<u>Alternative def^N</u> A <u>matrix factorisation</u> of $f \in \mathbb{C}[[x]]$ is a $\mathbb{Z}_2$-graded f.g. free $S = \mathbb{C}[[x]]$-module $X = X_0 \oplus X_1$ with an odd S-linear map $d_X : X \longrightarrow X$ such that $d_X^2 = f \cdot 1_X$.

$$d_X = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} : X_0 \oplus X_1 \longrightarrow X_0 \oplus X_1.$$

$$\therefore \quad d_X^2 = \begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix}$$

In conclusion, for our original example $R = \mathbb{C}[[u,v]]/(uv)$

$$\begin{array}{cccc} \mathrm{Vect}_{\mathbb{C}}^{\mathbb{Z}_2} \cong & \mathrm{hmf}(\mathbb{C}[[u,v]], uv) \cong & \underline{\mathrm{MCM}}(R) \cong & \mathbb{D}^b_{sg}(R) \\ \mathbb{C} & \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix} & R/u & R/u \\ \mathbb{C}[1] & \begin{pmatrix} 0 & v \\ u & 0 \end{pmatrix} & R/v & R/v \\ \mathbb{C} \oplus \mathbb{C}[1] & \begin{pmatrix} 0 & 0 & u & 0 \\ 0 & 0 & 0 & v \\ v & 0 & 0 & 0 \\ 0 & u & 0 & 0 \end{pmatrix} & R/u \oplus R/v & \mathbb{C} \end{array}$$

<u>Theorem</u> (Knörrer periodicity) For any $f \in \mathbb{C}[[x_1, \dots, x_n]]$ with an isolated singularity

$$\mathrm{hmf}(\mathbb{C}[[\underline{x}]], f) \cong \mathrm{hmf}(\mathbb{C}[[\underline{x}, u, v]], f + uv)$$

<u>Example</u> $\mathrm{hmf}(\mathbb{C}[[u,v]], uv) \cong \mathrm{hmf}(\mathbb{C}, 0) = \mathrm{Vect}_{\mathbb{C}}^{\mathbb{Z}_2}$.

<u>Def^n</u> We say $f \in \mathbb{C}[[x_1, \dots, x_n]]$ has an isolated singularity if

$$\dim_{\mathbb{C}} \left(\mathbb{C}[[\underline{x}]] / (\partial_{x_1} f, \dots, \partial_{x_n} f) \right) < \infty.$$

Theorem Let $f \in \mathbb{C}[[x]]$ have an isolated singularity. Then with $\mathcal{T} = \mathrm{hmf}(\mathbb{C}[[x]], f)$,

- $\mathcal{T}$ has finite-dimensional Hom-spaces
- $\mathcal{T}$ is idempotent complete
- $\mathcal{T}$ is Krull-Remak-Schmidt, i.e.
  - every object is a direct sum of indecomposables
  - if $\bigoplus_{i=1}^{n} N_i \cong \bigoplus_{j=1}^{m} M_j$ with N_i, M_j all indecomposable then $m=n$ and after renumbering $N_i \cong M_i$ for all i.

Example $\mathrm{hmf}(\mathbb{C}[[x,y,z]], x^{n+1}+y^2+z^2) \cong \mathrm{hmf}(\mathbb{C}[[x,u,v]], x^{n+1}+uv)$

A_n surface singularity

$$\overset{\text{Knörrer}}{\cong} \mathrm{hmf}(\mathbb{C}[[x]], x^{n+1})$$

$$\overset{\text{Buchweitz-Orlov}}{\cong} \underline{\mathrm{MCM}}(\mathbb{C}[[x]]/x^{n+1})$$

But $R = \mathbb{C}[[x]]/x^{n+1}$ has $\dim(R) = 0$, so every f.g. R-module M is MCM. By the fundamental theorem for modules over a PID, we have in $\mathrm{mod}(R)$

$$M \cong R^{\oplus a} \oplus (R/x)^{\oplus a_1} \oplus \cdots \oplus (R/x^n)^{\oplus a_n}$$

