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7/9/16

A_∞-algebras from hypersurface singularities

The aim of this final lecture is to explain how to obtain "finite-dimensional" models of matrix factorisation categories, in the language of A_∞-algebras. As an example, and an attempt to tie together some of the themes of the workshop, I will propose an interpretation in this setting of the cocyclic object $F^\bullet$ of Toby's lectures in terms of semiuniversal deformations of A_n-singularities.

Heuristically we want a map

$$\left\{ \begin{array}{c} W : \mathbb{C}^n \to \mathbb{C} \text{ with isolated} \\ \text{singularities} \end{array} \right\} \longrightarrow \left\{ A_\infty\text{-algebras} \right\}$$

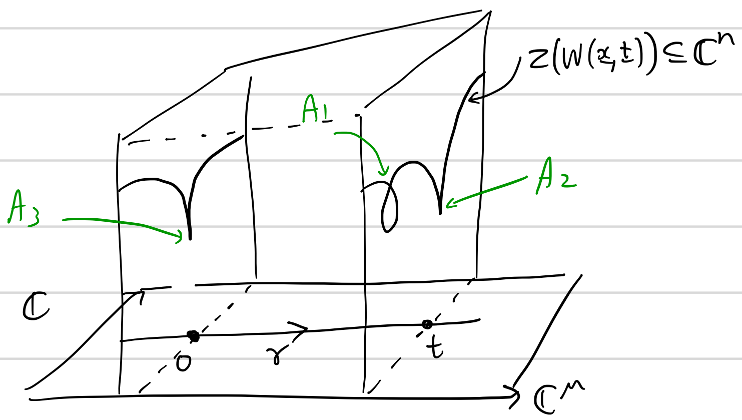
$$W \longmapsto \mathcal{A}_W$$

$\mathcal{A}_W$ ← f.d. vector space with higher operations

such that $\mathrm{mod}(\mathcal{A}_W) \cong \mathrm{hmf}(W)$, for some notion of modules, and

$$\left\{ \begin{array}{c} \text{family } W(\underline{x},\underline{t}) : \mathbb{C}^n \times \mathbb{C}^m \longrightarrow \mathbb{C} \\ \text{of isolated singularities} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{sheaf of } A_\infty\text{-algebras} \\ \text{on } \mathbb{C}^m \end{array} \right\}$$

The sheaf $\underline{t} \longmapsto \mathcal{A}_{W(\underline{x},\underline{t})}$ is too naive, but a small modification works. Then we can consider in e.g. the semiuniversal deformation of the A_3-singularity, a path with parameter t, and a sheaf of A_∞-categories on $\mathbb{C}[t]$ which is $\mathcal{A}_{A_3}$ over 0 and a mix of $\mathcal{A}_{A_1}, \mathcal{A}_{A_2}$ over a generic point. This tangent vector γ gives a A_∞ $\mathcal{A}_{A_1}$-$\mathcal{A}_{A_2}$-bimodule $B(\gamma)$.



1 A_∞-algebras

Let k be a commutative $\mathbb{Q}$-algebra, $\otimes = \otimes_k$

Def^N An A_∞-algebra over k is a $\mathbb{Z}$-graded f.g. projective k-module

$$A = \bigoplus_{n \in \mathbb{Z}} A_n$$

with operations $m_n : A^{\otimes n} \longrightarrow A$, $n \geq 1$, k-linear, degree $2-n$.

$$\begin{array}{ll} m_1 : A \longrightarrow A & \deg +1 \\ m_2 : A \otimes A \longrightarrow A & \deg 0 \\ m_3 : A^{\otimes 3} \longrightarrow A & \deg -1 \\ \vdots & \end{array}$$

$$\boxed{\begin{array}{c} A^{\otimes n} \\ \shortparallel \\ A^{\otimes r} \otimes A^{\otimes s} \otimes A^{\otimes t} \\ \downarrow 1 \otimes m_s \otimes 1 \\ A^{\otimes r} \otimes A \otimes A^{\otimes t} \\ \downarrow m_{r+1+t} \\ A \end{array}}$$

such that for $n \geq 1$

$$(*) \quad \sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t} (\mathbb{1}^{\otimes r} \otimes m_s \otimes \mathbb{1}^{\otimes t}) = 0$$

Def^N A morphism $f : A \to B$ is $f_n : A^{\otimes n} \longrightarrow B$ s.t. $m_1 f_1 = f_1 m_1, \ldots$

Example $m_n = 0$, $n \geq 3$, $(*)$ says (A, m_1, m_2) satisfies

write $ab = m_2(a \otimes b)$

$(n=1)$ $m_1^2 = 0$

$(n=2)$ $m_1(ab) = m_1(a)b + (-1)^{|a|} a\, m_1(b)$

$(n=3)$ m_2 is associative.

$\therefore (A, m_1, m_2)$ is a DG-algebra

⌜strict unit is $e \in A^0$, $m_1(e) = 0$, e a unit for m_2 and m_n vanishes for $n > 2$ as soon as any entry is e.

homological unit is a unit for H^*A, we say A is h-unital⌟

Def^n A is minimal if $m_1 = 0$.

Example For $d > 2$, $|\varepsilon| = 1$, $A^{(d)} = k[\varepsilon]/\varepsilon^2 = k \oplus k\varepsilon$

$$\left.\begin{array}{l} m_n = 0 \text{ for } n \notin \{2, d\} \\ m_2 = \text{multiplication} \\ m_d(\varepsilon \otimes \cdots \otimes \varepsilon) = (-1)^{d-1} \cdot 1 \end{array}\right\} A^{(d)} \text{ is a } \mathbb{Z}_2\text{-graded } A_\infty\text{-algebra}$$

Where do A_∞-algebras come from?

