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7/9/16

A_∞-algebras from hypersurface singularities

The aim of this final lecture is to explain how to obtain "finite-dimensional" models of matrix factorisation categories, in the language of A_∞-algebras. As an example, and an attempt to tie together some of the themes of the workshop, I will propose an interpretation in this setting of the cocyclic object $F^\bullet$ of Toby's lectures in terms of semiuniversal deformations of A_n-singularities.

Heuristically we want a map

$$\left\{ \begin{array}{c} W : \mathbb{C}^n \to \mathbb{C} \text{ with isolated} \\ \text{singularities} \end{array} \right\} \longrightarrow \left\{ A_\infty\text{-algebras} \right\}$$

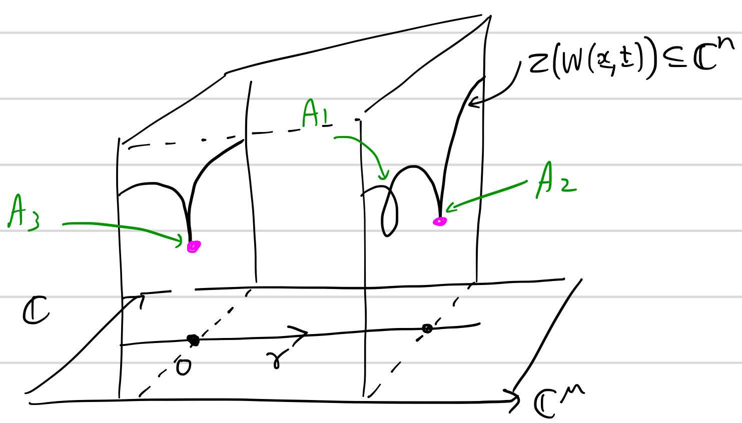
$$W \longmapsto \mathcal{A}_W$$

f.d. vector space with higher operations

such that $\operatorname{mod}(\mathcal{A}_W) \cong \operatorname{hmf}(W)$, for some notion of modules, and

$$\left\{ \begin{array}{c} \text{unfolding } W(\underline{x},\underline{t}) : \mathbb{C}^n \times \mathbb{C}^\mu \longrightarrow \mathbb{C} \\ \text{of an isolated sing. } W = W(\underline{x},0) \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{sheaf of } A_\infty\text{-algebras} \\ \text{on } \mathbb{C}^\mu \end{array} \right\}$$

The sheaf $\underline{t} \longmapsto \mathcal{A}_{W(\underline{x},\underline{t})}$ is too naive, but a small modification works. Then we can consider in e.g. the semiuniversal deformation of the A_3-singularity, a path $\gamma = \gamma(t)$ in $\mathbb{C}^\mu$, and a sheaf of A_∞-categories on $\mathbb{C}[t]$ which is $\mathcal{A}_{A_3}$ over 0 and a mix of $\mathcal{A}_{A_1}, \mathcal{A}_{A_2}$ over a generic point. From γ we obtain an $\mathcal{A}_{A_2}$-$\mathcal{A}_{A_3}$ bimodule $B(\gamma)$.



1 A_∞-algebras

Let k be a commutative $\mathbb{Q}$-algebra, $\otimes = \otimes_k$

<u>Def^N</u> An A_∞-algebra over k is a $\mathbb{Z}$-graded f.g. projective k-module

$$A = \bigoplus_{n\in\mathbb{Z}} A_n$$

with <u>operations</u> $m_n : A^{\otimes n} \longrightarrow A$, $n \geq 1$, k-linear, degree $2-n$.

$$m_1 : A \longrightarrow A \qquad \deg +1$$
$$m_2 : A\otimes A \longrightarrow A \qquad \deg 0$$
$$m_3 : A^{\otimes 3} \longrightarrow A \qquad \deg -1$$
$$\vdots$$

such that for $n \geq 1$

$$(*) \quad \sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t}(\mathbb{1}^{\otimes r} \otimes m_s \otimes \mathbb{1}^{\otimes t}) = 0$$

$$\begin{tikzcd} A^{\otimes n} \\ A^{\otimes r}\otimes A^{\otimes s}\otimes A^{\otimes t} \arrow[u, phantom, "\shortparallel"] \arrow[d, "1\otimes m_s \otimes 1"] \\ A^{\otimes r}\otimes A \otimes A^{\otimes t} \arrow[d, "m_{r+1+t}"] \\ A \end{tikzcd}$$

<u>Def^N</u> A <u>morphism</u> $f : A \to B$ is $f_n : A^{\otimes n} \longrightarrow B$ s.t. $m_1 f_1 = f_1 m_1, \ldots$

<u>Example</u> $m_n = 0$, $n \geq 3$, $(*)$ says (A, m_1, m_2) satisfies

write $ab = m_2(a\otimes b)$

$(n=1)$ $\quad m_1^2 = 0$

$(n=2)$ $\quad m_1(ab) = m_1(a)b + (-1)^{|a|} a\, m_1(b)$

$(n=3)$ $\quad m_2$ is associative.

$\therefore (A, m_1, m_2)$ is a DG-algebra

⌜<u>strict unit</u> is $e \in A^0$, $m_1(e) = 0$, e a unit for m_2 and m_n vanishes for $n > 2$ as soon as any entry is e.
<u>homological unit</u> is a unit for H^*A, we say A is <u>h-unital</u>⌟

Def^n A is minimal if $m_1 = 0$.

Example For $d>2$, $|\varepsilon| = 1$, $A^{(d)} = k[\varepsilon]/\varepsilon^2 = k \oplus k\varepsilon$

$$\left.\begin{array}{l} m_n = 0 \text{ for } n \notin \{2,d\} \\ m_2 = \text{multiplication} \\ m_d(\varepsilon \otimes \cdots \otimes \varepsilon) = (-1)^{d-1} \cdot 1 \end{array}\right\} A^{(d)} \text{ is a } \mathbb{Z}_2\text{-graded } A_\infty\text{-algebra}$$

Where do A_∞-algebras come from?

$\mathcal{T}$ — triangulated category of interest, e.g. $D^b(\mathrm{coh}\, X)$, $\mathrm{hmf}(W)$, ...