Hence in $\underline{\mathrm{MCM}}(R)$, $M \cong \bigoplus_{i=1}^{n} (R/x^i)^{\oplus a_i}$, so the indecomposables are R/x^i for $1 \le i \le n$. Observe that we have an exact sequence over R

$$\begin{tikzcd}
\cdots \arrow[r, "x^{n+1-i}"] & R \arrow[rr, "x^i"] \arrow[dr] & & R \arrow[rr, "x^{n+1-i}"] \arrow[dr] & & R \arrow[r, "x^i"] & \cdots \\
& & R/x^{n+1-i} \arrow[ur] & & R/x^i \arrow[ur] & &
\end{tikzcd}$$

Hence a triangle $R/x^{n+1-i} \longrightarrow R \longrightarrow R/x^i \longrightarrow R/x^{n+1-i}[1]$ in $\mathbb{D}^b(\mathrm{mod}\,R)$, hence

$$R/x^i \cong R/x^{n+1-i}[1] \text{ in } \mathbb{D}^b_{sg}(R)$$

We can now complete the earlier table for $f = x^{n+1}$, $R = \mathbb{C}[[x]]/x^{n+1}$

$\mathrm{hmf}(\mathbb{C}[[x]], x^{n+1})$	$\cong$	$\underline{\mathrm{MCM}}(R)$	$\cong$	$\mathbb{D}^b_{sg}(R)$	
$Y_i := \begin{pmatrix} 0 & x^i \\ x^{n+1-i} & 0 \end{pmatrix}$		R/x^i		R/x^i	$1 \le i \le n$

<u>Example</u> Let us compute Homs in the different categories, $S = \mathbb{C}[[x]]$

$$\mathrm{Hom}_{\mathrm{hmf}}(Y_i, Y_j) = \left\{ \begin{tikzcd} S \arrow[r, "x^i"] \arrow[d, "\varphi"] & S \arrow[r, "x^{n+1-i}"] \arrow[d, "\psi"] & S \arrow[d, "\varphi"] \\ S \arrow[r, "x^j"'] & S \arrow[r, "x^{n+1-j}"'] & S \end{tikzcd} \right\} \Big/ \mathrm{htpy}$$

$i \le j$

$$= \left\{ \varphi, \psi \in \mathbb{C}[[x]] \;\middle|\; \begin{array}{l} x^{n+1-i}\varphi = x^{n+1-j}\psi \\ x^j\varphi = x^i\psi \end{array} \right\} \Big/ \mathrm{htpy}$$

$$= \{ \varphi, \psi \in \mathbb{C}[[x]] \mid \psi = x^{j-i}\varphi \} / \mathrm{htpy}$$

where $(\varphi, x^{j-i}\varphi) \sim (\varphi', x^{j-i}\varphi')$ iff. there exist $g, h \in S$ such that $x^{j-i}(\varphi - \varphi') = x^j g + h x^{n+1-i}$, $\varphi - \varphi' = g x^i + x^{n+1-j} h$. Now $j \le n$ so that $j - i < n+1-i$ and these conditions are equivalent to $\varphi - \varphi' = x^i g + h x^{n+1-j}$ some g, h. Hence to

$$\varphi - \varphi' \in (x^i, x^{n+1-j}) = (x^{\min(i, n+1-j)})$$

$$\therefore \quad \mathrm{Hom}(Y_i, Y_j) \cong \mathbb{C}[[x]] / (x^{\min(i, n+1-j)})$$

End prep

Recap of Part I

<u>Def^n</u> (Buchweitz, Orlov) The <u>singularity category</u> of R is $D^b_{sg}(R) := D^b(\mathrm{mod}R)/\mathrm{Perf}(R)$.