$\mathcal{T}$	triangulated category of interest, e.g. $D^b(\mathrm{coh}\,X)$, $\mathrm{hmf}(W)$, ...
$\uparrow H^0$	
$\mathcal{D}$	DG enhancement of $\mathcal{T}$, e.g. by K-injective res.
$\rotatebox{90}{$\in$}$	
E	Generator (all objects "built from" E)
$\downarrow$	
$\mathrm{End}_{\mathcal{D}}(E)$	DG-algebra (usually ∞-dim$/\mathbb{C}$)
$\downarrow$ taking the minimal model	
$(H^*\mathrm{End}_{\mathcal{D}}(E), \{m_n\}_{n \geq 2})$	A_∞-algebra quasi-iso to $\mathrm{End}_{\mathcal{D}}(E)$. (f.dim$/\mathbb{C}$) Knows "everything" about $\mathcal{T}$.

Why find minimal models?

- To understand dependence of categories on moduli.
- Topological string theory (boundary sector)
  = minimal, cyclic strictly unital A_∞-categories
  (Herbst–Lazaroiu–Lerche, Costello)

A_∞-modules An A_∞-module over an A_∞-algebra $(A, \{m_n\}_{n\geq 1})$ is a $\mathbb{Z}$-graded f.g. proj k-module M with operations $(n\geq 1)$

$$m_n^M : A^{\otimes(n-1)} \otimes M \longrightarrow M$$

of degree $2-n$ satisfying the same identities ⊛. A <u>morphism</u> of A_∞-modules $\varphi: M \longrightarrow N$ is a collection of linear maps $\varphi_n : A^{\otimes(n-1)} \otimes M \longrightarrow N$ of degree $1-n$ such that

$$(u=r+1+t) \qquad \sum_{r+s+t=n} \pm\, \varphi_u(\mathbb{1}^{\otimes r} \otimes m_s \otimes \mathbb{1}^{\otimes t}) = \sum_{r+s=n} \pm\, m_u^N(\mathbb{1}^{\otimes r} \otimes \varphi_s)$$

this is an eq. of maps $A^{\otimes(n-1)} \otimes M \longrightarrow N$

The (ordinary) category of A_∞-modules and these morphisms is denoted $\mathrm{Mod}A$ (note H^*M is a H^*A-module).

says $\varphi_1 m_1 = m_1 \varphi_1$ and φ_1 commutes with the action of A "up to htpy", etc...

The derived category A a h-unital A_∞-algebra.

- There is an A_∞-category of (h-unital) A_∞-modules $\mathrm{Mod}_\infty(A)$, such that $\mathrm{Mod}A = Z^0(\mathrm{Mod}_\infty A)$.

  This is a triangulated A_∞-cat (see Seidel [S])

  More concretely, $\hom^a(M,N)$ is the space of $\{t^n\}_{n\geq 1}$ with each

  $$t^n : A^{\otimes(n-1)} \otimes M \longrightarrow N \qquad (\text{of degree } a-n+1)$$

  and only m_1, m_2 are nonzero in $\mathrm{Mod}_\infty(A)$ (i.e. this is a <u>DG-category</u>).

- The <u>perfect derived category</u> $\mathrm{per}(A)$ is the smallest triangulated subcategory of $H^0(\mathrm{Mod}_\infty A) = \mathrm{Mod}A/\sim$ containing A.

<u>Example</u> $A = A^{(d)}$ from above, $d>2$ (i.e. $m_n = 0$ $n \notin \{2,d\}$).

Given $2 \leq i \leq d-2$, $i < d-i$ we define an A_∞-module over $A^{(d)}$ by

$\mathbb{Z}_2$-graded

$$M_{(i)} := \Lambda(k\xi) = k \oplus k\xi$$

with operations $\alpha_n = 0$ unless $n \in \{2, i+1, d-i+1\}$

$$\alpha_n : A^{\otimes(n-1)} \otimes M_{(i)} \longrightarrow M_{(i)}$$

$$\alpha_2(1,-) = id,$$
$$\alpha_{i+1}(\varepsilon,\varepsilon,\dots,\varepsilon,-) = \pm \xi^* \lrcorner (-) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$\alpha_{d-i+1}(\varepsilon,\varepsilon,\dots,\varepsilon,-) = \pm \xi \wedge (-) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The minimal model theorem. $\left(\overset{\infty.d.}{A} \underset{qis}{\cong} \overset{f.d.}{B} \right)$

Let (A, ∂, m) be a DG-algebra (suspended forward product).
i.e. s.t. $(A, \{\partial, m\})$ sat. $(*)$

A <u>strict homotopy retraction</u> of A is a $\mathbb{Z}$-graded f.g. projective k-module B and linear maps

$$H \circlearrowright A \underset{i}{\overset{p}{\rightleftarrows}} B$$

such that

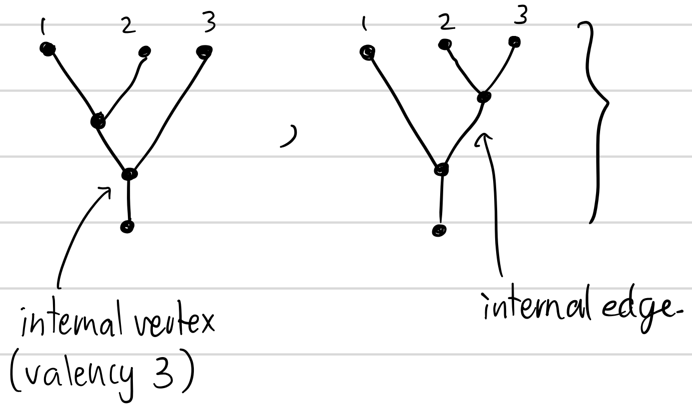
(i) p, i are degree zero morphisms of cpxs
(where B is given zero differential).

