Introduction to the LG/CFT correspondence

The aim of this lecture is to sketch the Landau-Ginzburg (LG) / Conformal Field Theory (CFT) correspondence, which has its origin in mathematical physics but which has inspired interesting pure mathematics. The correspondence is far from fully understood, mathematically or physically. On the LG side the main mathematical ingredient is the theory of <u>matrix factorisations</u>, and we begin with a (slightly nonstandard) introduction to this theory, which we hope is helpful in trying to understand the relation to vertex algebras via Drinfeld-Sokolov reduction.

	LG	CFT
SYMMETRY	super + ···	superconformal + ···
ALGEBRA	——	super vertex algebra V
CATEGORIES	matrix factorisations	Rep(V)

matrix factorisations $\xleftrightarrow{\text{LG/CFT}}$ Rep(V)

algebraic geometry

representation theory

Outline

1. Matrix factorisations & LG models
2. Simple singularities and minimal models in CFT
3. The LG/CFT correspondence

① Matrix factorisations & LG models

One way to understand the theory of matrix factorisations is as a generalisation (in a sense that can be made precise) of the theory of Clifford algebras, so let us begin with a reminder on the latter. Let k be a field, char $k=0$.

Def^N Given a f.d. vector space V and symmetric bilinear form $B: V\times V \to k$, the Clifford algebra of (V, B) is the $\mathbb{Z}_2$-graded algebra

$$Cl(V,B) := T(V)\Big/\big(v\otimes w + w\otimes v - 2B(v,w)\big)_{v,w}$$

(i.e generated by a basis $e_1,\dots,e_n$ of V subject to $e_ie_j + e_je_i = 2B(e_i,e_j)$.)

Example $Cl(V,0) \cong \bigwedge V$ as $\mathbb{Z}_2$-graded algebras

Lemma There is always an iso of $\mathbb{Z}_2$-graded vector spaces

$$\bigwedge V \xrightarrow{\cong} Cl(V,B)$$

$\Longrightarrow$ $Cl(V,B)$ is a deformation (as an algebra) of $\bigwedge V$.

$\boxed{Q}$ Are there other deformations of $\bigwedge V$ as a $\mathbb{Z}_2$-graded algebra?

The question is best phrased in terms of the Hochschild cohomology $HH^*_{\mathbb{Z}_2}(\bigwedge V)$ (in its $\mathbb{Z}_2$-graded form) of $\bigwedge V$, or what is the same, the deformations of $\bigwedge V$ within the space of $\mathbb{Z}_2$-graded A_∞-algebras (= vector spaces A equipped with operations $m_n: A^{\otimes n} \to A$ for $n \geq 1$).

$$\{\mathbb{Z}_2\text{-graded algebras}\} \subseteq \{\mathbb{Z}_2\text{-graded DG-algebras}\} \subseteq \{\mathbb{Z}_2\text{-graded } A_\infty\text{-algs}\}$$

$\boxed{A}$ Yes. Say $V=\mathbb{C}^n$. Every polynomial $W \in \mathbb{C}[z_1,\dots,z_n]$ with an isolated singularity at $\underline{0}$ (i.e. $W(\underline{0})=0$ and $\dim_{\mathbb{C}} \mathbb{C}[z]/(\partial_{z_1}W,\dots,\partial_{z_n}W) < \infty$) gives rise to an A_∞-algebra

$$\mathcal{A}_W = \left(\Lambda\mathbb{C}^n, m_2, m_3, \dots, m_n, \dots\right)$$

whose underlying vector space is $\Lambda\mathbb{C}^n$ (existence due to Dyckerhoff).

Example $W=\sum_{i=1}^n z_i^2$ is the quadratic form of $B: V\times V \to \mathbb{C}$, $B(e_i,e_j)=\delta_{ij}$ and $\mathcal{A}_W \cong Cl(V,B)$ as A_∞-algebras

Example $n=1$, so $\Lambda\mathbb{C} = \mathbb{C}[\varepsilon]/\varepsilon^2$, for $W=z^N \in \mathbb{C}[z]$ there is a Hochschild cocycle $(N\geq 3)$

$$c \in HH^N(\Lambda\mathbb{C}),\quad c: (\Lambda\mathbb{C})^{\otimes N} \longrightarrow \Lambda\mathbb{C}$$
$$c(\varepsilon\otimes\cdots\otimes\varepsilon)=1 \quad \text{(zero on other basis elts.)}$$