$\uparrow H^0$

$\mathcal{D}$ — DG enhancement of $\mathcal{T}$, e.g. by K-injective res.

$\cup\!\!\!\!\!\!\;\shortmid$

E — Generator (all objects "built from" E)

$\downarrow$

$\mathrm{End}_{\mathcal{D}}(E)$ — DG-algebra (usually ∞-dim$/\mathbb{C}$)

$\downarrow$ taking the minimal model

$(H^*\mathrm{End}_{\mathcal{D}}(E), \{m_n\}_{n\geq 2})$ — A_∞-algebra quasi-iso to $\mathrm{End}_{\mathcal{D}}(E)$. (f.dim$/\mathbb{C}$) Knows "everything" about $\mathcal{T}$.

Why find minimal models?

- To understand dependence of categories on moduli.
- Topological string theory (boundary sector) = minimal, cyclic strictly unital A_∞-categories (Herbst–Lazaroiu–Lerche, Costello)

A_∞-modules An A_∞-module over an A_∞-algebra $(A, \{m_n\}_{n\geq 1})$ is a $\mathbb{Z}$-graded f.g. proj k-module M with operations $(n\geq 1)$

$$m_n^M : A^{\otimes(n-1)} \otimes M \longrightarrow M$$

of degree $2-n$ satisfying the same identities ⊛. A *morphism* of A_∞-modules $\varphi : M \longrightarrow N$ is a collection of linear maps $\varphi_n : A^{\otimes(n-1)} \otimes M \longrightarrow N$ of degree $1-n$ such that

$$(u = r+1+t) \qquad \sum_{r+s+t=n} \pm\, \varphi_u(\mathbb{1}^{\otimes r} \otimes m_s \otimes \mathbb{1}^{\otimes t}) = \sum_{r+s=n} \pm\, m_u^N(\mathbb{1}^{\otimes r} \otimes \varphi_s)$$

this is an eq. of maps $A^{\otimes(n-1)} \otimes M \longrightarrow N$

The (ordinary) category of A_∞-modules and these morphisms is denoted $\mathrm{Mod}A$ (note H^*M is a H^*A-module).

says $\varphi_1 m_1 = m_1 \varphi_1$ and φ_1 commutes with the action of A "up to htpy", etc...

The derived category A a h-unital A_∞-algebra.

- There is an A_∞-category of (h-unital) A_∞-modules $\mathrm{Mod}_\infty(A)$, such that $\mathrm{Mod}A = Z^0(\mathrm{Mod}_\infty A)$.

  This is a triangulated A_∞-cat (see Seidel [S])

  More concretely, $\hom^a(M, N)$ is the space of $\{t^n\}_{n\geq 1}$ with each

$$t^n : A^{\otimes(n-1)} \otimes M \longrightarrow N \qquad (\text{of degree } a-n+1)$$

  and only m_1, m_2 are nonzero in $\mathrm{Mod}_\infty(A)$ (i.e. this is a DG-category).

- The *perfect derived category* $\mathrm{per}(A)$ is the smallest triangulated subcategory of $H^0(\mathrm{Mod}_\infty A) = \mathrm{Mod}A/\sim$ containing A.

<u>Example</u> $A = A^{(d)}$ from above, $d>2$ (i.e. $m_n = 0$ $n \notin \{2,d\}$).
Given $2 \leq i \leq d-2$, $i < d-i$ we define an A_∞-module over $A^{(d)}$ by

↑ $\mathbb{Z}_2$-graded

$$M_{(i)} := \Lambda(k\xi) = k \oplus k\xi$$

with operations $\alpha_n = 0$ unless $n \in \{2, i+1, d-i+1\}$

$$\alpha_n : A^{\otimes(n-1)} \otimes M_{(i)} \longrightarrow M_{(i)}$$

$$\alpha_2(1,-) = id,$$
$$\alpha_{i+1}(\varepsilon, \varepsilon, \dots, \varepsilon, -) = \pm \xi^* \lrcorner (-) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$\alpha_{d-i+1}(\varepsilon, \varepsilon, \dots, \varepsilon, -) = \pm \xi \wedge (-) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The minimal model theorem. $\left(\underset{\infty.\mathrm{d.}}{A} \underset{\mathrm{qis}}{\cong} \underset{\mathrm{f.d.}}{B} \right)$

Let (A, ∂, m) be a DG-algebra (suspended forward product).
i.e. s.t. $(A, \{\partial, m\})$ sat. $(*)$

A <u>strict homotopy retraction</u> of A is a $\mathbb{Z}$-graded f.g. projective k-module B and linear maps

$$\begin{tikzcd} A \arrow[loop left, "H"] \arrow[r, shift left, "p"] & B \arrow[l, shift left, "i"] \end{tikzcd}$$

such that

(i) p, i are degree zero morphisms of cpxs
(where B is given zero differential).

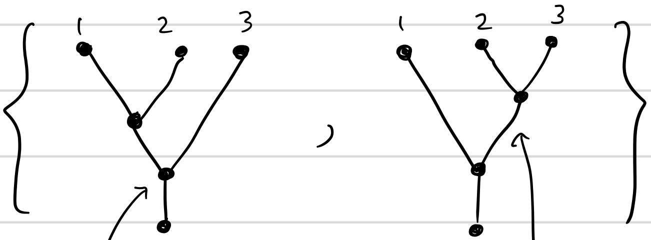
(ii) $p \circ i = 1_B$

(iii) $1_A - i \circ p = H\partial + \partial H$ (i.e. $i \circ p \simeq 1_A$)

$\Longrightarrow$ $B \cong H^*(A, \partial)$, with a particular choice of how to project elements in A onto cocycles ($\partial i p(a) = i \partial p(a) = 0$).

