Generalised orbifolding of simple singularities II

In Part II and III of the minicourse we will expand on various points of the first overview talk. In this talk we focus on the abstract bicategorical framework for generalised orbifolding, following Carqueville & Runkel "Orbifold completion of defect bicategories". In Part III we return to the concrete examples, but in a more thoroughgoing way.

In summary: the right context for orbifold equivalence is bicategorical.

Outline for Part II

[everything given without citation is from either Carqueville-Murfet '12, Carqueville-Runkel '12]

① Bicategories

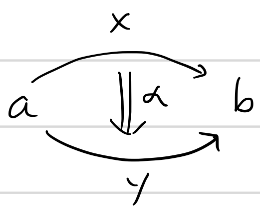
② Orbifold completion $\mathcal{B} \longmapsto \mathcal{B}_{eq}$

③ Landau-Ginzburg models $\mathcal{LG}$, $\mathcal{LG}^{gr}$

① Bicategories

A bicategory $\mathcal{B}$ has

- a class $ob(\mathcal{B})$ of objects $a, b, c, \ldots$
- for each pair $a, b \in ob(\mathcal{B})$ a small category $\mathcal{B}(a,b)$
  - objects are called 1-morphisms $X : a \longrightarrow b$
  - morphisms are called 2-morphisms



- for each triple $a, b, c \in ob(\mathcal{B})$ a functor

$$\mathcal{B}(b,c) \times \mathcal{B}(a,b) \longrightarrow \mathcal{B}(a,c)$$

  - on 1-morphisms $(Y: b \to c,\ X: a \to b) \mapsto Y \otimes X : a \to c$
  - on 2-morphisms $\left(b \underset{Y_2}{\overset{Y_1}{\Downarrow \alpha}} c,\ a \underset{X_2}{\overset{X_1}{\Downarrow \beta}} b \right) \mapsto \alpha \otimes \beta : Y_1 \otimes X_1 \to Y_2 \otimes X_2$

- for each $a \in ob(\mathcal{B})$ a 1-morphism $\Delta_a : a \to a$.

- for each composable triple X, Y, Z of 1-morphisms a natural 2-iso

$$a_{XYZ} : (Z \otimes Y) \otimes X \xrightarrow{\cong} Z \otimes (Y \otimes X)$$

- for each 1-morphism $X : a \to b$ a natural 2-iso

$$\lambda_X : \Delta_b \otimes X \xrightarrow{\cong} X, \quad \rho_X : X \otimes \Delta_a \xrightarrow{\cong} X$$

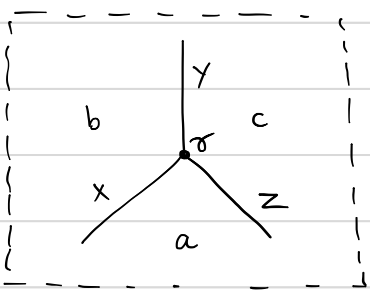
subject to the usual coherence axioms for monoidal categories.

<u>Example</u> (a) <u>Cat</u> = small categories, functors, nat. trans

(b) $ob(\mathcal{B}) = \{\bullet\}$, $\mathcal{B}(\bullet, \bullet)$ monoidal cat.

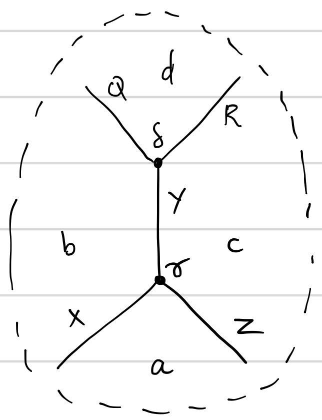
(c) $\underline{Bim}_k$ = k-algebras, bimodules