The corresponding deformation of $\Lambda\mathbb{C}$ is the A_∞-algebra

$$\mathcal{A}_{z^N} = \left(\Lambda\mathbb{C}, \underbrace{m_2}_{\text{usual product on } \Lambda\mathbb{C}}, 0, \dots, 0, \overset{(N)}{c}, 0, \dots\right)$$

The above raises the question: <u>what are the categories of A_∞-modules over these deformations</u>? And how are the m_n <u>derived from W</u>?

$\boxed{\mathrm{per}_\infty(\mathcal{A}_W) = ?}$ $\boxed{m_n = ?}$

triangulated cat. of perfect A_∞-modules

generally the underlying v.-space of these modules are again Fock spaces

The first question is best-addressed using <u>matrix factorisations</u> (finally), but the second is beyond the scope of this talk (and only recently worked out).

<u>Def^n</u> Let $W \in \mathbb{C}[z_1, \dots, z_n]$. A <u>matrix factorisation</u> X of W is described equivalently as either

(i) a pair $A, B \in M_k(\mathbb{C}[z])$ such that $AB = BA = W \cdot I_k$.

(ii) a f.g. free $\mathbb{Z}_2$-graded $\mathbb{C}[z]$-module $X = X^0 \oplus X^1$ with odd $d_X : X \longrightarrow X$ (the <u>differential</u>) s.t. $d_X^2 = W \cdot 1_X$.

<u>Def^n</u> A morphism $\phi : (X, d_X) \longrightarrow (Y, d_Y)$ is degree zero, $\mathbb{C}[z]$-linear map $\phi : X \to Y$ with $d_Y \phi = \phi d_X$. We say $\phi \sim \psi$ are <u>homotopic</u> if there is $h : X \longrightarrow Y$ odd with $\phi - \psi = d_Y h + h d_X$.

<u>Def^n</u> $\mathrm{hmf}(W) := \{$ matrix factorisations with homotopy equivalence classes of morphisms $\}$

<u>Theorem</u> (Eisenbud, Buchweitz–Orlov, Auslander) $hmf(W)$ is a Calabi–Yau triangulated category and

$$hmf(W) \cong \mathbb{D}^b(\operatorname{coh} Z(W))/\operatorname{Perf} Z(W)$$

where $Z(W) = \{\underline{a} \in \mathbb{C}^n \mid W(\underline{a}) = 0\}$.

<u>Theorem</u> (Dyckerhoff) There exist linear maps $m_k : (\Lambda\mathbb{C}^n)^{\otimes k} \to \Lambda\mathbb{C}^n$ such that $(\Lambda\mathbb{C}^n, \{m_k\}_{k \geq 2})$ is an A_∞-algebra and

$$\operatorname{per}_\infty(\mathcal{A}_W) \cong hmf(W)$$

<u>Example</u> For $W = \sum_i z_i^2$, $\operatorname{per}_\infty(\mathcal{A}_W) \cong$ f.d. representations of $Cl(V, B)$, as $\mathcal{A}_W = Cl(V, B)$ (recovers old result of Buchweitz-Eisenbud-Herzog).

<u>Example</u> $W = z^N$, $\quad d_X = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} = \begin{pmatrix} 0 & z^i \\ z^{N-i} & 0 \end{pmatrix} \quad 1 \leq i \leq N.$

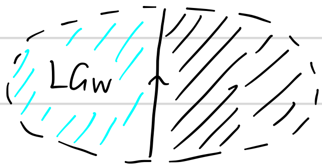
i.e. $\operatorname{per}_\infty(\mathcal{A}_{z^N}) \cong hmf(z^N)$.
(with m_k as defined above).