$\mathcal{T}_n = \{$ oriented and connected planar trees, with $n+1$ leaves $\}$
↑ inc. root

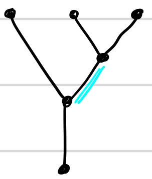
i.e. $\mathcal{T}_3 =$



internal vertex (valency 3)

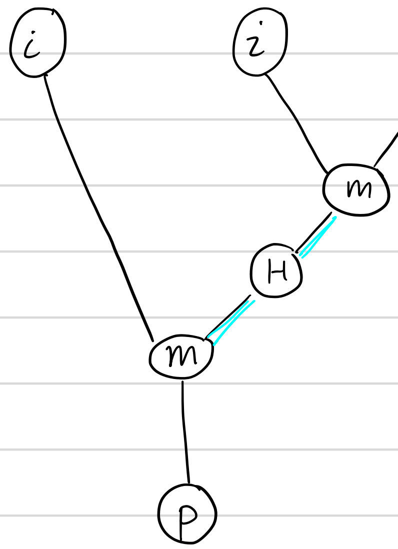
internal edge.

<u>Def^N</u> Given $T \in \mathcal{T}_n$ we define $\rho_T : B^{\otimes n} \longrightarrow B$ by example:



T

$\rightsquigarrow$



$$B \otimes B \otimes B \xrightarrow{1 \otimes i \otimes i} B \otimes A \otimes A \xrightarrow{1 \otimes m} B \otimes A \xrightarrow{i \otimes H} A \otimes A \xrightarrow{m} A \xrightarrow{p} B$$

$$\rho_T = (-1)^{\#\text{int. edges}} \; p \circ m \circ (i \otimes H) \circ (1_B \otimes m) \circ (1_B \otimes i \otimes i)$$

$$\boxed{\rho_n := \sum_{T \in \mathcal{T}_n} \rho_T \; : \; B[1]^{\otimes n} \longrightarrow B[1]}$$

<u>Theorem</u> (Minimal model) $(B, \{\rho_n\}_{n \geq 2})$ is an A_∞-algebra (with suspended forward products) and there is an A_∞-quasi-isomorphism

$$(A, m, \partial) \longrightarrow (B, \{\rho_n\}_{n \geq 2})$$

called the <u>minimal model</u>
(recall $B \cong H^*A$)

② Singularities

Def^n $W \in k[x_1, \dots, x_n]$ is a *potential* (over k) if (with $f_i = \partial_{x_i} W$)

(i) $f_1, \dots, f_n$ is a quasi-regular sequence
(ii) $k[\underline{x}]/(f_1, \dots, f_n)$ is a f.g. projective k-module
(iii) The Koszul complex of $f_1, \dots, f_n$ is exact except in degree 0.

Example (1) $k = \mathbb{C}$, all critical pts isolated

(2) Consider $k = \mathbb{C}[t]$, $W(x,y,t) = x^2 + y^3 - 3t^2 y + 2t^3 \in k[x,y]$ is the semi-universal deformation of the cusp, restricted to the discriminant. Observe $\partial_x W = 2x$, $\partial_y W = 3y^2 - 3t^2$ so

$$k[x,y]/(\partial_x W, \partial_y W) = \mathbb{C}[t,x,y]/(x, y^2 - t^2) \cong \mathbb{C}[t] \oplus \mathbb{C}[t] y$$

so W is a potential over k.

(3) The usual theory of LG models (at least anything which can be encoded into the bicategories $\mathcal{LG}$, $\mathcal{LG}^{gr}$ of the previous lectures) works for these "relative" potentials.

Want Potential $W \rightsquigarrow$ DG category $mf(W) \xrightarrow{\text{min. model}} A_\infty$-category $/ k$

But usually this A_∞-category is constructed by taking cohomology, which is terrible if k is not a field. So we do something different.

Let $W \in k[x_1,\dots,x_n]$ be a potential and $X \in mf(k[\underline{x}], W)$. Then

$$\mathrm{End}(X) := \left(\mathrm{Hom}_{k[\underline{x}]}(X,X),\ d_{\mathrm{Hom}}(\alpha) = d_X\alpha - (-1)^{|\alpha|}\alpha d_X\right)$$

is a DG-algebra. Write $i : k[\underline{x}] \longrightarrow k[\underline{x}]/(f_1,\dots,f_n)$ for the projection (recall $f_i = \partial_{x_i} W$, or in fact any other sequence with properties (i)-(iii) s.t. each f_i acts null-homotopically on $\mathrm{End}(X)$). We further write

$$S = \bigwedge(k\theta_1 \oplus \cdots \oplus k\theta_n). \qquad |\theta_i| = 1$$

<u>Theorem</u> (Dyckerhoff-M '09, M '15) There is a strict homotopy retract of $\mathbb{Z}_2$-graded complexes / k

$$H \circlearrowright S \otimes_k \mathrm{End}(X) \underset{i}{\overset{p}{\rightleftarrows}} i^*\mathrm{End}(X)$$

defined in terms of a k-linear connection $k[\underline{x}] \xrightarrow{\nabla} k[\underline{x}] \otimes_{k[\underline{f}]} \Omega^1_{k[\underline{f}]/k}$

cpx of f.g. proj k-modules

<u>Remarks</u>

- The minimal model theorem is useful precisely to the extent that you have a good homotopy. The above H is good.

- Get a (possibly non-min) A_∞-algebra $(i^*\mathrm{End}(X), \{m_n\}_{n\geq 2})$ quasi-iso to $S \otimes_k \mathrm{End}(X)$, together with a Clifford action which picks out a subalgebra q.is to $\mathrm{End}(X)$.

- In many cases, can promote this to a <u>minimal</u> model of a sub-DG-category of $S \otimes_k mf(W)$.