<u>Notation</u>



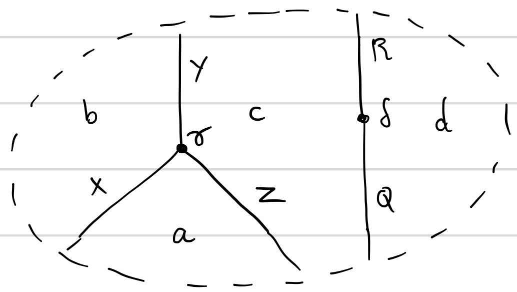
$X : a \to b$, $Y : c \to b$, $Z : c \to a$

denotes $\gamma : X \otimes Z \longrightarrow Y$



denotes the composition of 2-morphisms

$$X \otimes Z \xrightarrow{\gamma} Y \xrightarrow{\delta} Q \otimes R$$

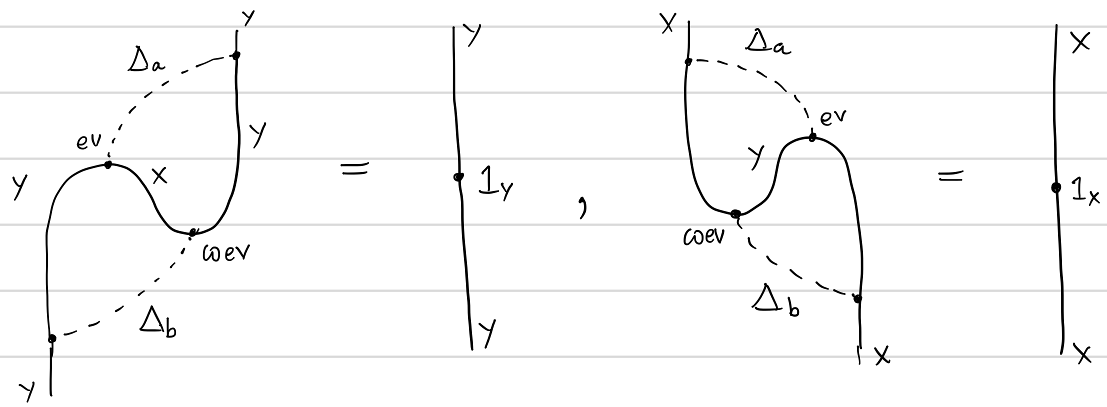


denotes $(X \otimes Z) \xrightarrow{\gamma \otimes \delta} Y \otimes R$

⌈ Made precise by Joyal, Street "geometry of tensor calculus" ⌋

Defⁿ

Def^n An <u>adjunction</u> $X \dashv Y$ between $X : a \to b$ and $Y : b \to a$ in $\mathcal{B}$ is a pair of 2-morphisms $ev : Y \otimes X \to \Delta_a$, $coev : \Delta_b \to X \otimes Y$ such that the following identities hold



We write $^{\dagger}X$ for the left and $X^{\dagger}$ for the right adjoint, if they exist. There are canonical isomorphisms

$$\mathcal{R} : (Y \otimes X)^{\dagger} \xrightarrow{\cong} X^{\dagger} \otimes Y^{\dagger} \qquad \mathcal{L} : {}^{\dagger}(Y \otimes X) \xrightarrow{\cong} {}^{\dagger}X \otimes {}^{\dagger}Y$$

<u>Def^n</u> A <u>pivotal structure</u> on a bicategory $\mathcal{B}$ with left and right adjoints for every 1-morphism is a natural family of 2-isomorphisms

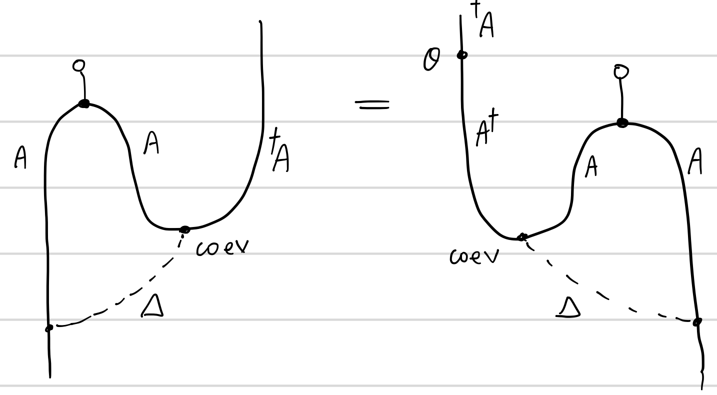
$$\{ \theta_X : X^+ \xrightarrow{\cong} {}^+X \}_{X \text{ a 1-morphism in } \mathcal{B}}$$

with the property that for every composable pair Y, X

$$\begin{array}{ccc} (Y \otimes X)^+ & \xrightarrow{\theta_{Y\otimes X}} & {}^+(Y \otimes X) \\ \downarrow \mathcal{R} & & \downarrow \mathcal{L} \\ X^+ \otimes Y^+ & \xrightarrow{\theta_X \otimes \theta_Y} & {}^+X \otimes {}^+Y \end{array}$$

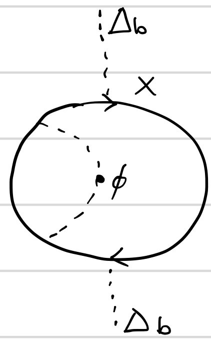
commutes.