⌜related to A_{N-1} quiver, since
$\mathbb{D}^b(\operatorname{rep} A_{N-1})$
$\cong$
$hmf^{gr}(z^N + x^2 + y^2)$⌟

Landau-Ginzburg models

LG_W is an $N=2$ supersymmetric QFT whose Lagrangian involves the polynomial $W \in \mathbb{C}[z_1, \dots, z_n]$. The topological B-twist of LG_W is described in various aspects by rigorous mathematics:

PHYSICS	MATH
• Bulk observables	$J_W = \mathbb{C}[z_1,\dots,z_n] / (\partial_{z_1} W, \dots, \partial_{z_n} W)$
• Boundary conditions	Matrix factorisations of W



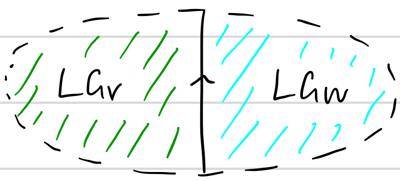
(similar to the role of vector bundles in nonlinear sigma models)

• Boundary observables .... bosonic $\mathrm{Hom}(X,Y)$, fermionic $\mathrm{Hom}(X, Y[1])$ — in $hmf(W)$

• Defect conditions .... Matrix factorisations of

$$V(z') - W(z)$$

different variables.



(may be composed $\Rightarrow$ $\otimes$-structure)

More precisely

$$\left[\begin{array}{ccc}\text{Open-closed 2D TFT} & \longleftrightarrow & \text{Calabi-Yau triangulated category} \\ LG_W & \text{- - - - - - -} & hmf(W)\end{array}\right]$$

$$\left[\begin{array}{ccc}\text{2D defect TFT} & \xleftrightarrow{\text{(conjecturally)}} & \text{Pivotal bicategory} \\ \{LG_W\}_W & \text{- - - - - - - - - - - -} & \mathcal{LG}\end{array}\right]$$

Lazaroiu-McNamee, Carqueville-Runkel, Carqueville-M

② <u>Simple singularities and minimal CFTs</u>

Given $W \in \mathbb{C}[z_1, \dots, z_n]$ topologise $\mathbb{C}\{\underline{z}\}$ such that $\mathbb{C}[\underline{z}] \longrightarrow \mathbb{C}\{\underline{z}\}/\mathfrak{m}^k$ is continuous for all $k \geq 1$. Then provided

$$f = \text{germ}_{\underline{0}} W \in \mathfrak{m} \subseteq \mathbb{C}\{\underline{z}\}$$

defines an isolated singularity there is an open neighborhood $f \in U \subseteq \mathfrak{m}$ in which every pt. is either <u>smooth</u> or also an <u>isolated singularity</u>. We say f is a <u>simple singularity</u> if $\exists U$ containing only finitely many "distinct" singularities $(=$ orbits of $\text{Aut}\,\mathbb{C}\{\underline{x}\})$.

<u>Theorem</u> (Arnold '72) The simple singularities have an ADE classification:

$A_k : x_1^{k+1} + x_2^2 + \dots + x_n^2 \qquad k \geq 1$

$D_k : x_1(x_2^2 + x_1^{k-2}) + x_3^2 + \dots + x_n^2 \qquad k \geq 4$

$E_6 : x_1^3 + x_2^4 + \text{squares}$

$E_7 : x_1(x_1^2 + x_2^3) + \text{squares}$

$E_8 : x_1^3 + x_2^5 + \text{squares}$

- 2D <u>conformal field theories</u> are QFTs which are covariant w.r.t. local conformal transformations. Infinitesimally these transformations are generated by two copies of the Virasoro algebra (called the left and right <u>chiral algebras</u>). The <u>central charge</u> of the CFT is the c appearing in these Virasoro algebras. A given CFT may have an extended chiral algebra $\mathcal{A}$ containing Virasoro, such as affine Kac-Moody algebras or the superconformal algebras.

- An $N=2$ superconformal field theory is a CFT whose chiral algebra contains a particular Lie superalgebra called the "$N=2$ superconformal algebra". The <u>minimal</u> $N=2$ superCFTs have an ADE classification, see [ADE], with e.g.

$$\left[A_k\text{-type } N=2 \text{ minimal CFT}, \qquad c = 3 - \frac{6}{k+1} \right]$$

$$\left(\underset{d=k+1}{\text{by coset construction}} \quad \frac{\widehat{su}(2)_{d-2} \times \widehat{u}(1)_4}{\widehat{u}(1)_{2d}} \right)$$