<u>Def^n</u> The <u>stable category of MCM R-modules</u> $\underline{MCM}(R)$ has MCM R-modules as objects

<u>Def^n</u> The <u>homotopy category of matrix factorisations</u> $hmf(\mathbb{C}[\![x]\!], f)$ has as objects pairs (A, B) of polynomial matrices satisfying $AB = f \cdot Id$, $BA = f \cdot Id$.

<u>Theorem</u> For any hypersurface ring $R = \mathbb{C}[\![x]\!]/f$ there are equivalences of triangulated categories

$$hmf(\mathbb{C}[\![x]\!], f) \xrightarrow[\cong]{\Lambda} \underline{MCM}(R) \xrightarrow[\cong]{\Phi} D^b_{sg}(R)$$

A_1-singularity $R = \mathbb{C}[\![u,v]\!]/uv$

$$\mathrm{Vect}^{\mathbb{Z}_2}_{\mathbb{C}} \cong \underline{MCM}(R) \cong D^b_{sg}(R)$$

$\mathbb{C}$	R/u	R/u
$\mathbb{C}[1]$	R/v	R/v
$\mathbb{C} \oplus \mathbb{C}[1]$	$R/u \oplus R/v$	$\mathbb{C}$

A_n-singularity $R = \mathbb{C}[\![x]\!]/x^{n+1}$

?

Theorem For any hypersurface ring there are equivalences of triangulated categories

$$\mathrm{hmf}(\mathbb{C}[[\underline{x}]], f) \xrightarrow[\cong]{\Lambda} \underline{\mathrm{MCM}}(R) \xrightarrow[\cong]{\Phi} \mathbb{D}^b_{sg}(R)$$

Def^n We say $f \in \mathbb{C}[[x_1, \dots, x_n]]$ has an isolated singularity if $\dim_{\mathbb{C}} \left(\mathbb{C}[[\underline{x}]] / (\partial_{x_1} f, \dots, \partial_{x_n} f) \right) < \infty$.

Theorem Let $f \in \mathbb{C}[[\underline{x}]]$ have an isolated singularity. Then with $\mathcal{T} = \mathrm{hmf}(\mathbb{C}[[\underline{x}]], f)$,

- $\mathcal{T}$ has finite-dimensional Hom-spaces
- $\mathcal{T}$ is idempotent complete
- $\mathcal{T}$ is Krull-Remak-Schmidt, i.e.
  - every object is a direct sum of indecomposables
  - if $\bigoplus_{i=1}^n N_i \cong \bigoplus_{j=1}^m M_j$ with N_i, M_j all indecomposable then $m = n$ and after renumbering $N_i \cong M_i$ for all i.

Theorem For any hypersurface ring there are equivalences of triangulated categories

$$\mathrm{hmf}(\mathbb{C}[[x]], f) \xrightarrow[\cong]{\Lambda} \underline{\mathrm{MCM}}(R) \xrightarrow[\cong]{\Phi} \mathbb{D}^b_{sg}(R)$$

Theorem (Knörrer periodicity) For any $f \in \mathbb{C}[[x_1, \dots, x_n]]$ with an isolated singularity

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Example $\mathrm{hmf}(\mathbb{C}[[x, y, z]], x^{n+1} + y^2 + z^2) \cong \mathrm{hmf}(\mathbb{C}[[x]], x^{n+1})$

An surface singularity

$$\cong \underline{\mathrm{MCM}}(\mathbb{C}[[x]]/x^{n+1})$$

Theorem For any hypersurface ring there are equivalences of triangulated categories

$$\mathrm{hmf}(\mathbb{C}[[x]], f) \xrightarrow[\cong]{\Lambda} \underline{\mathrm{MCM}}(R) \xrightarrow[\cong]{\Phi} \mathbb{D}^b_{sg}(R)$$

$$\mathrm{hmf}(\mathbb{C}[[x,y,z]], x^{n+1}+y^2+z^2) \cong \underline{\mathrm{MCM}}(\mathbb{C}[[x]]/x^{n+1})$$