<u>Def^n</u> A Frobenius algebra A in a pivotal monoidal category $\mathcal{C}$ (e.g. $\mathcal{B}(a,a)$) is <u>symmetric</u> if



Notation

From now on, $\mathcal{B}$ is a pivotal bicategory, i.e. it is equipped with a pivotal structure, and we assume $\mathcal{B}$ is $\mathbb{C}$-linear with all spaces $\mathrm{Hom}_{\mathcal{B}(a,b)}(X,Y)$ finite-dimensional. We denote the common left and right adjoint of X by $X^\vee \cong {}^+X \cong X^+$.
We further assume all idempotent 2-morphisms split. Call all these conditions ⊛



Def^n Given a 1-morphism $X : a \to b$ define the $\mathbb{C}$-linear map

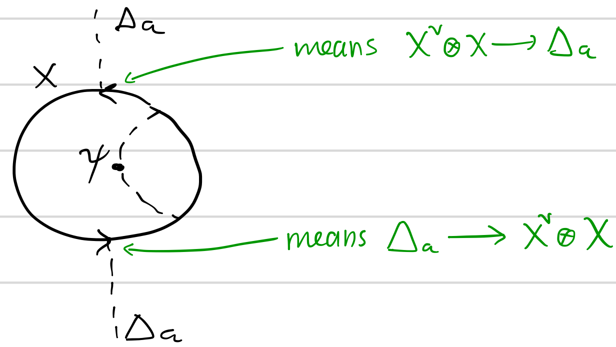
$$\mathcal{D}_r(X) : \mathrm{End}(\Delta_a) \to \mathrm{End}(\Delta_b), \text{ by}$$

$$\mathcal{D}_r(X)(\phi) := \Delta_b \xrightarrow{\mathrm{coev}} X \otimes X^\vee \xrightarrow{\cong} X \otimes \Delta_a \otimes X^\vee \xrightarrow{1 \otimes \phi \otimes 1} X \otimes \Delta_a \otimes X^\vee \xrightarrow{\cong} X \otimes X^\vee \xrightarrow{\mathrm{ev}} \Delta_b$$

Similarly we define

$$\mathcal{D}_\ell(X) : \mathrm{End}(\Delta_b) \to \mathrm{End}(\Delta_a),$$

$\mathcal{D}_\ell(X)(\psi) :=$



Lemma
(i) $\mathcal{D}_\ell(X) = \mathcal{D}_r(X^\vee)$, and vice versa.
(ii) $\mathcal{D}_\ell(Y \otimes X) = \mathcal{D}_\ell(Y) \circ \mathcal{D}_\ell(X)$, same for r.
(iii) $\mathcal{D}_\ell(\Delta_a) = \mathcal{D}_r(\Delta_a) = 1$.
(iv) In a triangulated setting $\mathcal{D}_\ell(X[1]) = -\mathcal{D}_\ell(X)$, same for r.

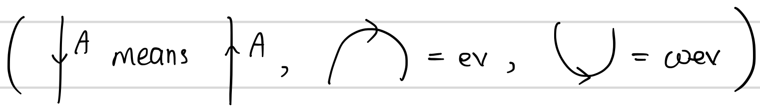
Def^n The <u>left and right quantum dimensions</u> of $X : a \to b$ are