3 Calculations

$W \in k[x_1, \dots, x_n]$ a potential. For $P \in \mathrm{Sing}(W)$,

$$k(P)^{stab} := \Big(k[\underline{x}] \otimes_k \Lambda(k\psi_1 \oplus \cdots \oplus k\psi_n),\ \sum_{i=1}^n (x_i - P_i)\psi_i^* + \sum_{i=1}^n W_P^i \psi_i \Big)$$

where we choose $W = \sum_{i=1}^n (x_i - P_i) W_P^i$, some $W_P^i \in \mathfrak{m}_P^2$. In the case W has local quadratic terms there is a simple modification to the following. For the following take $P = 0$, and write

$$\mathcal{A}_W := \text{minimal model of } \mathrm{End}(k^{stab})$$

Def^N The underlying algebra of $\mathcal{A}_W$ is

$$\mathcal{A}_W = \Lambda(k\psi_1 \oplus \cdots \oplus k\psi_n) \qquad |\psi_i| = 1$$

To define $m_r : \mathcal{A}_W^{\otimes n} \longrightarrow \mathcal{A}_W$ we introduce an auxiliary space

$$\mathcal{H} := \mathcal{A}_W \otimes \Lambda(k\theta_1 \oplus \cdots \oplus k\theta_n) \otimes k[\underline{x}]$$

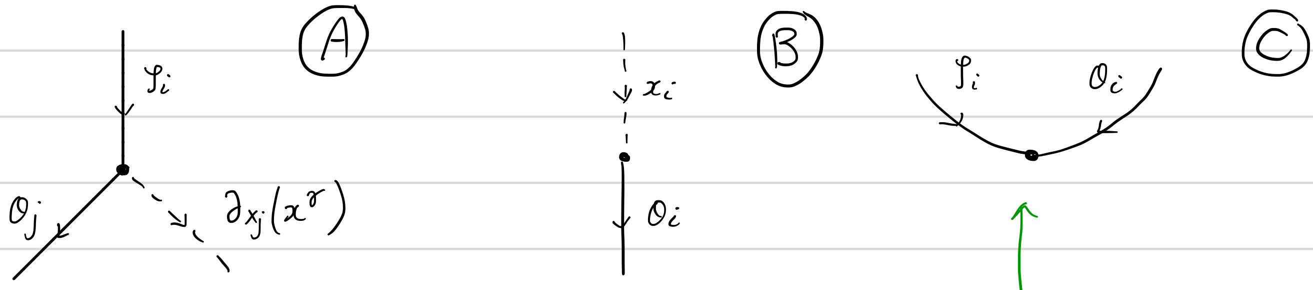
write m for the product in $\mathcal{H}$

$$\begin{tikzcd} \mathcal{H} \arrow[r, bend left, "p = \text{projection } (\theta, x \mapsto 0)"] & \mathcal{A}_W \arrow[l, bend left, "i = \text{embedding}"] \end{tikzcd}$$

Standard operations

$$\mathcal{H} \circlearrowleft \underbrace{\overset{\text{wedge}}{\psi_i}, \overset{\text{contraction}}{\psi_i^*}, \theta_i, \theta_i^*}_{\substack{\text{fermion} \\ \text{creation/annihilation}}}, \underbrace{x_i, \partial_{x_i}}_{\substack{\text{boson creation/} \\ \text{annihilation}}}$$

Interactions $\quad W = \sum_i x_i W^i \qquad W^i = \sum_{\gamma \in \mathbb{N}^m} W^i(\gamma) x^\gamma \qquad W^i(\gamma) \in k$



$-\frac{1}{|\gamma|} W^i(\gamma)$

(for all i, j and $\gamma \in \mathbb{N}^m$)

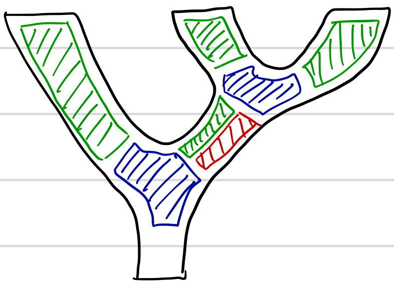
$-\frac{1}{|\gamma|} W^i(\gamma)\, \theta_j\, \partial_{x_j}(x^\gamma)\, \varphi_i^* \in \mathcal{H} \qquad \theta_i \partial_{x_i} \in \mathcal{H} \qquad \varphi_i^* \otimes \theta_i^* \in \mathcal{H}^{\otimes 2}$

The Feynman calculus now describes the structure constants of the m_r's, for $\gamma_1, \ldots, \gamma_r, \delta \in A_W$ (products of φ's) by the formula

$$m_r(\gamma_1 \otimes \cdots \otimes \gamma_r)_\delta = \sum_{\substack{\text{binary} \\ \text{trees} \\ T}} \sum_{\substack{\text{Feynman} \\ \text{diagrams } D, \\ \gamma \text{ incoming} \\ \delta \text{ outgoing}}} \text{amplitude}(D)$$

where the amplitude is an element of k defined by

<u>Rough def^N</u> A <u>Feynman diagram</u> D for a tree $T \in \mathcal{T}_n$ is an oriented graph in the thickening of T with lines labelled by $\mathcal{S}_i, \mathcal{O}_i, x_i$ $1 \le i \le n$ and nodes of type A, B, C above, subject to the constraints



<u>A nodes</u>: may occur at input edges and int. edges

<u>B nodes</u>: precisely one at each internal edge

<u>C nodes</u>: may occur at internal vertices

Finally, there is a <u>boundary condition</u> : graph edges incident at the boundary of T (i.e. input edges or the root) may only be labelled $\mathcal{S}_i$, $1 \le i \le n$.

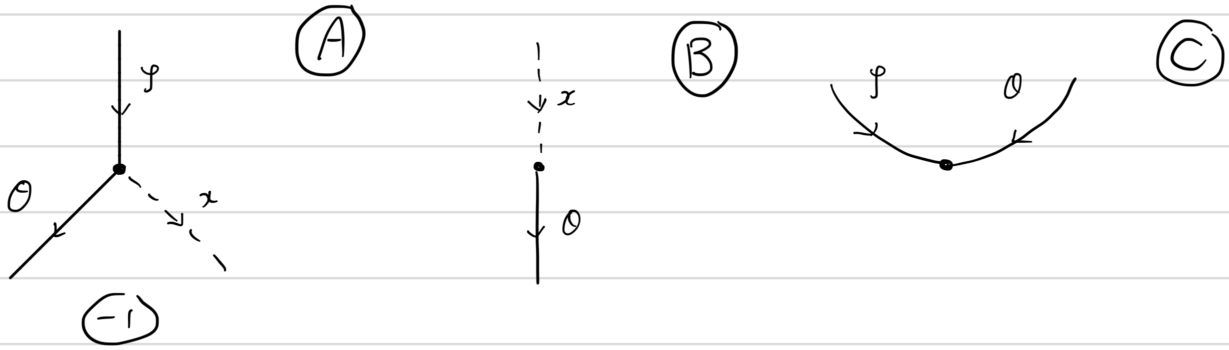
<u>Def^n</u> The <u>amplitude</u> of a Feynman diagram D is