<u>Theorem</u> For any hypersurface ring there are equivalences of triangulated categories

$$\mathrm{hmf}(\mathbb{C}[[x]], f) \xrightarrow[\cong]{\Lambda} \underline{\mathrm{MCM}}(R) \xrightarrow[\cong]{\Phi} \mathbb{D}^b_{sg}(R)$$

$$\mathrm{hmf}(\mathbb{C}[[x,y,z]], x^{n+1}+y^2+z^2) \cong \underline{\mathrm{MCM}}\left(\mathbb{C}[[x]]/x^{n+1}\right)$$

$$\mathrm{hmf}(\mathbb{C}[[x]], x^{n+1}) \cong \underline{\mathrm{MCM}}(R) \cong \mathbb{D}^b_{sg}(R)$$

$$Y_i := \begin{pmatrix} 0 & x^i \\ x^{n+1-i} & 0 \end{pmatrix} \qquad R/x^i \qquad R/x^i \qquad 1 \le i \le n$$

<u>Example</u> Let us compute Homs $S = \mathbb{C}[[x]]$

$$\underset{i \le j}{\mathrm{Hom}_{\mathrm{hmf}}(Y_i, Y_j)} = \left\{ \begin{tikzcd} S \arrow[r, "x^i"] \arrow[d, "\varphi"] & S \arrow[r, "x^{n+1-i}"] \arrow[d, "\psi"] & S \arrow[d, "\varphi"] \\ S \arrow[r, "x^j"'] & S \arrow[r, "x^{n+1-j}"'] & S \end{tikzcd} \right\} / \mathrm{htpy}$$

Equivariant matrix factorisations

- Let $\mathcal{T}$ be a triangulated category and $G \longrightarrow \mathrm{Aut}(\mathcal{T})$ an action of a finite group by auto-equivalences. There is a separable monad on $\mathcal{T}$

$$A_G : \mathcal{T} \longrightarrow \mathcal{T}, \qquad A_G(x) := \bigoplus_{g \in G} gx$$

and we define a <u>G-equivariant object of $\mathcal{T}$</u> to be an A_G-module

$$\mathcal{T}^G := \mathrm{mod}(A_G) = \{ (x, \eta) \mid x \in \mathrm{ob}(\mathcal{T}),\ \eta : A_G(x) \longrightarrow x \}$$

$$\cong \Big\{ (x, \{\eta_g\}_{g \in G}) \;\Big|\; \eta_g : gx \longrightarrow x \text{ is an iso},$$

$$\eta_e = 1_x \text{ and for } g, h \in G \text{ the diagram}$$

$$\begin{tikzcd} ghx \arrow[rr, "\eta_{gh}"] \arrow[dr, "g(\eta_h)"'] & & x \\ & gx \arrow[ur, "\eta_g"'] & \end{tikzcd} \quad \text{commutes} \Big\}$$

Equivariant matrix factorisations

- Let $f \in \mathbb{C}[[\underline{x}]]$ define an isolated hypersurface singularity, $R := \mathbb{C}[[\underline{x}]]/f$.
- Let G be a finite group acting linearly on $\mathbb{C}[[\underline{x}]]$ by $\mathbb{C}$-algebra automorphisms

$$\varphi_g : \mathbb{C}[[\underline{x}]] \longrightarrow \mathbb{C}[[\underline{x}]]$$

such that $\varphi_g(f) = f$ for all $g \in G$.