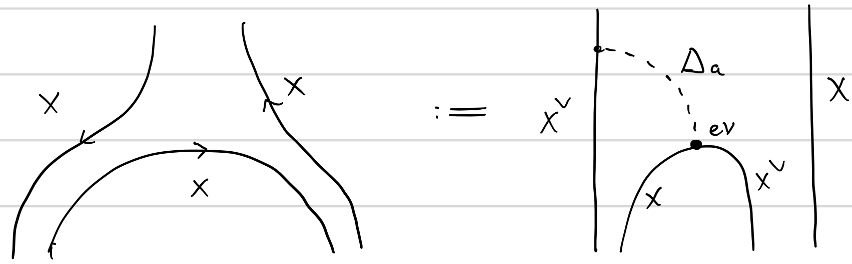
$$\dim_\ell(X) := \mathcal{D}_\ell(X)(1_{\Delta_b}) \in \mathrm{End}(\Delta_a)$$
$$\dim_r(X) := \mathcal{D}_r(X)(1_{\Delta_a}) \in \mathrm{End}(\Delta_b)$$

<u>Def^n</u> We say X has <u>invertible quantum dimensions</u> (IVD) if $\dim_r(X)$, $\dim_\ell(X)$ are nonzero scalar multiples of 1_{Δ_b}, 1_{Δ_a} resp. (note X IVD $\iff$ $X^\vee$ IVD).

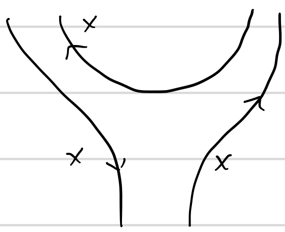
<u>Lemma</u> If X has invertible quantum dimensions, $A := X^\vee \otimes X$ is a separable symmetric Frobenius algebra in $\mathcal{B}(a,a)$.

<u>Proof</u> (Sketch) $\mu : A \otimes A \longrightarrow A$ is

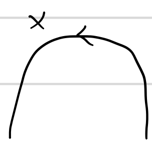




and comultiplication, counit are scaled:

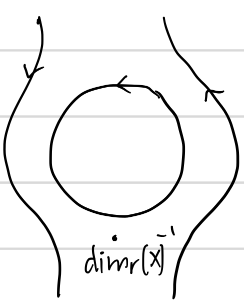
$\Delta :=$ 

$\cdot \dim_r(X)^{-1}$

$\varepsilon :=$ 

$\cdot \dim_r(X)$

To verify separability observe

$\mu \circ \Delta =$ 

$= \dim_r(X) \cdot \dim_r(X)^{-1} \cdot$ 

$= 1_A$. □

Tensor products

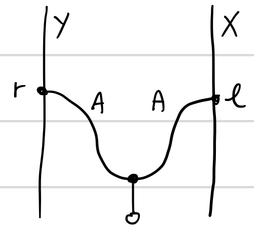
Let $A \in \mathcal{B}(b,b)$ be an algebra, and let $Y \in \mathcal{B}(b,c)$, $X \in \mathcal{B}(a,b)$ be A-modules (resp right and left), i.e. we have 2-morphisms $Y \otimes A \xrightarrow{r} Y$, $A \otimes X \xrightarrow{\ell} X$ satisfying the usual identities.

Lemma If A is separable Frobenius the pair

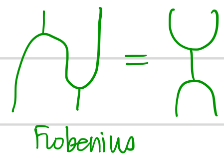
$$Y \otimes A \otimes X \underset{1 \otimes \ell}{\overset{r \otimes 1}{\rightrightarrows}} Y \otimes X \overset{\pi}{\dashrightarrow} Y \otimes_A X$$

has a coequaliser in $\mathcal{B}(a,c)$, denoted $(Y \otimes_A X, \pi)$.