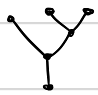
$$\text{amplitude}(D) = (-1)^{f(D)}\, T_D \prod_{A \text{ nodes}} \Big(- \underbrace{\frac{\gamma_j}{|\gamma|} W^i(\gamma)}_{j,\, i,\, \gamma \text{ depend on the node}} \Big) \in k$$

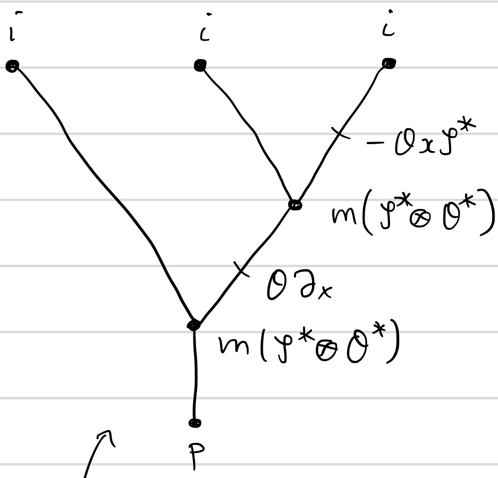
symmetry factor $(\in \mathbb{Q})$

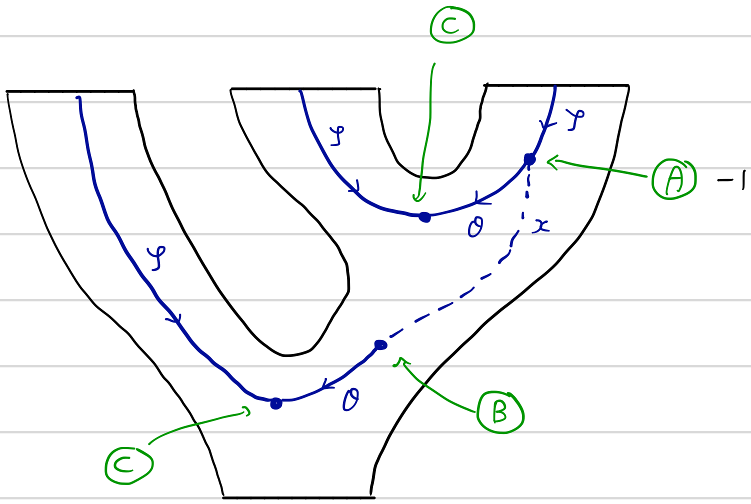
<u>Example</u> In the special case $W = x^3 = x \cdot x^2$, so $W' = x^2$

$$A_W = \Lambda(k\psi) = k \oplus k\psi \qquad \mathcal{H} = \Lambda(k\psi) \otimes \Lambda(k\theta) \otimes k[x]$$



A Feynman diagram for T =  is:





in = ψ, ψ, ψ , out = 1

amplitude = −1

denotes the linear map $A_W^{\otimes 3} \to A_W$ defined by

$$p\, m(\psi^* \otimes \theta^*)\Big(i(-) \otimes \theta \partial_x m(\psi^* \otimes \theta^*)\big(i(-) \otimes (-\theta x \psi^*) i(-) \big) \Big)$$

$$\psi \otimes \psi \otimes \psi \longmapsto -1$$

<u>In fact</u> this is the only nontrivial Feynman diagram, so $A_W = \Lambda(k\psi)$ has m_2 = usual product, $m_3(\psi \otimes \psi \otimes \psi) = -1$ otherwise zero, $m_n = 0$ $n \notin \{2, 3\}$.

Lemma For $d > 2$, $\mathcal{C}_{x^d} = A^{(d)}$ defined earlier (i.e. $m_n = 0$ $n \notin \{2, d\}$).

minimal models for MFs:

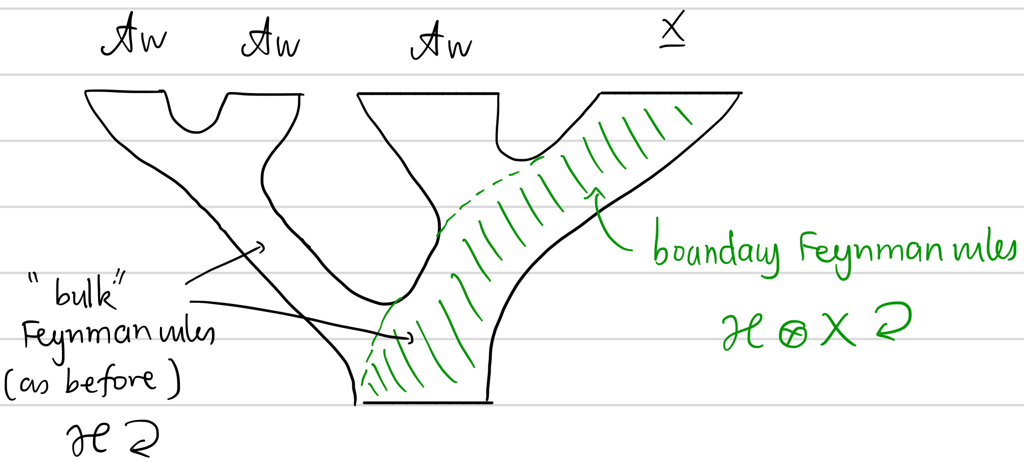
Q/ What is the A_∞-module corresponding to $X \in hmf(W)$?