(ii) $p \circ i = 1_B$

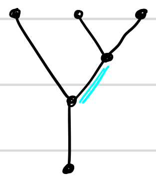
(iii) $1_A - i \circ p = H\partial + \partial H$ (i.e. $i \circ p \simeq 1_A$)

$\Rightarrow$ $B \cong H^*(A, \partial)$, with a particular choice of how to project elements in A onto cocycles ($\partial i p(a) = i \partial p(a) = 0$).

$\mathcal{T}_n = \{$ oriented and connected planar trees, with $n+1$ leaves $\}$
inc. root

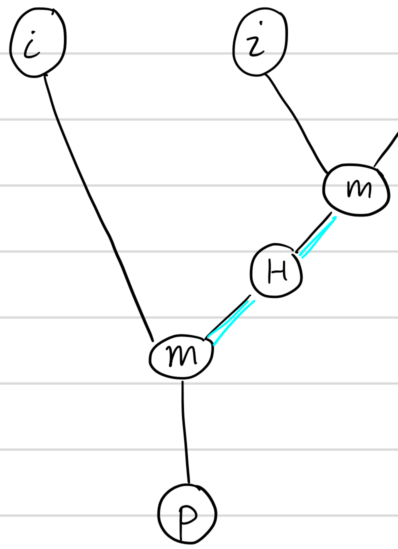
i.e. $\mathcal{T}_3 = \{$  $\}$

<u>Def^N</u> Given $T \in \mathcal{J}_n$ we define $\rho_T : B^{\otimes n} \longrightarrow B$ by example:



T

$\rightsquigarrow$



$$\begin{array}{c} B\otimes B\otimes B \\ \downarrow {\scriptstyle 1\otimes i\otimes i} \\ B\otimes A\otimes A \\ \downarrow {\scriptstyle 1\otimes m} \\ B\otimes A \\ \downarrow {\scriptstyle i\otimes H} \\ A\otimes A \\ \downarrow {\scriptstyle m} \\ A \xrightarrow[p]{} B \end{array}$$

$$\rho_T = (-1)^{\#\text{int. edges}} \, p \circ m \circ (i \otimes H) \circ (1_B \otimes m) \circ (1_B \otimes i \otimes i)$$

$$\boxed{\rho_n := \sum_{T \in \mathcal{J}_n} \rho_T \; : \; B[1]^{\otimes n} \longrightarrow B[1]}$$

<u>Theorem</u> (Minimal model) $(B, \{\rho_n\}_{n\geq 2})$ is an A_∞-algebra (with suspended forward products) and there is an A_∞-quasi-isomorphism

$$(A, m, \partial) \longrightarrow (B, \{\rho_n\}_{n \geq 2})$$

called the <u>minimal model</u>
(recall $B \cong H^*A$)

2 Singularities

<u>Def^n</u> $W \in k[x_1, \dots, x_n]$ is a <u>potential</u> (over k) if (with $f_i = \partial_{x_i} W$)

(i) $f_1, \dots, f_n$ is a quasi-regular sequence
(ii) $k[\underline{x}]/(f_1, \dots, f_n)$ is a f.g. projective k-module
(iii) The Koszul complex of $f_1, \dots, f_n$ is exact except in degree 0.

<u>Example</u> (1) $k = \mathbb{C}$, all critical pts isolated
(2) Consider $k = \mathbb{C}[t]$, $W(x,y,t) = x^2 + y^3 - 3t^2 y + 2t^3 \in k[x,y]$ is the semi-universal deformation of the cusp, restricted to the discriminant. Observe $\partial_x W = 2x$, $\partial_y W = 3y^2 - 3t^2$ so

$$k[x,y]/(\partial_x W, \partial_y W) = \mathbb{C}[t,x,y]/(x, y^2 - t^2) \cong \mathbb{C}[t] \oplus \mathbb{C}[t]y$$

so W is a potential over k.

(3) The usual theory of LG models (at least anything which can be encoded into the bicategories $\mathcal{LG}$, $\mathcal{LG}^{gr}$ of the previous lectures) works for these "relative" potentials.

<u>Want</u> Potential $W \rightsquigarrow$ DG category $mf(W) \xrightarrow{\text{min. model}} A_\infty$-category $/ k$

But usually this A_∞-category is constructed by taking cohomology, which is terrible if k is not a field. So we do something different.

Let $W \in k[x_1, \dots, x_n]$ be a potential and $X \in mf(k[\underline{x}], W)$. Then

$$\mathrm{End}(X) := \left(\mathrm{Hom}_{k[\underline{x}]}(X, X),\ d_{\mathrm{Hom}}(\alpha) = d_X \alpha - (-1)^{|\alpha|} \alpha d_X \right)$$

is a DG-algebra. Write $i : k[\underline{x}] \longrightarrow k[\underline{x}]/(f_1, \dots, f_n)$ for the projection (recall $f_i = \partial_{x_i} W$, or in fact any other sequence with properties (i)-(iii) s.t. each f_i acts null-homotopically on $\mathrm{End}(X)$). We further write

$$S = \bigwedge (k\theta_1 \oplus \cdots \oplus k\theta_n). \qquad |\theta_i| = 1$$

<u>Theorem</u> (Dyckerhoff-M '09, M'15) There is a strict homotopy retract of $\mathbb{Z}_2$-graded complexes / k

$$H \circlearrowright S \otimes_k \mathrm{End}(X) \underset{i}{\overset{p}{\rightleftarrows}} i^* \mathrm{End}(X)$$

H: defined in terms of a k-linear connection $k[\underline{x}] \xrightarrow{\nabla} k[\underline{x}] \otimes_{k[\underline{f}]} \Omega^1_{k[\underline{f}]/k}$

$i^*\mathrm{End}(X)$: cpx of f.g. proj k-modules

<u>Remarks</u>

- The minimal model theorem is useful precisely to the extent that you have a good homotopy. The above H is good.