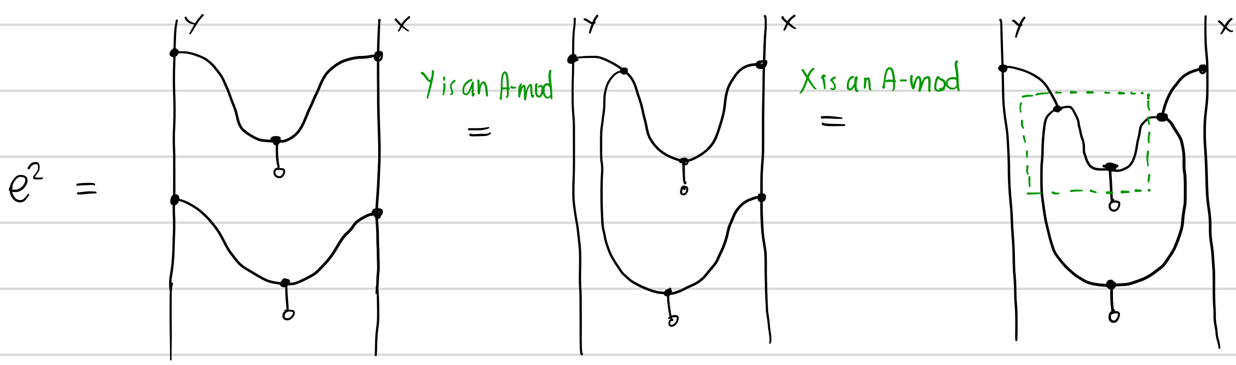
Proof We may define

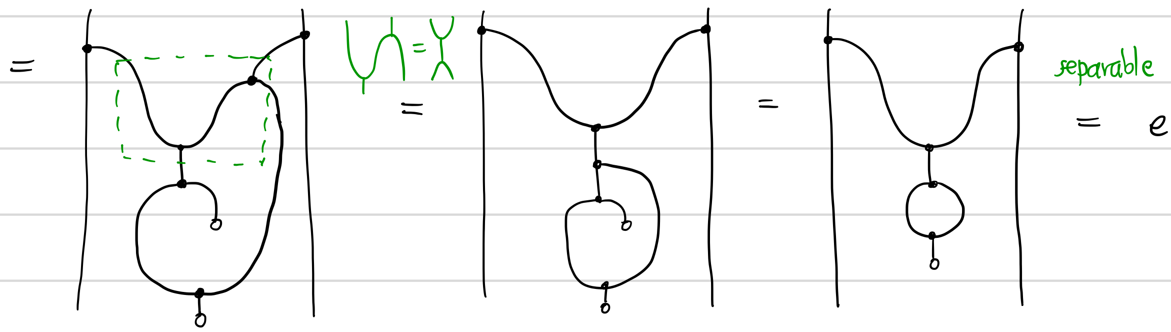
$e :=$ 

$\in \mathrm{End}_{\mathcal{B}(a,c)}(Y \otimes X)$

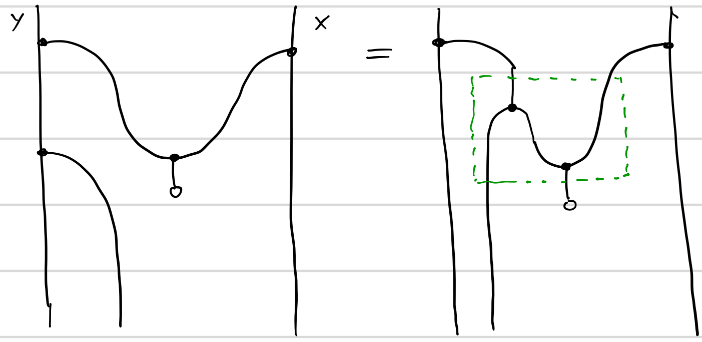


This is an idempotent:

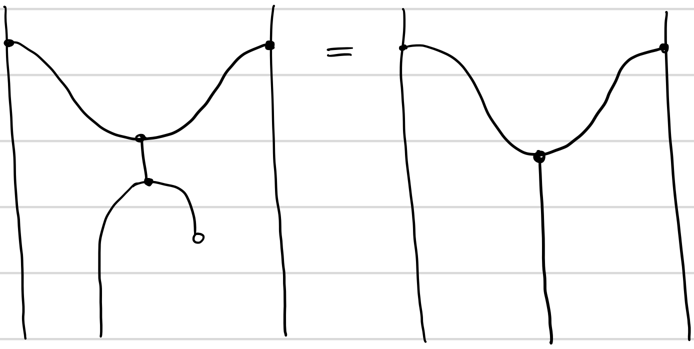




And $e\circ(r\otimes 1) =$



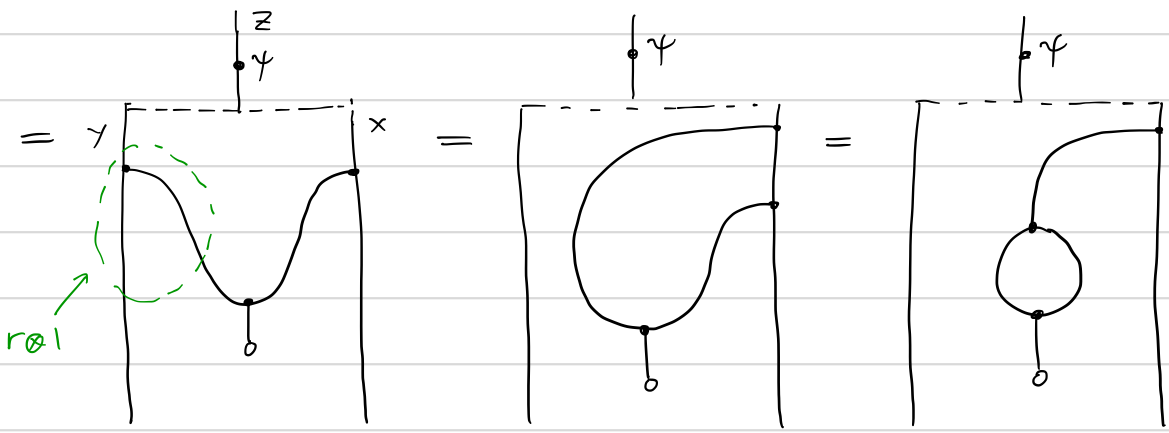
$=$



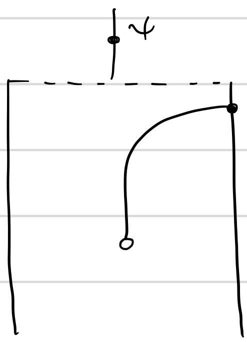
$= \cdots$
$= e\circ(1\otimes \ell)$

If $\psi : Y\otimes X \longrightarrow Z$ is such that $\psi\circ(r\otimes 1) = \psi\circ(1\otimes \ell)$ then we claim $\psi\circ(1-e) = 0$, so ψ factors uniquely via $Y\otimes X \overset{\pi}{\twoheadrightarrow} \mathrm{Im}(e) =: Y\otimes_A X$

$\psi\circ e =$



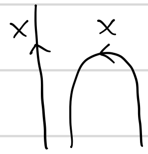
$=$

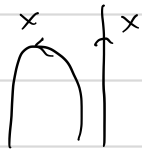


$= \psi$ as claimed. □

<u>Proposition</u> If $X : a \to b$ has IVD then with $A = X^\vee \otimes X$ we have $X \otimes_A X^\vee \cong \Delta_b$.

<u>Proof</u> The counit $X \otimes X^\vee \longrightarrow \Delta_b$ is the coequaliser of $r \otimes 1$, $1 \otimes \ell$, where

$r : X \otimes A \longrightarrow X$ is 

$\ell : A \otimes X \longrightarrow X$ is 

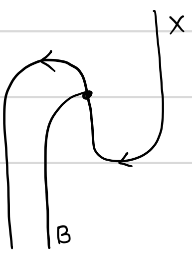
. □

<u>Def^n</u> The <u>equivariant completion</u> $\mathcal{B}_{eq}$ of $\mathcal{B}$ has

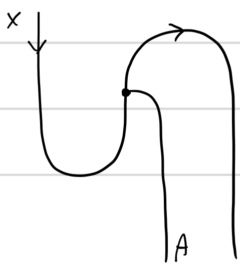
- <u>objects</u> (a, A) where $A \in \mathcal{B}(a,a)$ is a separable Frobenius algebra
- <u>1-morphisms</u> $(a,A) \longrightarrow (b,B)$ are B-A-bimodules
- <u>2-morphisms</u> are bimodule morphisms
- <u>composition</u> of $(a,A) \xrightarrow{X} (b,B) \xrightarrow{Y} (c,C)$ is $Y \otimes_B X$.
- <u>unit</u> at (a,A) is A as an A-A-bimodule.

<u>Theorem</u> (Carqueville-Runkel) $\mathcal{B}_{eq}$ is a bicategory which satisfies ⊛ from earlier, i.e. all 1-morphisms have adjoints and $\mathcal{B}_{eq}$ is pivotal.