Assume for simplicity that $d_X(X) \subseteq \mathfrak{m}^2 X$, then the underlying v.space is

$$\underline{X} := X \otimes_{k[x]} k[n]$$

$$\alpha_n : \mathcal{A}_W^{\otimes(n-1)} \otimes \underline{X} \longrightarrow \underline{X}$$

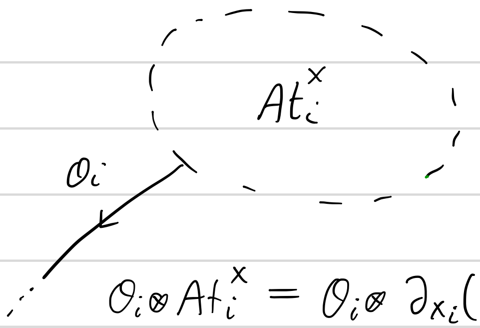
is computed by Feynman rules on diagrams of operators on $\mathcal{H} \otimes_{k[x]} X$, e.g.



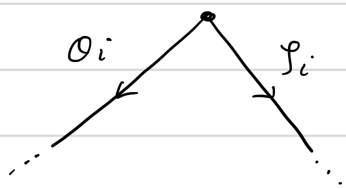
Boundary Feynman rules (in addition to Ⓐ, Ⓑ, Ⓒ)

Ⓓ, Ⓔ vertices allowed on any edge

Ⓓ



Ⓔ



$$\theta_i \otimes At_i^X = \theta_i \otimes \partial_{x_i}(d_X) \circlearrowleft \mathcal{H} \otimes X$$

"Atiyah class"

Note that the Atiyah classes completely determine the A_∞-module structure.

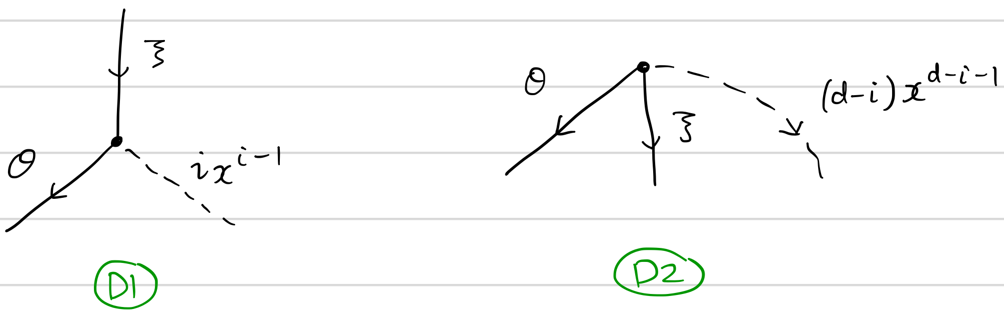
<u>Example</u> Consider $W = x^d$, $d \geq 3$ and $\mathcal{A}_W = (\Lambda(k\varphi), m_2, m_d)$

$$X = \left(\Lambda(k\xi),\ x^i \xi^* + x^{d-i} \xi \right) \qquad |\xi| = 1$$

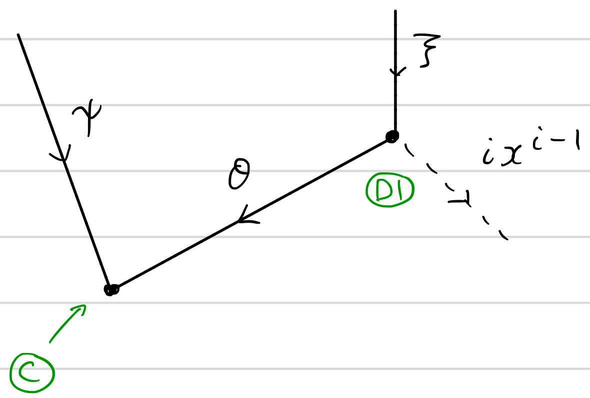
and assume $2 \leq i \leq d-2$. Then

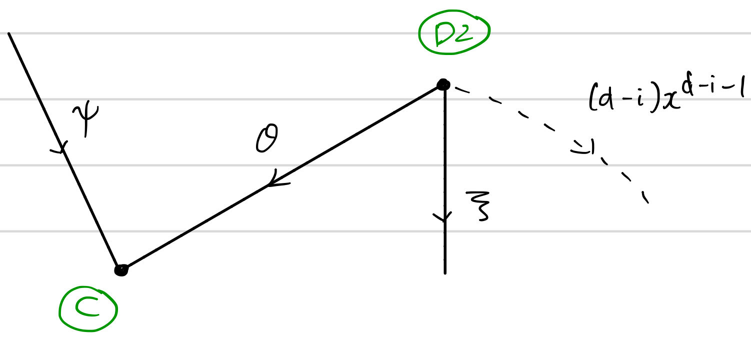
$$\underline{X} = (k \oplus k\xi)[1] \quad \text{and} \quad \partial_x(dx) = i x^{i-1} \xi^* + (d-i) x^{d-i-1} \xi$$

Hence the "Atiyah" interaction is actually two interactions:



Since $\underline{X}$ is an A_∞-module over $\mathcal{A}_W = \Lambda(k\varphi)$ we want to know how φ "acts" on ξ. The only interactions are the ones mediated by a θ:





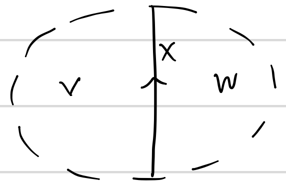
Lemma The A_∞-module $\underline{X}$ for $X = \begin{pmatrix} 0 & x^i \\ x^{d-i} & 0 \end{pmatrix}$ is $M_{(i)}$ from earlier.

Proof (11.1) gives rise to the operation $\varphi^{\otimes i+1} \longmapsto \xi^*$, (11.2) to $\varphi^{\otimes d-i+1} \mapsto \xi$. $\square$

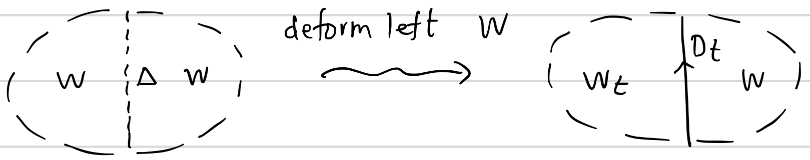
(4) From flows to A_∞-bimodules

As an application of the above, we propose an implementation of the following idea of Brunner-Roggenkamp "Defects and bulk perturbations of LG models" '08.