Aside

Given $W \in \mathbb{C}[x_1, \dots, x_n]$ and a decomposition $W = \sum_{i=1}^{k} a_i b_i$, we have the <u>Koszul factorisation</u> of W:

$$X = \left(\Lambda(\mathbb{C}\psi_1 \oplus \cdots \oplus \mathbb{C}\psi_k) \otimes_{\mathbb{C}} \mathbb{C}[\underline{x}] \;,\; \sum_i a_i \psi_i^* + \sum_i b_i \psi_i \right)$$

These are the most commonly encountered MFs in practice (e.g. in knot homology, LG/CFT correspondence, ...) and the underlying vector space of the corresp. A_∞ $\mathcal{A}_W$-module is again an exterior algebra

3 The LG/CFT correspondence (based on [R])

Based on:

natural length scale $\to \infty$

UV $\longrightarrow$ RG flow $\longrightarrow$ IR

N=2 LG model with potential W (e.g. $x_1^{k+1}+x_2^2+x_3^2$)

N=2 SCFT $c = 3\sum_i (1-q(x_i))$ (e.g. A_k minimal model)

Preservation of topological quantities under RG flow "implies"

$\boxed{\text{LG side}}$ $\boxed{\text{CFT side}}$

Bulk observables $J_W \xrightarrow{\cong}$ Chiral primary fields

Martinec '89, Vafa, Warner '89, Lerche, Vafa, Warner '89

(B-type) Boundary conditions $\supseteq \mathcal{A}$ (MFs of W) $\xrightarrow{\cong}$ Boundary conditions $\supseteq \mathcal{A}'$ (reps. of chiral algebra, SVOA)

known in some cases

Kapustin-Li '02, '03
Brunner, Herbst, Lerche
Scheuner '03

A_k — N=2 minimal SVOA V for A_k

Defect conditions $\supseteq \mathcal{B}$ (MFs of $W_1 - W_2$) $\xrightarrow{\cong}$ Defect conditions $\supseteq \mathcal{B}'$ (bimodules over chiral algebra)

known in some cases

Brunner-Roggenkamp '07
Carqueville-Runkel '09, '10
Davydov, Ros Camacho, Runkel, '14

(Ex: natural geometric defects between A-type singularities correspond to defects / flows between minimal CFTs).

<u>Theorem</u> (Davydov–Ros Camacho–Runkel '14) Let k be even. There is an equivalence of <u>tensor</u> categories

$$\mathcal{P}^{gr}_{k+1} \xrightarrow{\cong} \mathcal{C}(N=2, k+1)_{NS} \qquad (9.1)$$

$\cap$ $\cap$

{graded MFs of $y^{k+1} - x^{k+1}$} (i.e. defects between two copies of the A_k-type LG model)

{irred. highest weight reps. of A_k-type minimal SVOA}.

<u>Example</u> $\mathcal{P}^{gr}_{k+1}$ is generated by $P_{a;\lambda} = \begin{pmatrix} 0 & \prod_{i=a}^{a+\lambda}(y-\eta^i x) \\ \prod_{\text{rest}}(y-\eta^i x) & 0 \end{pmatrix}$ $\eta = e^{\frac{2\pi i}{k+1}}$

using $y^{k+1} - x^{k+1} = \prod_{i \in \mathbb{Z}_{k+1}} (y - \eta^i x)$

So far all rigorous checks of the LG/CFT correspondence are "bespoke" (e.g. (9.1) is proved by a direct comparison of both sides)

<u>OPEN QUESTION</u>: What is a conceptual mathematical explanation/proof of the LG/CFT correspondence? of the kind

$$\left\{ \begin{array}{c} \text{geometry of singular} \\ \text{hypersurface } W=0 \end{array} \right\} \overset{?}{\dashrightarrow} \left\{ \text{super VOA} \right\}$$

<u>Summary</u> A simple singularity $W(x_1, x_2, x_3)$ gives rise to

$$\mathrm{rep}_\infty(\Lambda\mathbb{C}^3, m_2, m_3, \ldots) \cong \mathrm{hmf}(W) \overset{LG/CFT}{\longleftarrow\!\!-\!-\!\!\longrightarrow} \mathrm{Rep}(V)$$

A_∞-modules

A_∞-algebra, defo of exterior alg.

