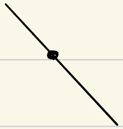
Alternative def^N A matrix factorisation of $f \in \mathbb{C}[[x]]$ is a $\mathbb{Z}_2$-graded f.g. free $S = \mathbb{C}[[x]]$-module $X = X_0 \oplus X_1$ with an odd S-linear map $d_X : X \longrightarrow X$ such that $d_X^2 = f \cdot 1_X$.

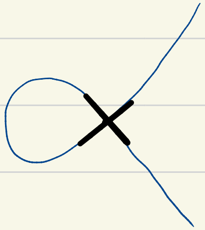
$$d_X = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} : X_0 \oplus X_1 \longrightarrow X_0 \oplus X_1 . \qquad \therefore \; d_X^2 = \begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix}$$

In conclusion, for our original example $R = \mathbb{C}[[u,v]]/(uv)$

$R/u \cong \mathbb{C}[[v]]$, $R/v \cong \mathbb{C}[[u]]$







$$\text{Vect}_{\mathbb{C}}^{\mathbb{Z}_2} \cong \text{hmf}(\mathbb{C}[[u,v]], uv) \cong \underline{\text{MCM}}(R) \cong D^b_{sg}(R)$$

Theorem For any hypersurface ring there are equivalences of triangulated categories

$$\mathrm{hmf}(\mathbb{C}[\![x]\!], f) \xrightarrow[\cong]{\Lambda} \underline{\mathrm{MCM}}(R) \xrightarrow[\cong]{\Phi} \mathbb{D}^b_{sg}(R)$$

Def^n We say $f \in \mathbb{C}[\![x_1, \dots, x_n]\!]$ has an isolated singularity if $\dim_{\mathbb{C}} \left(\mathbb{C}[\![x]\!] / (\partial_{x_1} f, \dots, \partial_{x_n} f) \right) < \infty$.

Theorem Let $f \in \mathbb{C}[\![x]\!]$ have an isolated singularity. Then with $\mathcal{T} = \mathrm{hmf}(\mathbb{C}[\![x]\!], f)$,

- $\mathcal{T}$ has finite-dimensional Hom-spaces
- $\mathcal{T}$ is idempotent complete
- $\mathcal{T}$ is Krull-Remak-Schmidt, i.e.
  - every object is a direct sum of indecomposables
  - if $\bigoplus_{i=1}^n N_i \cong \bigoplus_{j=1}^m M_j$ with N_i, M_j all indecomposable then $m = n$ and after renumbering $N_i \cong M_i$ for all i.

Theorem For any hypersurface ring there are equivalences of triangulated categories

$$\mathrm{hmf}(\mathbb{C}[[\underline{x}]], f) \xrightarrow[\cong]{\Lambda} \underline{\mathrm{MCM}}(R) \xrightarrow[\cong]{\Phi} \mathbb{D}^b_{sg}(R)$$

Theorem (Knörrer periodicity) For any $f \in \mathbb{C}[[x_1, \dots, x_n]]$ with an isolated singularity

$$\mathrm{hmf}(\mathbb{C}[[\underline{x}]], f) \cong \mathrm{hmf}(\mathbb{C}[[\underline{x}, u, v]], f + uv)$$

Example $\mathrm{hmf}(\mathbb{C}[[u,v]], uv) \cong \mathrm{hmf}(\mathbb{C}, 0) = \mathrm{Vect}^{\mathbb{Z}_2}_{\mathbb{C}}$.

Example $\mathrm{hmf}(\mathbb{C}[[x,y,z]], x^{n+1} + y^2 + z^2) \cong \mathrm{hmf}(\mathbb{C}[[x]], x^{n+1})$

A_n surface singularity

$$\cong \underline{\mathrm{MCM}}\left(\mathbb{C}[[x]]/x^{n+1}\right)$$

$$\begin{array}{c} \mathrm{hmf}(\mathbb{C}[[u,v]], uv) \\ \wr\wr \\ \mathrm{Vect}^{\mathbb{Z}_2}_{\mathbb{C}} \end{array} \qquad \mathrm{hmf}(\mathbb{C}[[x,y,z]], x^2+y^2+z^2) \cong \underline{\mathrm{MCM}}\left(\overset{R}{\overset{\shortparallel}{\mathbb{C}[[x]]/x^2}}\right)$$

$$\cancel{R}, \quad R/x = \mathbb{C}$$

- $R = \mathbb{C}[[x_1,\dots,x_n]]/(f_1,\dots,f_c)$ $(f_1,\dots,f_c)$ regular

$\underline{MCM}(R) \cong D^b_{sg}(R)$

$y^m - x^n$

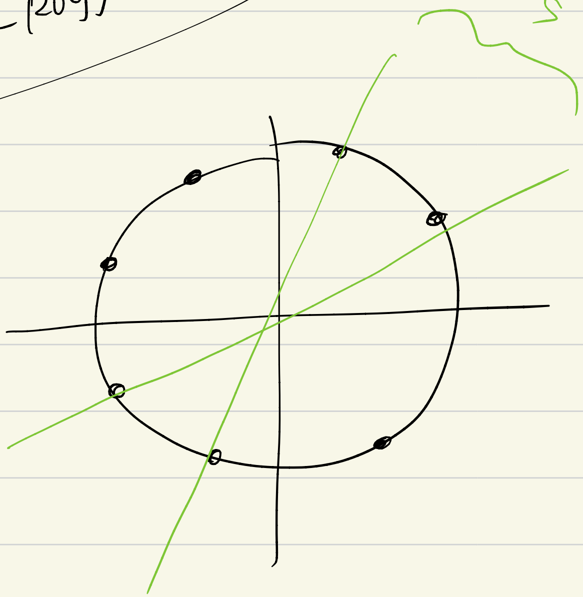
Brunner–Roggenkamp.