- We have an adjoint pair of functors

$$\begin{tikzcd} \mathrm{Mod}\,\mathbb{C}[[\underline{x}]] \arrow[r, bend right, "\varphi_{g*}"'] & \mathrm{Mod}\,\mathbb{C}[[\underline{x}]] \arrow[l, bend right, "\varphi_g^*"'] \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{hmf}(\mathbb{C}[[\underline{x}]], f) \arrow[loop below, "\varphi_g^*"] & \cong & \underline{\mathrm{CM}}(R) \arrow[loop below, "\varphi_g^*"] & \cong & \mathbb{D}^b_{sg}(R) \arrow[loop below, "\varphi_g^*"] \end{tikzcd}$$

Equivariant matrix factorisations

- We have equivalences of triangulated categories (see Balmer "Separability and triangulated cats")

$$\mathrm{hmf}(\mathbb{C}[[x]], f)^G \simeq \underline{CM}(R)^G \simeq \mathbb{D}^b_{sg}(R)^G$$

- A G-equivariant matrix factorisation is a matrix factorisation $X := (A, B)$ together with isomorphisms $\eta_g : \varphi_g^*(X) \longrightarrow X$ in the homotopy category for $g \in G$ such that $\eta_e = 1_X$ and for $g, h \in G$ the diagram below commutes

$$\begin{tikzcd}
\varphi_h^* \varphi_g^*(X) \arrow[r, "\varphi_h^*(\eta_g)"] \arrow[d, phantom, "\cong" rotate=90] & \varphi_h^*(X) \arrow[d, "\eta_h"] \\
\varphi_{hg}^*(X) \arrow[r, "\eta_{hg}"'] & X
\end{tikzcd}$$

$G \longrightarrow \mathrm{Aut}(\mathbb{C}[[x]]),\ g \longmapsto \varphi_g \qquad S := \mathbb{C}[[x]],\ R := S/f.$

A G-equivariant MF is $\left(X \in \mathrm{hmf}(\mathbb{C}[[x]], f),\ \{ \eta_g : \varphi_g^*(X) \xrightarrow{\cong} X \}_{g \in G} \right)$

Remark Let $P := S$ as an S-module and $a : P \longrightarrow P$

$1 \otimes 1$

$$\begin{tikzcd}
\varphi_g^*(P) = P \otimes_S S \arrow[r, "a \otimes 1"] & P \otimes_S S = \varphi_g^*(P) \\
P \arrow[u, "\cong"'] \arrow[r, "\varphi_g(a)"] & P \arrow[u, "\cong"']
\end{tikzcd}$$

1

$a \otimes 1 = 1 \otimes \varphi_g(a)$

$= 1 \otimes 1 \cdot \varphi_g(a)$

$\varphi_g(a)$

Hence

$$\varphi_g^*(X) = \varphi_g^*\left(S^{\oplus d} \xrightarrow{A} S^{\oplus d} \xrightarrow{B} S^{\oplus d} \right) \cong S^{\oplus d} \xrightarrow{\varphi_g A} S^{\oplus d} \xrightarrow{\varphi_g B} S^{\oplus d}$$

$$\varphi_g(A)\varphi_g(B) = \varphi_g(AB) = \varphi_g(f) = f.$$

$G \longrightarrow \mathrm{Aut}(\mathbb{C}[[x]]),\ g \longmapsto \varphi_g \qquad S := \mathbb{C}[[x]],\ R := S/f.$

A G-equivariant MF is $\left(X \in hmf(\mathbb{C}[[x]], f),\ \{\eta_g : \varphi_g^*(X) \xrightarrow{\cong} X\}_{g \in G} \right)$

$\varphi_g^*(A, B) = (\varphi_g A, \varphi_g B)$

<u>Example</u> $f = x^{n+1}$, $G = \mathbb{Z}/(n+1)\mathbb{Z} \circlearrowright \mathbb{C}[[x]]$ $\quad x \overset{\varphi}{\longmapsto} \xi x \quad \xi = e^{\frac{2\pi i}{n+1}}$

$$\varphi^*(Y_i) = \varphi^*\left(S \xrightarrow{x^i} S \xrightarrow{x^{n+1-i}} S\right) \cong$$