<u>Proof</u> (Sketch) Let $X : (a,A) \longrightarrow (b,B)$. We claim $X^\vee : b \to a$ is an A-B-bimodule via the morphisms



and

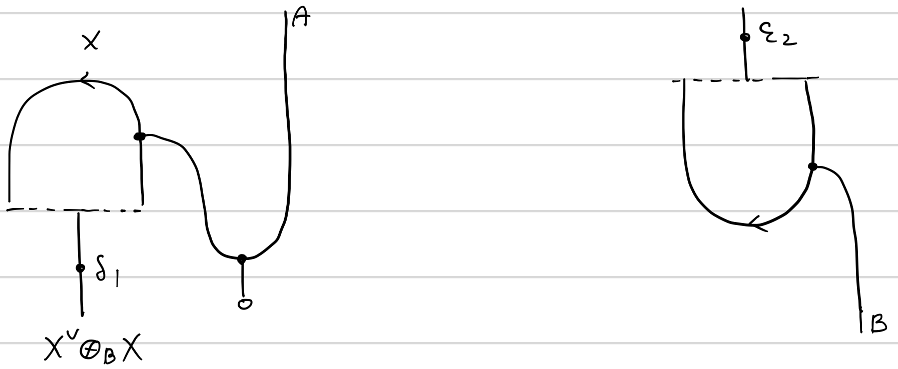


To show $X^\vee \dashv X \dashv X^\vee$ in $\mathcal{B}_{eq}$, let

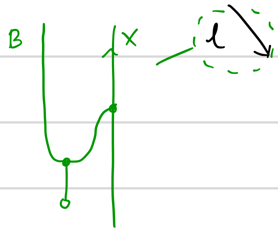
$$X^\vee \otimes X \underset{\delta}{\overset{\pi}{\rightleftarrows}} X^\vee \otimes_B X \qquad \pi\circ\delta = 1$$

$$X \otimes X^\vee \underset{\delta}{\overset{\pi}{\rightleftarrows}} X \otimes_A X^\vee \qquad \delta\circ\pi = e$$

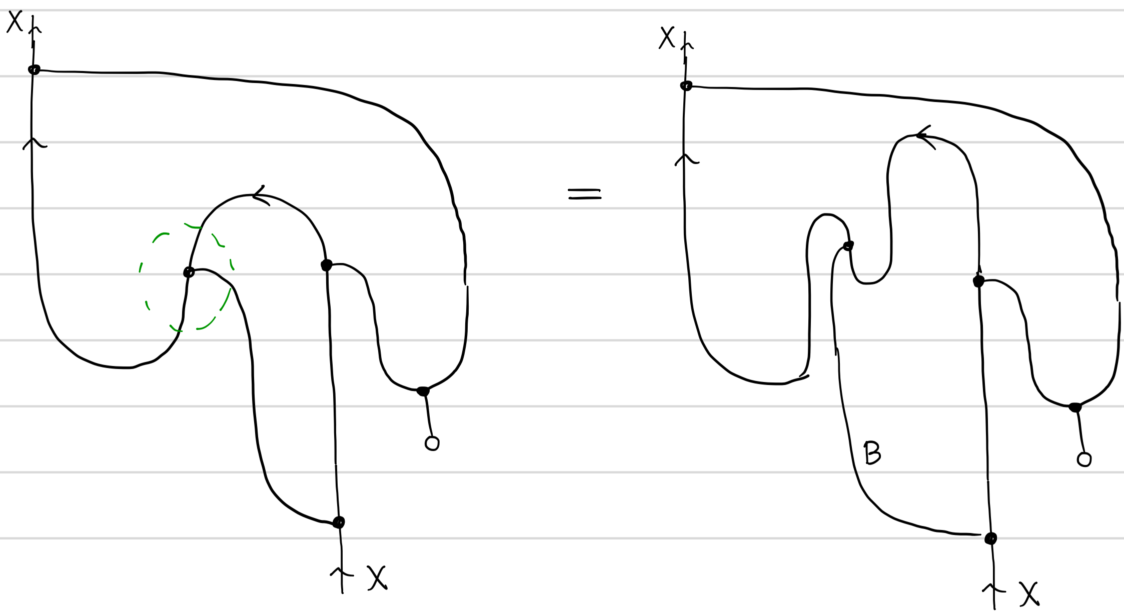
Define $X^\vee \otimes_B X \xrightarrow{\varepsilon} A$ and $B \xrightarrow{\eta} X \otimes_A X^\vee$ by