- Get a (possibly non-min) A_∞-algebra $(i^*\mathrm{End}(X), \{m_n\}_{n \geq 2})$ quasi-iso to $S \otimes_k \mathrm{End}(X)$, together with a Clifford action which picks out a subalgebra qis to $\mathrm{End}(X)$.

- In many cases, can promote this to a <u>minimal</u> model of a sub-DG-category of $S \otimes_k mf(W)$.

3 Calculations

$W \in k[x_1,\dots,x_n]$ a potential. For $P \in \mathrm{Sing}(W)$,

$$k(P)^{stab} := \left(k[\underline{x}] \otimes_k \Lambda(k\psi_1 \oplus \cdots \oplus k\psi_n),\ \sum_{i=1}^n (x_i - P_i)\psi_i^* + \sum_{i=1}^n W_P^i \psi_i \right)$$

where we choose $W = \sum_{i=1}^n (x_i - P_i) W_P^i$, some $W_P^i \in \mathfrak{m}_P^2$. In the case W has local quadratic terms there is a simple modification to the following. For the following take $P=0$, and write

$$\mathcal{A}_W := \text{minimal model of } \mathrm{End}(k^{stab})$$

Def^N The underlying algebra of $\mathcal{A}_W$ is

$$A_W = \Lambda(k\psi_1 \oplus \cdots \oplus k\psi_n) \qquad |\psi_i| = 1$$

To define $m_r : A_W^{\otimes n} \longrightarrow A_W$ we introduce an auxiliary space

$$\mathcal{H} := A_W \otimes \Lambda(k\theta_1 \oplus \cdots \oplus k\theta_n) \otimes k[\underline{x}]$$

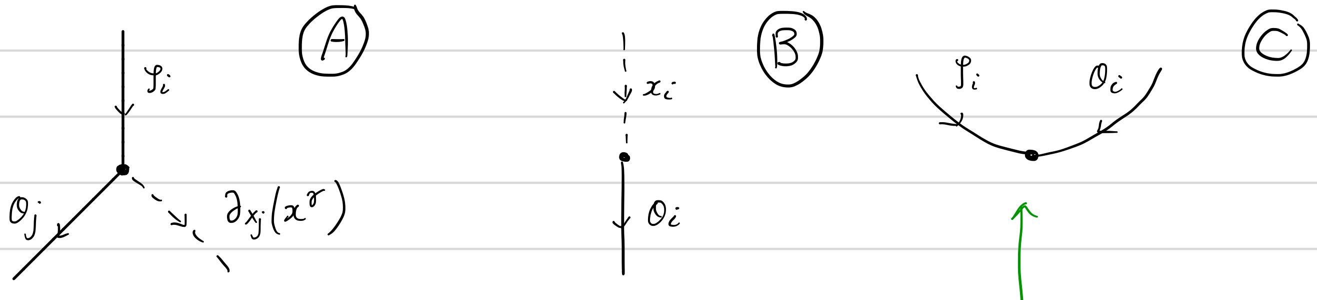
write m for the product in $\mathcal{H}$

$$\begin{tikzcd} \mathcal{H} \arrow[r, bend left, "p = \text{projection } (\theta, x \mapsto 0)"] & A_W \arrow[l, bend left, "i = \text{embedding}"] \end{tikzcd}$$

Standard operations

$$\mathcal{H} \circlearrowleft \underbrace{\overset{\text{wedge}}{\psi_i}, \overset{\text{contraction}}{\psi_i^*}, \theta_i, \theta_i^*}_{\text{fermion creation/annihilation}}, \underbrace{x_i, \partial_{x_i}}_{\text{boson creation/annihilation}}$$

<u>Interactions</u> $W = \sum_i x_i W^i \qquad W^i = \sum_{\gamma \in \mathbb{N}^m} W^i(\gamma) x^\gamma \qquad W^i(\gamma) \in k$



$-\frac{1}{|\gamma|} W^i(\gamma)$

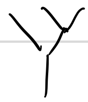
(for all i,j and $\gamma \in \mathbb{N}^m$)

$$-\frac{1}{|\gamma|} W^i(\gamma)\, \theta_j\, \partial_{x_j}(x^\gamma)\, \varphi_i^* \in \mathcal{H} \qquad \theta_i \partial_{x_i} \in \mathcal{H} \qquad \varphi_i^* \otimes \theta_i^* \in \mathcal{H}^{\otimes 2}$$

The Feynman calculus now describes the structure constants of the m_r's, for $\gamma_1, \ldots, \gamma_r, \delta \in A_W$ (products of Ψ's) by the formula

$$m_r(\gamma_1 \otimes \cdots \otimes \gamma_r)_\delta = \sum_{\substack{\text{binary} \\ \text{trees} \\ T}} \ \sum_{\substack{\text{Feynman} \\ \text{diagrams } D, \\ \gamma \text{ incoming} \\ \delta \text{ outgoing}}} \text{amplitude}(D)$$

where the amplitude is an element of k defined by

"Def^N" A <u>Feynman diagram</u> D for a binary tree T (e.g. )
is an oriented graph embedded in the thickening of T, with <u>lines</u> labelled φ_i, θ_i, x_i $1 \leq i \leq m$ and <u>nodes</u> of type A, B, C, with the following constraints:

- A nodes may only occur along <u>edges</u> (not adjacent to root)
- B nodes " " " at internal vertices (i.e. )
- There is <u>precisely one</u> C node on every internal edge (and no other C nodes)
- The only lines incident at the boundary (of T) are φ-lines.