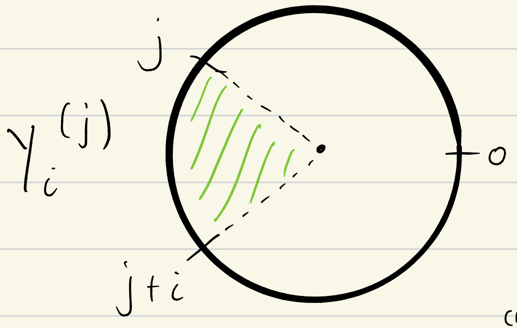
$$\begin{tikzcd}
(S \arrow[r, "\xi^i x^i"] \arrow[d, "\alpha"'] & S \arrow[r, "\xi^{n+1-i} x^{n+1-i}"] \arrow[d, "\beta"] & S) \arrow[d, "\alpha"] \\
\cong (S \arrow[r, "x^i"'] & S \arrow[r, "x^{n+1-i}"'] & S)
\end{tikzcd}$$

$\xi^i \beta x^i = x^i \alpha$

$\Longrightarrow \quad \alpha = \xi^i \beta$

There are $n+1$ distinct equivariant structures on Y_i, given by

$$Y_i^{(j)} := \left(Y_i,\ (\alpha, \beta) = (\xi^{i+j}, \xi^j) \right)$$



$1 \leq i \leq n$

$1 \leq j \leq n$

Brunner-Roggenkamp
"Defects and bulk pertur..."

Theorem For any hypersurface ring $R = S/f$, $S = \mathbb{C}[\![x]\!]$ there are equivalences of triangulated categories

$$\mathrm{hmf}(\mathbb{C}[\![x]\!], f) \xrightarrow{\Lambda} \underline{\mathrm{MCM}}(R) \xrightarrow{\Phi} D^b_{sg}(R)$$

where $\Lambda(A,B) = \operatorname{coker} A$.

Graded matrix factorisations

Let $A = \bigoplus_{n \geq 0} A_n$ with $A_0 = k$ a field be a f.g. commutative graded Gorenstein k-algebra with an isolated singularity, $\mathfrak{m} = A_{\geq 1}$ and $\hat{A} := \mathfrak{m}$-adic completion.

Example $A = \mathbb{C}[x_1, \dots, x_n]/f$ with f homogeneous (possibly $|x_i| \neq 1$).

Def$^{\text{n}}$ A f.g. A-module M is graded maximal Cohen-Macaulay if $M_{\mathfrak{m}}$ is MCM over $A_{\mathfrak{m}}$.

$$\underline{\mathrm{MCM}}^{\mathbb{Z}}(A) := \text{stable category of graded MCM modules}$$

Theorem (Keller–van den Bergh–M) Let (1) denote the $\mathbb{Z}$-grading shift, then

$$\underline{\mathrm{MCM}}^{\mathbb{Z}}(A)/(1) \xrightarrow{\cong} \underline{\mathrm{MCM}}(\hat{A})$$

Graded matrix factorisations

Def^n A quasi-homogeneous potential is $f \in \mathbb{C}[x_1, \ldots, x_n]$ and $|x_i| \in \mathbb{Q}_{>0}$ s.t.

- $\dim_{\mathbb{C}}\left(\mathbb{C}[x_1, \ldots, x_n]/f \right) < \infty$
- $|f| = 2$ in the $\mathbb{Q}$-grading

Define $G_f \subseteq \mathbb{Z}$ to be $\langle \{ |x_i| \mid 1 \leq i \leq n \} \rangle \cong \mathbb{Z}$.

Example (A_n-singularity) $f_{A_n} = x^{n+1} + yz$ $\qquad |x| = \frac{2}{n+1}$ $\quad |y| = |z| = 1.$

(All ADE singularities are quasi-homogeneous) $\qquad G_f = \frac{2}{n+1}\mathbb{Z}$

Def^n A graded matrix factorisation of a quasi-homogeneous potential f is a $\mathbb{Z}_2 \times \mathbb{Q}$-graded f.g. free $\mathbb{C}[\underline{x}]$-module X and a bidegree $(1,1)$ $\mathbb{C}[\underline{x}]$-linear map $d_X : X \longrightarrow X$ such that $d_X^2 = f \cdot 1_X$.