$N=2$ super VOA with modular invariant of the ADE type of W

Some observations

- Following [C90], we can try to construct representations of the $N=2$ superconformal algebra using Hochschild cocycles of exterior algebras. Recall the loop algebra of a Lie algebra $\mathfrak{g}$ is $\mathfrak{g} \otimes_{\mathbb{C}} C^\infty(S^1)$ where $C^\infty(S^1)$ are smooth complex-valued functions. If we pass to Fourier coefficients and ignore convergence (Q: does this change the cyclic cohomology?) this is

$$\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[[t, t^{-1}]].$$

The super-loop-algebra is $(G_N = \Lambda \mathbb{C}^N$ the exterior algebra$)$

$$\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[[t, t^{-1}]] \otimes_{\mathbb{C}} G_N. \qquad (*)$$

Kac-Moody (resp super Kac-Moody) Lie algebras have underlying vector space $(\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[[t,t^{-1}]]) \oplus \mathbb{C}R$ with R central and the new bracket determined by a Lie algebra 2-cocycle on $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[[t,t^{-1}]]$, call it ω, by

$$[(\xi,\alpha),(\eta,\beta)]_{new} = \big([\xi,\eta], \omega(\alpha,\beta)\big)$$

(for the super case, we need a $\mathbb{Z}_2$-graded cocycle on $(*)$). The super case is richer, since there are more relevant cocycles to twist by:

$$\{\text{cyclic Hochschild 1-cocycles of } G_N\} \longrightarrow \{\text{cyclic Lie 2-cocycles of } \mathbb{C}[[t,t^{-1}]] \otimes_{\mathbb{C}} G_N\}$$

large!

Let C be a Lie superalgebra, and

$$\mathrm{Der}(C) = \{ \mathcal{L} : C \to C \mid \mathcal{L} \text{ is } \mathbb{C}\text{-linear, homog. and for all } X, Y \in C \quad \mathcal{L}([X,Y]) = [\mathcal{L}X, Y] + (-1)^{|X||\mathcal{L}|}[X, \mathcal{L}Y] \}$$

This is a Lie superalgebra with the bracket from $\mathrm{End}_{\mathbb{C}}(C)$. (graded). Given a $\mathbb{Z}_2$-graded 2-cocycle on C, we can form the central extension $C \oplus \mathbb{C}R$ with the bracket $[-,-]_\omega$ from the cocycle ω, and given $\mathcal{L} \in \mathrm{Der}(C)$ ask if $\tilde{\mathcal{L}} : C \oplus \mathbb{C}R \to C \oplus \mathbb{C}R$ defined by $\tilde{\mathcal{L}}|_C = \mathcal{L}$, $\tilde{\ }(R) = 0$ is a graded derivation for the new bracket. This amounts to

$$\omega(\mathcal{L}X, Y) + (-1)^{|X||\mathcal{L}|}\omega(X, \mathcal{L}Y) = 0 \quad \forall X, Y. \qquad (*).$$

<u>Def^N</u> $\mathrm{Der}_\omega(C)$ is the Lie subalgebra of $\mathrm{Der}(C)$ given by all $\mathcal{L}$ satisfying $(*)$.

⌜The way [C90] define Hochschild cohomology is to identify n-cocycles of A as $\mathbb{C}$-linear forms $A^{\otimes n+1} \to \mathbb{C}$, probably as follows: we would say n-cocycles are $\mathbb{C}$-linear $\bar{A}^{\otimes n} \to A$, i.e. $A^* \otimes \bar{A}^{\otimes n} \to \mathbb{C}$, and if A is self-dual, this is the same as $A^{\otimes n+1} \to \mathbb{C}$.⌟

<u>Theorem</u> For $N=1$ there exists a cyclic Hocschild cocycle α of G_N with associated Lie algebra 2-cocycle ω of $C = \mathbb{C}[[t,t^{-1}]] \otimes_{\mathbb{C}} G_N$, such that $\mathrm{Der}_\omega(C)$ is the $N=1$ superconformal algebra.

<u>Theorem</u> For $N=2$ there is α, ω with $\mathrm{Der}_\omega(C)$ containing a copy of the $N=2$ superconformal algebra.

related by [C95] to a degenerate LG model. ($j=-1 \Rightarrow$ central charge -1)

$\boxed{Q}$ role for higher Hochschild cocycles of the Grassmann alg. G_N, via L_∞-algebras?

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