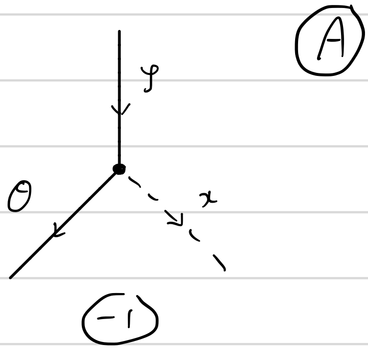
The <u>amplitude</u> of D is

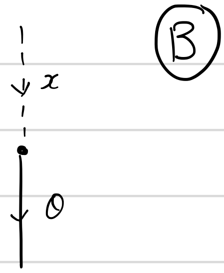
$$\text{amplitude}(D) = (-1)^{f(D)} T_D \prod_{A \text{ nodes}} \Bigg(\underbrace{-\frac{\gamma_j}{|\gamma|} W^i(\gamma)}_{j,\, i,\, \gamma \text{ depend on the node}} \Bigg)$$

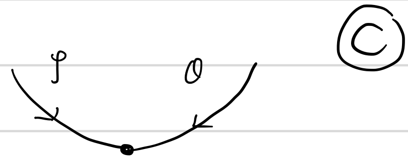
symmetry factor ($\in \mathbb{Q}$)

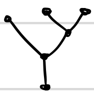
Example In the special case $W = x^3 = x\cdot x^2$, so $W' = x^2$

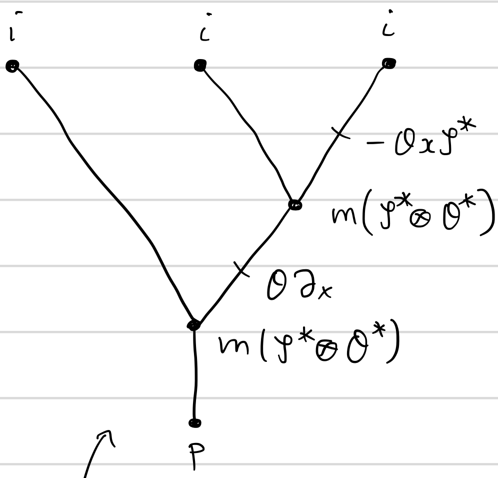
$$A_W = \Lambda(k\psi) = k \oplus k\psi \qquad \mathcal{H} = \Lambda(k\psi)\otimes\Lambda(k\theta)\otimes k[x]$$

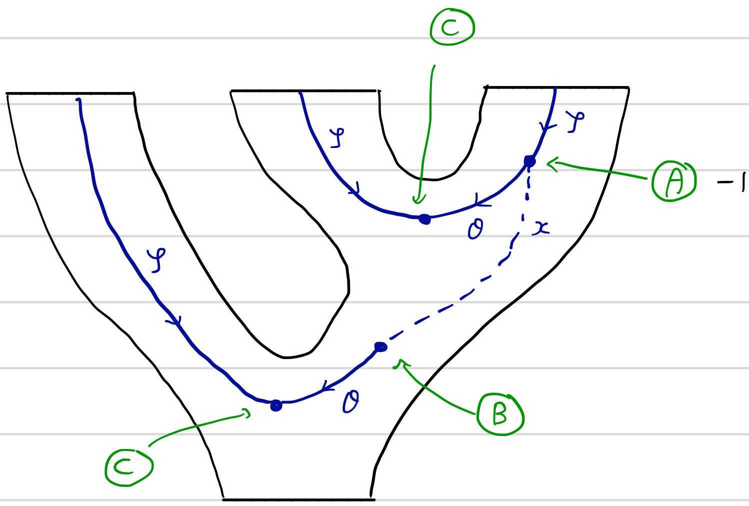






A Feynman diagram for T=  is:





in $= \psi, \psi, \psi$, out $= 1$

amplitude $= -1$

denotes the linear map $A_W^{\otimes 3} \to A_W$ defined by

$$p\, m(\psi^*\otimes\theta^*)\Big(i(-)\otimes \theta\partial_x m(\psi^*\otimes\theta^*)\big(i(-)\otimes(-\theta x\psi^*)i(-)\big)\Big)$$

$$\psi\otimes\psi\otimes\psi \longmapsto -1$$

In fact this is the only nontrivial Feynman diagram, so $A_W = \Lambda(k\psi)$ has $m_2 =$ usual product, $m_3(\psi\otimes\psi\otimes\psi) = -1$ otherwise zero, $m_n = 0$ $n \notin \{2,3\}$.

<u>Lemma</u> For $d>2$, $\mathcal{C}_{x^d} = A^{(d)}$ defined earlier (i.e. $m_n = 0$ $n \notin \{2,d\}$).

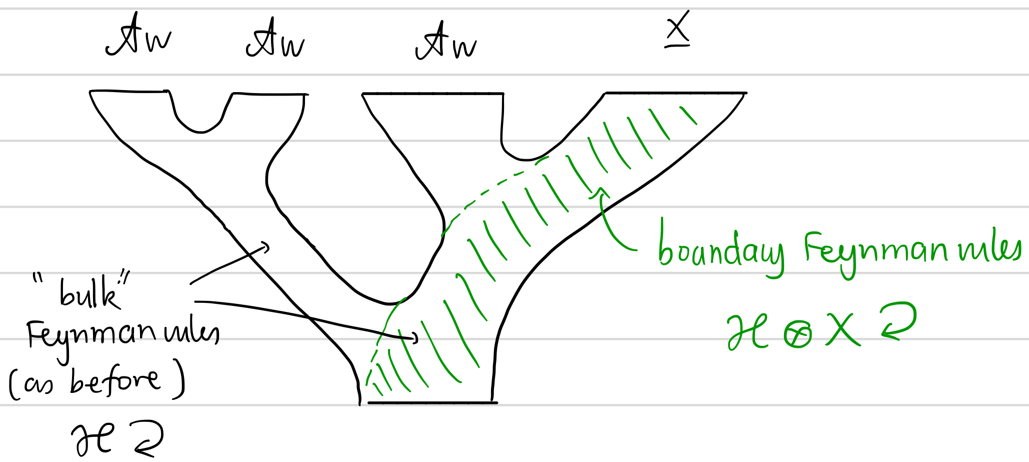
<u>minimal models for MFs</u>:

Q/ What is the A_∞-module corresponding to $X \in hmf(W)$?
Assume for simplicity that $d_X(X) \subseteq m^2 X$, then the underlying v.space is

$$\underline{X} := X \otimes_{k[x]} k[n]$$

$$\alpha_n : \mathcal{A}_W^{\otimes(n-1)} \otimes \underline{X} \longrightarrow \underline{X}$$

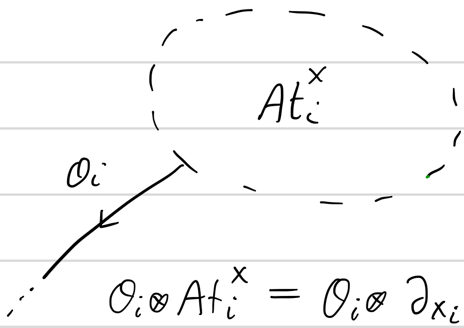
is computed by Feynman rules on diagrams of operators on $\mathcal{H} \otimes_{k[\underline{x}]} X$, e.g.



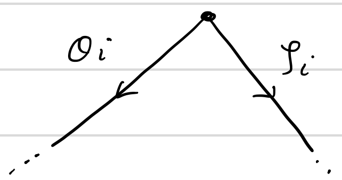
<u>Boundary Feynman rules</u> (in addition to Ⓐ, Ⓑ, Ⓒ)

Ⓓ, Ⓔ vertices allowed on any edge

Ⓓ



Ⓔ



$$\mathcal{O}_i \otimes At_i^X = \mathcal{O}_i \otimes \partial_{x_i}(d_X) \circlearrowleft \mathcal{H} \otimes X$$

"Atiyah class"

Note that the Atiyah classes completely determine the A_∞-module structure.

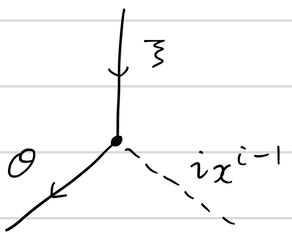
<u>Example</u> Consider $W = x^d$, $d \geq 3$ and $A_W = (\Lambda(k\rho), m_2, m_d)$

$$X = (\Lambda(k\xi), x^i\xi^* + x^{d-i}\xi) \qquad |\xi| = 1$$

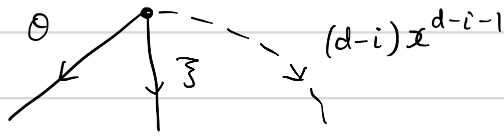
and assume $2 \leq i \leq d-2$. Then

$$\underline{X} = (k \oplus k\xi)[1] \quad \text{and} \quad \partial_x(dx) = i x^{i-1}\xi^* + (d-i)x^{d-i-1}\xi$$

Hence the "Atiyah" interaction is actually two interactions:

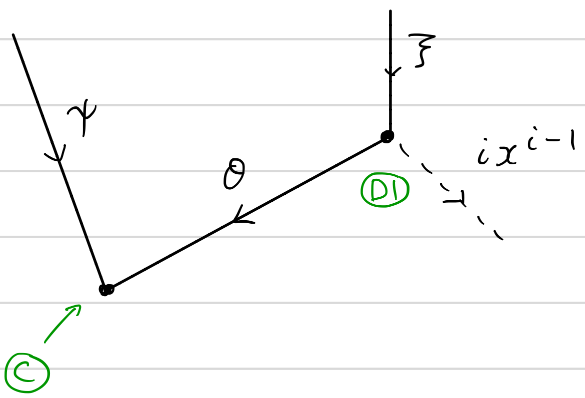


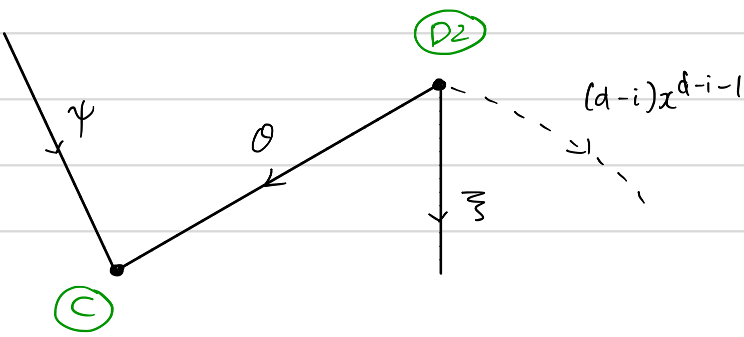
D1



D2

Since $\underline{X}$ is an A_∞-module over $A_W = \Lambda(k\rho)$ we want to know how ρ "acts" on ξ. The only interactions are the ones mediated by a θ:





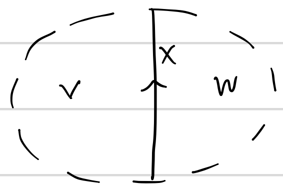
<u>Lemma</u> The A_∞-module $\underline{X}$ for $X = \begin{pmatrix} 0 & x^i \\ x^{d-i} & 0 \end{pmatrix}$ is $M_{(i)}$ from earlier.