(A_n-singularity) $f_{A_n} = x^{n+1} + yz \qquad |x| = \frac{2}{n+1} \qquad |y| = |z| = 1.$

<u>Def^n</u> A <u>graded matrix factorisation</u> of a quasi-homogeneous potential f is a $\mathbb{Z}_2 \times \mathbb{Q}$-graded f.g. free $\mathbb{C}[\underline{x}]$-module X and a bidegree $(1,1)$ $\mathbb{C}[\underline{x}]$-linear map $d_X : X \longrightarrow X$ such that $d_X^2 = f \cdot 1_X$.

The data of such a matrix factorisation is pair of polynomial matrices $(S = \mathbb{C}[\underline{x}])$ (A, B) with homogeneous entries such that $AB = BA = f \cdot I_d$, some d.

$$\begin{tikzcd}
\bigoplus_{i=1}^d S(a_i) \arrow[r, "A"] \arrow[d, equal] & \bigoplus_{i=1}^d S(b_i) \arrow[r, "B"] \arrow[d, equal] & \bigoplus_{i=1}^d S(a_i) \arrow[d, equal] \\
X_0 & X_1 & X_0
\end{tikzcd} \qquad a_i, b_i \in \mathbb{Q}$$

$\leftarrow \mathbb{Z}_2$-degree

<u>Def^n</u> We call a graded MF <u>pure</u> if $a_i, b_i \in G_f \subseteq \mathbb{Z}$ for $1 \le i \le d$.

<u>Def^n</u> $\mathrm{hmf}^{\mathbb{Z}}(\mathbb{C}[\underline{x}], f)$ is the homotopy category of pure graded MFs with bidegree $(0,0)$ maps, modulo homotopy.

(A_n-singularity) $f_{A_n} = x^{n+1} + yz$ $\qquad |x| = \frac{2}{n+1} \quad |y| = |z| = 1.$

Def^n A <u>graded matrix factorisation</u> of a quasi-homogeneous potential f is a $\mathbb{Z}_2 \times \mathbb{Q}$-graded f.g. free $\mathbb{C}[\underline{x}]$-module X and a bidegree $(1,1)$ $\mathbb{C}[\underline{x}]$-linear map $d_X : X \longrightarrow X$ such that $d_X^2 = f \cdot 1_X$.

<u>Example</u> $f = f_{A_n}$, for $1 \leq i \leq n$ (recall $\mathrm{hmf}(\mathbb{C}[x,y,z], x^{n+1} + y^2 + z^2) \simeq \mathrm{hmf}(\mathbb{C}[x], x^{n+1})$)

$$S(a_1) \oplus S(a_2) \xrightarrow{\begin{pmatrix} y & x^i \\ x^{n+1-i} & -z \end{pmatrix}} S(b_1) \oplus S(b_2) \xrightarrow{\begin{pmatrix} z & x^i \\ x^{n+1-i} & -y \end{pmatrix}} S(a_1) \oplus S(a_2)$$

$a_1 = b_1 \quad \frac{2i}{n+1} = a_1 - b_2 + 1$

$2 - \frac{2i}{n+1} = a_2 - b_1 + 1$

$b_2 = a_2$

$a_1 = b_1$

$a_2 = b_2$

$\frac{2i}{n+1} = a_1 - a_2 + 1$

$2 - \frac{2i}{n+1} = a_2 - a_1 + 1$

$-\frac{2i}{n+1} = a_2 - a_1 - 1$

$a_1 - a_2 = \frac{2i}{n+1} - 1$

$a_1 = b_1 = 0, \qquad a_2 = b_2 = 1 - \frac{2i}{n+1}$ will work.