(1) 1-morphisms $W \xrightarrow{X} V$ are defect conditions



(2) A deformation of W should be implemented by a defect D_t



$\Delta : W \longrightarrow W$ $\qquad\qquad$ $D_t : W \longrightarrow W_t$

Example The semiuniversal unfolding of the A_3-singularity is $(k = \mathbb{C}[a,b,c])$

$$W(x,y,a,b,c) = x^4 + y^2 + ax^2 + bx + c \in k[x,y]$$

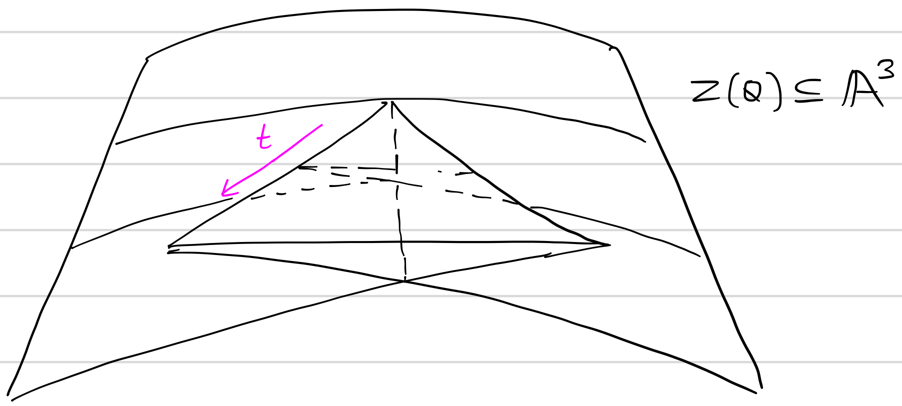
which is a potential $/ k$. Consider

$$\begin{tikzcd} \mathrm{Sing}(W) \arrow[r, phantom, "\subseteq"] \arrow[dr, "\pi"'] & \mathrm{Spec}(k[x,y]) \arrow[d] \\ & \mathrm{Spec}\, k \end{tikzcd}$$

$\mathrm{Im}(\pi)$ is the discriminant, with equation

$$Q = 256c^3 - 27b^4 + 144ab^2c - 128a^2c^2 - 4a^3b^2 + 16a^4c$$

which is the swallowtail surface.



Choose a parametrised pair of points P_t, Q_t in $\pi^{-1}(t)$ with P_t an A_2 and Q_t an A_1-singularity. Taking the fiber product with $\mathbb{C}[t]$ we may take

$$k(P_t)^{stab},\ k(Q_t)^{stab} \in mf(\mathbb{C}[x,y,t], W_t)$$

and look at the DG-category consisting of these two objects and their two mapping complexes. Call this $\mathcal{C}$. For $t=0$, $P_t = Q_t$ is an A_3-singularity.

The above allows us to compute a minimal A_∞-category structure on the f.g. projective $\mathbb{C}[t]$-module $\left(i_t : \mathbb{C}[x,y,t] \longrightarrow \mathbb{C}[x,y,t]/(\partial_x W_t, \partial_y W_t) \right)$

$$i_t^* \mathcal{C} = \left\{ i_t^* k(P_t)^{stab} \rightleftarrows i_t^* k(Q_t)^{stab} \right\}$$

i.e. a vector bundle of A_∞-algebras and bimodules on $\mathbb{A}^1 = \mathrm{Spec}(\mathbb{C}[t])$. Over a point $t = \lambda \in \mathbb{C}$, $i_t^* k(P_t)^{stab}|_{t=\lambda}$ is quasi-iso to $\wedge \mathbb{C}^n \otimes_{\mathbb{C}} H^* \mathrm{End}(k(P_\lambda)^{stab})$.

For $t \neq 0$, $P_t \neq Q_t$ so $\mathrm{Hom}(i_t^* k(P_t)^{stab}, i_t^* k(Q_t)^{stab})$ is contractible, and $i_t^*\mathcal{C}$-modules are just a pair of $i_t^* k(P_t)^{stab}$, $i_t^* k(Q_t)^{stab}$-modules. This suggests a functor Φ_t which is "lift and project"

$$\begin{array}{c} \mathrm{hmf}(V^{(A_3)}) \cong \mathrm{per}(k(P_0)^{stab}) \\ \uparrow t \mapsto 0 \\ \mathrm{per}(i_t^* \mathcal{C}) \\ \downarrow t \neq 0 \\ \mathrm{per}(i_t^* k(P_t)^{stab}) \cong \mathrm{hmf}(V^{(A_2)}) \end{array}$$

lifting a module to a *family*

To implement the Brunner-Roggenkamp idea, do this with

$$\Delta_W \in \mathrm{hmf}(V^{(A_3)}(x', y') - V^{(A_3)}(x, y))$$

only deforming the left hand copy of $V^{(A_3)}$, to obtain a $\mathcal{A}_{A_2}$-$\mathcal{A}_{A_3}$-bimodule implementing the deformation. It is natural to expect these are the arrows in the Dyckerhoff-Kapranov cocyclic object $F^\bullet$ in $\mathcal{C}at^{(2)}_{dg}$

Appendix

Example $W = y^3 - x^3$, $\mathcal{C}_W = \Lambda(k\varphi_1 \oplus k\varphi_2)$, using forward suspended products $\rho_n : \mathcal{C}_W[1]^{\otimes n} \longrightarrow \mathcal{C}_W[1]$, only $\rho_2, \rho_3, \rho_4, \rho_6$ are nonzero, and for $\Lambda_1, \dots, \Lambda_6 \in \mathcal{C}_W$

$$\rho_6(\varphi_1\varphi_2 \otimes \cdots \otimes \varphi_1\varphi_2) = \frac{1}{4} \qquad \text{(only nonzero value)}$$

$$\rho_3(\Lambda_1 \otimes \Lambda_2 \otimes \Lambda_3) = \pm\Big(\varphi_2^*(\Lambda_1)\varphi_2^*(\Lambda_2)\varphi_2^*(\Lambda_3) - \varphi_1^*(\Lambda_1)\varphi_1^*(\Lambda_2)\varphi_1^*(\Lambda_3) \Big)$$

$$\rho_4(\Lambda_1 \otimes \cdots \otimes \Lambda_4) = \pm \tfrac{1}{2} \varphi_2^*(\Lambda_1) \cdot \varphi_2^*\varphi_1^*(\Lambda_2) \cdot \varphi_1^*(\Lambda_3) \cdot \varphi_2^*\varphi_1^*(\Lambda_4) + \cdots$$