<u>Proof</u> (11.1) gives rise to the operation $\varphi^{\otimes i+1} \longmapsto \xi^*$, (11.2) to $\varphi^{\otimes d-i+1} \mapsto \xi$. □

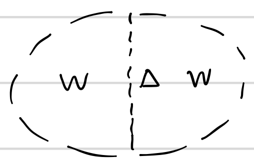
4) From flows to A_∞-bimodules

As an application of the above, we propose an implementation of the following idea of Brunner-Roggenkamp "Defects and bulk perturbations of LG models" '08.

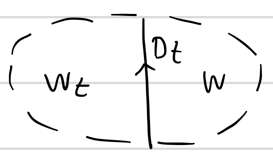
① 1-morphisms $W \xrightarrow{X} V$ are defect conditions



② A deformation of W should be implemented by a defect D_t



deform left W $\rightsquigarrow$



$\Delta : W \longrightarrow W$ $\qquad$ $D_t : W \longrightarrow W_t$

<u>Example</u> The semiuniversal unfolding of the A_3-singularity is $(k = \mathbb{C}[a,b,c])$

$$W(x,y,a,b,c) = x^4 + y^2 + ax^2 + bx + c \in k[x,y]$$

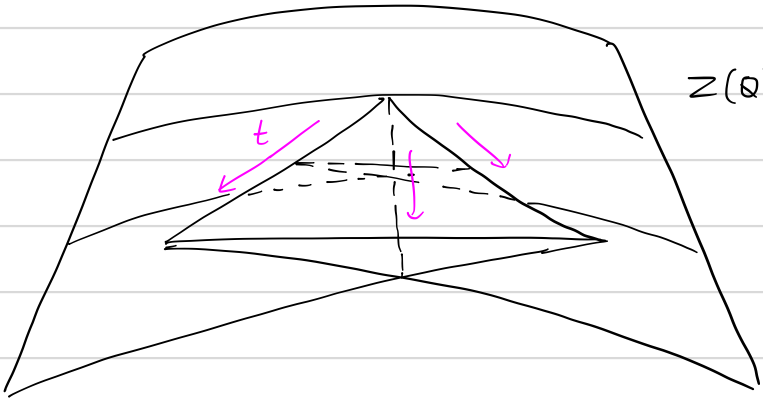
which is a potential / k. Consider

$$\begin{tikzcd} \mathrm{Sing}(W) \arrow[r, phantom, "\subseteq"] \arrow[dr, "\pi"'] & \mathrm{Spec}(k[x,y]) \arrow[d] \\ & \mathrm{Spec}\, k \end{tikzcd}$$

$\mathrm{Im}(\pi)$ is the discriminant, with equation

$$Q = 256c^3 - 27b^4 + 144ab^2c - 128a^2c^2 - 4a^3b^2 + 16a^4c$$

which is the swallowtail surface.



Let $i : k[x,y] \longrightarrow k[x,y]/(\partial_x W, \partial_y W)$ be as above, and choose a parametrised pair of points P_t, Q_t in the fiber $\pi^{-1}(t)$ with P_t an A_2-singularity and Q_t an A_1-singularity. Taking the fiber product with $\mathbb{C}[t]$ we may take

$$k(P_t)^{stab},\ k(Q_t)^{stab} \in mf(\mathbb{C}[x,y,t], W_t)$$

and look at the DG-category consisting of these two objects and their two mapping complexes. Call this $\mathcal{C}$. For $t=0$, $P_t = Q_t$ is an A_3-singularity.

The above allows us to compute a minimal A_∞-category structure on the f.g. projective $\mathbb{C}[t]$-module $\left(i_t : \mathbb{C}[x,y,t] \longrightarrow \mathbb{C}[x,y,t]/(\partial_x W_t, \partial_y W_t)\right)$

$$i_t^* \mathcal{C} = \left\{ i_t^* k(P_t)^{stab} \rightleftarrows i_t^* k(Q_t)^{stab} \right\}$$

i.e. a vector bundle of A_∞-algebras and bimodules on $\mathbb{A}^1 = \mathrm{Spec}(\mathbb{C}[t])$, which at a generic point gives an $\mathcal{A}_{A_2}$-$\mathcal{A}_{A_1}$-bimodule.

It is natural to guess the three cuspidal edges in the swallowtail represent three different A_∞-bimodules, from

$$\mathcal{A}_{A_1} \leftleftarrows\!\!\leftarrow \mathcal{A}_{A_2}$$

$$\begin{array}{ccc} \mathrm{per}(\mathcal{A}_{A_1}) & & \mathrm{per}(\mathcal{A}_{A_2}) \\ \wr\Vert & & \wr\Vert \\ \mathrm{hmf}(z^2+y^2) & \leftleftarrows\!\!\leftarrow & \mathrm{hmf}(z^3+y^2) \end{array}$$

and that this recovers geometrically the face maps in the Dyckerhoff-Kapranov cocyclic object F° in $\mathcal{C}at^{(2)}_{dg}$.