$\boxed{\text{Call this } Y_i}$

(A_n-singularity) $f_{A_n} = x^{n+1} + yz \qquad |x| = \frac{2}{n+1} \quad |y| = |z| = 1.$

Def^n A <u>graded matrix factorisation</u> of a quasi-homogeneous potential f is a $\mathbb{Z}_2 \times \mathbb{Q}$-graded f.g. free $\mathbb{C}[\underline{x}]$-module X and a bidegree $(1,1)$ $\mathbb{C}[\underline{x}]$-linear map $d_X : X \longrightarrow X$ such that $d_X^2 = f \cdot 1_X$.

We have a commutative diagram of triangulated categories

$$\begin{tikzcd}
\mathrm{hmf}^{\mathbb{Z}}(\mathbb{C}[\underline{x}], f) \arrow[rr, "\cong"] \arrow[dr] \arrow[dd, "- \otimes_{\mathbb{C}[\underline{x}]} \mathbb{C}[[\underline{x}]]"'] & & \underline{\mathrm{MCM}}^{\mathbb{Z}}(\underbrace{\mathbb{C}[\underline{x}]/f}_{A}) \arrow[dr] \arrow[dd] & \\
& \mathrm{hmf}^{\mathbb{Z}}(\mathbb{C}[\underline{x}], f)/(1) \arrow[rr, "\cong"] \arrow[dl, "\cong"] & & \underline{\mathrm{MCM}}^{\mathbb{Z}}(A)/(1) \arrow[dl, "\cong"] \\
\mathrm{hmf}(\mathbb{C}[[\underline{x}]], f) \arrow[rr, dash, "\cong"'] & & \underline{\mathrm{MCM}}(\underbrace{\mathbb{C}[[\underline{x}]]/f}_{\hat{A}}) &
\end{tikzcd}$$

KvdBM

- every MF is gradable, so to classify MFs of ADE singularities it suffices to classify graded MFs.

ADE singularities and the McKay correspondence

$$f(x,y,z) = \begin{cases} x^{n+1} + yz & A_n \quad (n \geq 1) \\ x^2 y + y^{n-1} + z^2 & D_n \quad (n \geq 4) \\ x^3 + y^4 + z^2 & E_6 \\ x^3 + xy^3 + z^2 & E_7 \\ x^3 + y^5 + z^2 & E_8 \end{cases}$$

<u>Theorem</u> The finite non-trivial subgroups $G \subseteq SL(2,\mathbb{C})$ are, up to conjugacy, given by an ADE classification with A_n $(n \geq 1)$ corresponding to

$$\left\langle \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \right\rangle \subseteq SL(2,\mathbb{C}) \qquad \xi = e^{\frac{2\pi i}{n+1}}$$

and $\mathbb{C}[[u,v]]^G \cong \mathbb{C}[[x,y,z]]/f$, f the corresponding ADE potential.

Theorem (McKay correspondence) There is a bijection between nontrivial irreducible G-representations and indecomposable matrix factorisations of f (where $G \subseteq SL(2,\mathbb{C})$ and f correspond via ADE), induced by

$$
\begin{tikzcd}
\mathrm{Rep}(G) \arrow[rr] \arrow[drr, bend right] & & \mathrm{hmf}^{\mathbb{Z}}(\mathbb{C}[x,y,z], f) \\
& & \underline{\mathrm{MCM}}^{\mathbb{Z}}(R) \arrow[u, "\cong"', "\Lambda^{-1}"]
\end{tikzcd}
\qquad R = \mathbb{C}[x,y,z]/f
$$

$$V \longmapsto (S \otimes_{\mathbb{C}} V)^G$$

where $S = \mathbb{C}[u,v]$ and $R := S^G \subseteq S$, with S as a graded MCM R-module

Example $G = \left\langle \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \right\rangle$, $f = x^{n+1} + y^2 + z^2$ $V = \mathbb{C}_i$ (g acts as ζ^i)

$$\Lambda^{-1}(S \otimes_{\mathbb{C}} \mathbb{C}_i)^G \cong Y_i$$