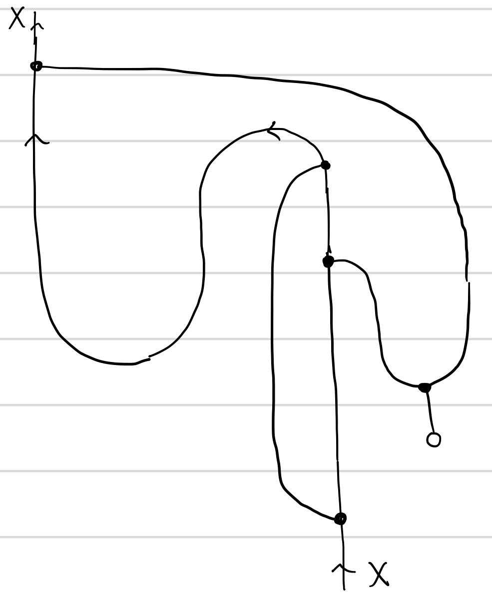


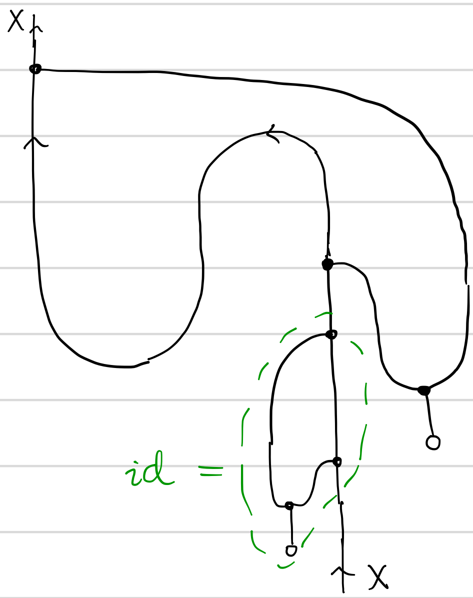
Then for example $X \cong B \otimes_B X \xrightarrow{\eta\otimes 1} X \otimes_A X^\vee \otimes_B X \xrightarrow{1\otimes\varepsilon} X \otimes_A A \cong X$ is

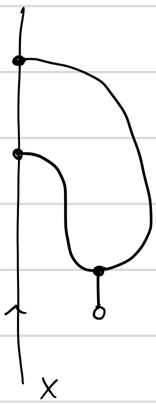


$$\begin{array}{ccccc}
B\otimes_B X & \xrightarrow{\eta\otimes 1} & X\otimes_A X^\vee \otimes_B X & \xrightarrow{1\otimes\varepsilon} & X\otimes_A A \cong X \\
\downarrow \delta & & \downarrow \delta\otimes 1 & & \downarrow \delta \quad \nearrow r \\
B\otimes X & \longrightarrow & X\otimes X^\vee \otimes_B X & \xrightarrow{1\otimes\varepsilon} & X\otimes A \\
 & & \downarrow 1\otimes\delta & \nearrow & \\
 & & X\otimes X^\vee \otimes X & &
\end{array}$$



$=$ 

$\overset{\text{bimod}}{=}$ 

$=$ 

$= \mathrm{id}_X \cdot \square$

<u>Lemma</u> There is an embedding $\mathcal{B} \hookrightarrow \mathcal{B}_{eq}$, $a \mapsto (a, \Delta_a)$.

<u>Proposition</u> If $X : a \to b$ has IVD in $\mathcal{B}$ then there is an isomorphism $(a, A) \cong (b, \Delta_b)$ in $\mathcal{B}_{eq}$ where $A = X^\vee \otimes X$.

This allows us to consistently rephrase everything about b as being about (a, A):

$$\mathcal{B}(c, b) \cong \mathcal{B}_{eq}(c, b) \cong \mathcal{B}_{eq}(c, (a, A)) = A\text{-modules in } \mathcal{B}(c, a)$$

$$\mathcal{B}(b, b) \cong \mathcal{B}_{eq}(b, b) \cong \mathcal{B}_{eq}((a, A), (a, A)) = A\text{-bimodules in } \mathcal{B}(a, a)$$