Appendix

Example $W = y^3 - x^3$, $\mathcal{C}_W = \Lambda(k\varphi_1 \oplus k\varphi_2)$, using forward suspended products $\rho_n : \mathcal{C}_W[1]^{\otimes n} \longrightarrow \mathcal{C}_W[1]$, only $\rho_2, \rho_3, \rho_4, \rho_6$ are nonzero, and for $\Lambda_1, \dots, \Lambda_6 \in \mathcal{C}_W$

$$\rho_6(\varphi_1\varphi_2 \otimes \cdots \otimes \varphi_1\varphi_2) = \frac{1}{4} \qquad \text{(only nonzero value)}$$

$$\rho_3(\Lambda_1 \otimes \Lambda_2 \otimes \Lambda_3) = \pm\Big(\varphi_2^*(\Lambda_1)\varphi_2^*(\Lambda_2)\varphi_2^*(\Lambda_3) - \varphi_1^*(\Lambda_1)\varphi_1^*(\Lambda_2)\varphi_1^*(\Lambda_3) \Big)$$

$$\rho_4(\Lambda_1 \otimes \cdots \otimes \Lambda_4) = \pm \tfrac{1}{2} \varphi_2^*(\Lambda_1) \cdot \varphi_2^*\varphi_1^*(\Lambda_2) \cdot \varphi_1^*(\Lambda_3) \cdot \varphi_2^*\varphi_1^*(\Lambda_4) + \cdots$$

Symmetry factor x an internal edge

$$\omega(x) = \sum_{y<x} \sum_{j \in J(y)} |\gamma_j(y)| - \sum_{z<x} m(z)$$

(y int. edge or inputs) (z = int. vertices)

$$F(x) = \frac{1}{\omega(x)} C^{un}_{\omega(x)}\left(\{ |\gamma_j(x)| \}_{j \in J(x)} \right)$$

$$C^{un}_\alpha(\ell_1, \dots, \ell_r) = \ell_1 \cdots \ell_r \sum_{\sigma \in S_r} \frac{1}{\alpha + \ell_{\sigma(r)}} \frac{1}{\alpha + \ell_{\sigma(r)} + \ell_{\sigma(r-1)}} \cdots \frac{1}{\alpha + \ell_1 + \cdots + \ell_r}$$

Cyclic A D-cyclic structure on A_∞-cat is $\langle\ \rangle_{ab} : \mathrm{Hom}(a,b) \otimes \mathrm{Hom}(b,a) \to \mathbb{C}[-D]$, $\langle u \otimes v \rangle = (-1)^{|u||v|} \langle v \otimes u \rangle$ and $\langle x_0 \otimes r(x_1 \otimes \cdots \otimes x_n) \rangle = \pm \langle x_1 \otimes r(x_2 \otimes \cdots \otimes x_n \otimes x_0) \rangle$.

(see [L] §3)

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Backgwound

Let us compute the Hochschild cohomology of $k[\varepsilon]/\varepsilon^2$ for a commutative ring k, following lipvri. The Bar complex is $(R = k[\varepsilon]/\varepsilon^2,\ A = k)$

$$\mathbb{B}_n = R^e \otimes_k (R/k)^{\otimes n} \qquad R/k = k\varepsilon \qquad (\otimes = \otimes_k)$$
$$= R \otimes \underbrace{k\varepsilon \otimes \cdots \otimes k\varepsilon}_{n} \otimes R$$

with differential $\partial_n : \mathbb{B}_n \longrightarrow \mathbb{B}_{n-1}$ defined by

$$\partial_n(r[a_1|\cdots|a_n]r') = ra_1[a_2|\cdots|a_n]r' + \sum_{i=1}^{n-1} r[a_1|\cdots|a_ia_{i+1}|\cdots|a_n]r' + (-1)^n r[a_1|\cdots|a_{n-1}]a_nr'$$

Now in this case $\mathbb{B}_n \cong R^e$ as R-bimodules and $a_ia_{i+1} = 0$ if $a_i = \varepsilon$ or $a_{i+1} = \varepsilon$, so the complex $\mathbb{B}$ is simply

$$\partial_n(r[\varepsilon|\cdots|\varepsilon]r') = r\varepsilon[\varepsilon|\cdots|\varepsilon]r' + (-1)^n r[\varepsilon|\cdots|\varepsilon]\varepsilon r'$$

$$\mathbb{B}_n \cong R^e \longrightarrow R^e \cong \mathbb{B}_{n-1}$$

$$r \otimes r' \longmapsto r\varepsilon \otimes r' + (-1)^n r \otimes \varepsilon r'$$

Now, it follows that $\mathrm{Hom}_{R^e}(\mathbb{B}, R)$ is simply the R-linear map

$$\begin{array}{c} 1\otimes 1 \\ \updownarrow \\ a \end{array}$$

$$\begin{array}{ccc} \mathrm{Hom}_{R^e}(\mathbb{B}_n, R) & \longleftarrow & \mathrm{Hom}_{R^e}(\mathbb{B}_{n-1}, R) \\ \wr\| & & \wr\| \\ R & & R \\ (1+(-1)^n)\varepsilon & \longleftarrow\!\shortmid & 1 \end{array}$$

That is, $\mathrm{Hom}_{R^e}(\mathbb{B}, R)$ is

$$\begin{tikzcd}
 & n & n-1 & & 2 & 1 & 0 \\
\cdots & R \arrow[l] & R \arrow[l, "(1+(-1)^n)\varepsilon"'] & \cdots \arrow[l, phantom, ""] & R \arrow[l, "0"'] & R \arrow[l, "2\varepsilon"'] & R \arrow[l, "0"']
\end{tikzcd}$$

$k\varepsilon$

Assuming $\frac{1}{2} \in k$, this allows us to compute that

$$HH^n(R) = H^n \mathrm{Hom}_{R^e}(\mathbb{B}, R) = \begin{cases} R & n = 0 \\ k\varepsilon & n > 0 \text{ odd} \\ k & n > 0 \text{ even} \end{cases}$$

For $n>0$ the generator of $HH^n(R)$ is the cocycle $(k\varepsilon)^{\otimes n} \longrightarrow R$ given by $\varepsilon^{\otimes n} \longmapsto \varepsilon$ if n is odd, and $\varepsilon^{\otimes n} \longmapsto 1$ if n is even.