Symmetry factor x an internal edge

$$\omega(x) = \sum_{y < x} \sum_{j \in J(y)} |\gamma_j(y)| - \sum_{z < x} m(z)$$

(y int. edge or inputs) (z = int. vertices)

$$F(x) = \frac{1}{\omega(x)} C^{un}_{\omega(x)}\left(\{ |\gamma_j(x)| \}_{j \in J(x)} \right)$$

$$C^{un}_{\alpha}(\ell_1, \dots, \ell_r) = \ell_1 \cdots \ell_r \sum_{\sigma \in S_r} \frac{1}{\alpha + \ell_{\sigma(r)}} \frac{1}{\alpha + \ell_{\sigma(r)} + \ell_{\sigma(r-1)}} \cdots \frac{1}{\alpha + \ell_1 + \cdots + \ell_r}$$

Cyclic A D-cyclic structure on A_∞-cat is $\langle\ \rangle_{ab} : \mathrm{Hom}(a,b) \otimes \mathrm{Hom}(b,a) \to \mathbb{C}[-D]$ $\langle u \otimes v \rangle = (-1)^{|u||v|} \langle v \otimes u \rangle$ and $\langle x_0 \otimes r(x_1 \otimes \cdots \otimes x_n) \rangle = \pm \langle x_1 \otimes r(x_2 \otimes \cdots \otimes x_n \otimes x_0) \rangle$.
(see [L] §3)

References

[C] K. Costello "Topological conformal field theories and Calabi-Yau categories", Advances 2007.

[K] B. Keller "A brief introduction to A_∞-algebras"

[L] C. Lazaroiu "Generating the superpotential on a D-brane category"

[D] T. Dyckerhoff "Compact generators in categories of matrix factorisations" Duke, 2011.

[M] D. Murfet "Cut systems and matrix factorisations I" arXiv: 1402.4541.

[S] P. Seidel "Fukaya categories and Picard-Lefschetz theory"

[Le] K. Lefèvre-Hasegawa "Sur les A_∞-catégories"

Backgwound

Let us compute the Hochschild cohomology of $k[\varepsilon]/\varepsilon^2$ for a commutative ring k, following (lipvrl). The Bar complex is $(R = k[\varepsilon]/\varepsilon^2,\ A = k)$

$$\mathbb{B}_n = R^e \otimes_k (R/k)^{\otimes n} \qquad R/k = k\varepsilon \qquad (\otimes = \otimes_k)$$

$$= R \otimes \underbrace{k\varepsilon \otimes \cdots \otimes k\varepsilon}_{n} \otimes R$$

with differential $\partial_n : \mathbb{B}_n \longrightarrow \mathbb{B}_{n-1}$ defined by

$$\partial_n(r[a_1|\cdots|a_n]r') = ra_1[a_2|\cdots|a_n]r'$$
$$+ \sum_{i=1}^{n-1} r[a_1|\cdots|a_ia_{i+1}|\cdots|a_n]r'$$
$$+ (-1)^n r[a_1|\cdots|a_{n-1}]a_nr'$$

Now in this case $\mathbb{B}_n \cong R^e$ as R-bimodules and $a_ia_{i+1} = 0$ if $a_i = \varepsilon$ or $a_{i+1} = \varepsilon$, so the complex $\mathbb{B}$ is simply

$$\partial_n(r[\varepsilon|\cdots|\varepsilon]r') = r\varepsilon[\varepsilon|\cdots|\varepsilon]r'$$
$$+ (-1)^n r[\varepsilon|\cdots|\varepsilon]\varepsilon r'$$

$$\mathbb{B}_n \cong R^e \longrightarrow R^e \cong \mathbb{B}_{n-1}$$

$$r \otimes r' \longmapsto r\varepsilon \otimes r' + (-1)^n r \otimes \varepsilon r'$$

Now, it follows that $\mathrm{Hom}_{R^e}(\mathbb{B}, R)$ is simply the R-linear map

$$\begin{array}{c} 1 \otimes 1 \\ \updownarrow \\ a \end{array}$$

$$\begin{array}{ccc} \mathrm{Hom}_{R^e}(\mathbb{B}_n, R) & \longleftarrow & \mathrm{Hom}_{R^e}(\mathbb{B}_{n-1}, R) \\ \wr\Vert & & \wr\Vert \\ R & & R \\ (1+(-1)^n)\varepsilon & \longleftarrow & 1 \end{array}$$

That is, $\mathrm{Hom}_{R^e}(\mathbb{B}, R)$ is

$$
\begin{array}{ccccccccccc}
 & & n & & n-1 & & & 2 & & 1 & & 0 \\
\cdots & \longleftarrow & R & \xleftarrow{(1+(-1)^n)\varepsilon} & R & \cdots & \xleftarrow{0} & R & \xleftarrow{2\varepsilon} & R & \xleftarrow{0} & R
\end{array}
$$

$k\varepsilon$

Assuming $\frac{1}{2} \in k$, this allows us to compute that

$$
HH^n(R) = H^n \mathrm{Hom}_{R^e}(\mathbb{B}, R) = \begin{cases} R & n = 0 \\ k\varepsilon & n > 0 \text{ odd} \\ k & n > 0 \text{ even} \end{cases}
$$

For $n > 0$ the generator of $HH^n(R)$ is the cocycle $(k\varepsilon)^{\otimes n} \longrightarrow R$ given by $\varepsilon^{\otimes n} \longmapsto \varepsilon$ if n is odd, and $\varepsilon^{\otimes n} \longmapsto 1$ if n is even.