- (X, W), relative singularity category

$$Z = Z(W) \subseteq X$$

$$D^b_{sg}(Z) \overset{?}{\longleftrightarrow} MF(X, W)$$

Positselski–Efimov.



- $hmf(\mathbb{C}[[x,y]], y^n - x^n)$ $y^n - x^n = \prod_{i=1}^{n} (y - \xi^i x)$ $\xi = e^{2\pi i/n}$

$S \subseteq \{1,\dots,n\}$

$$P_S := \left(S \xrightarrow{A = \prod_{i \in S}(y-\xi^i x)} S \xrightarrow{B = \prod_{i \notin S}(y - \xi^i x)} S \right)$$

permutation defects

Theorem For any hypersurface ring there are equivalences of triangulated categories

$$\mathrm{hmf}(\mathbb{C}[[x]], f) \xrightarrow[\cong]{\Lambda} \underline{\mathrm{MCM}}(R) \xrightarrow[\cong]{\Phi} \mathbb{D}^{b}_{sg}(R)$$

$$\mathrm{hmf}(\mathbb{C}[[x,y,z]], x^{n+1}+y^2+z^2) \cong \underline{\mathrm{MCM}}\left(\mathbb{C}[[x]]/x^{n+1}\right)$$

Theorem For any hypersurface ring there are equivalences of triangulated categories

$$\mathrm{hmf}(\mathbb{C}[[x]], f) \underset{\cong}{\xrightarrow{\Lambda}} \underline{\mathrm{MCM}}(R) \underset{\cong}{\xrightarrow{\Phi}} \mathbb{D}^{b}_{sg}(R)$$

$$\mathrm{hmf}(\mathbb{C}[[x,y,z]], x^{n+1}+y^2+z^2) \cong \underline{\mathrm{MCM}}(\mathbb{C}[[x]]/x^{n+1})$$

$$\mathrm{hmf}(\mathbb{C}[|x|], x^{n+1}) \cong \underline{\mathrm{MCM}}(R) \cong \mathbb{D}^b_{sg}(R)$$

$$Y_i := \begin{pmatrix} 0 & x^i \\ x^{n+1-i} & 0 \end{pmatrix} \qquad R/x^i \qquad R/x^i \qquad 1 \leq i \leq n$$

<u>Example</u> Let us compute Homs in the different categories, $S = \mathbb{C}[|x|]$

$$\mathrm{Hom}_{hmf}(Y_i, Y_j) = \left\{ \begin{tikzcd} S \arrow[r, "x^i"] \arrow[d, "\varphi"] & S \arrow[r, "x^{n+1-i}"] \arrow[d, "\psi"] & S \arrow[d, "\varphi"] \\ S \arrow[r, "x^j"'] & S \arrow[r, "x^{n+1-j}"'] & S \end{tikzcd} \right\} / \mathrm{htpy}$$

$i \leq j$

Conclusion

Def^n (Buchweitz, Orlov) The singularity category of R is $\mathbb{D}^b_{sg}(R) := \mathbb{D}^b(\mathrm{mod} R) / \mathrm{Perf}(R)$.

Def^n The stable category of MCM R-modules $\underline{\mathrm{MCM}}(R)$ has MCM R-modules as objects

Def^n The homotopy category of matrix factorisations $\mathrm{hmf}(\mathbb{C}[\![x]\!], f)$ has as objects pairs (A, B) of polynomial matrices satisfying $AB = f \cdot Id$, $BA = f \cdot Id$.

Theorem For any hypersurface ring there are equivalences of triangulated categories

$$\mathrm{hmf}(\mathbb{C}[\![x]\!], f) \xrightarrow[\cong]{\Lambda} \underline{\mathrm{MCM}}(R) \xrightarrow[\cong]{\Phi} \mathbb{D}^b_{sg}(R)$$

A_1-singularity $R = \mathbb{C}[\![u, v]\!]/uv$

$$\mathrm{Vect}^{\mathbb{Z}_2}_{\mathbb{C}} \cong \underline{\mathrm{MCM}}(R) \cong \mathbb{D}^b_{sg}(R)$$

$\mathbb{C}$	R/u	R/u
$\mathbb{C}[1]$	R/v	R/v
$\mathbb{C} \oplus \mathbb{C}[1]$	$R/u \oplus R/v$	$\mathbb{C}$

A_n-singularity $R = \mathbb{C}[\![x]\!]/x^{n+1}$

$$\mathrm{hmf}(\mathbb{C}[\![x]\!], x^{n+1}) \cong \underline{\mathrm{MCM}}(R) \cong \mathbb{D}^b_{sg}(R)$$

$Y_i := \begin{pmatrix} 0 & x^i \\ x^{n+1-i} & 0 \end{pmatrix}$	R/x^i	R/x^i

$1 \leq i \leq n$

Next ADE!