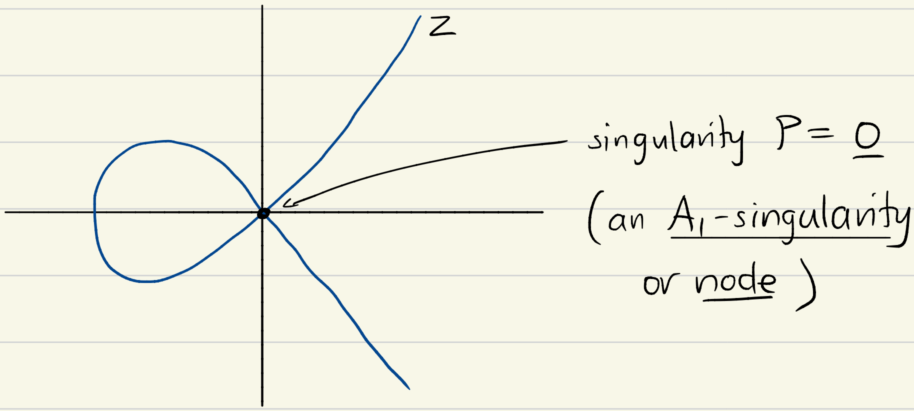
# Introduction to matrix factorisations Part I

① What is a singularity? a point $P$ of a curve $f(x,y)=0$ is nonsingular if $\nabla f(P) \neq 0$, so that locally near $P$, $Z = \{(x,y) \mid f(x,y)=0\}$ is a submanifold. Otherwise $P$ is a singularity of $Z$

$$f(x,y) = y^2 - x^2(x+1)$$




- Associated to the germ $(Z,P)$ is the local coordinate ring

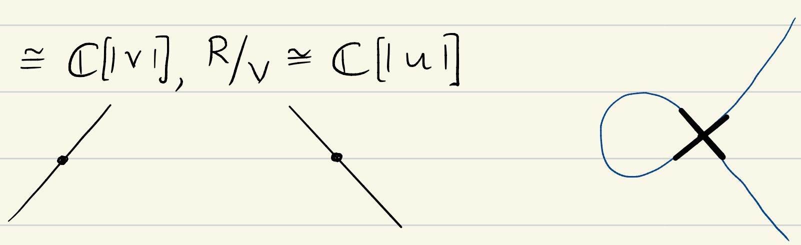
$$\mathbb{C}[[x,y]]/f(x,y) \cong \mathbb{C}[[u,v]]/(uv) =: R$$ (Hartshorne Ex. 5.6.3)

- Let $\mathrm{mod}(R)$ denote finitely generated $R$-modules. We can study the singularity $(Z,P)$ by understanding

$$\mathrm{mod}(R) \subseteq \mathbb{D}^b(\mathrm{mod}\,R)$$

morphisms are Ext classes

Example $R/u \cong \mathbb{C}[[v]]$, $R/v \cong \mathbb{C}[[u]]$



- Actually the "interesting" homological information about $R$-modules is concentrated in the <u>infinite tail of projective resolutions</u>.

<u>Example</u> $M = R/(u,v) \cong \mathbb{C}$ has (minimal) free resolution $F$

$$\cdots \longrightarrow \underbrace{R^{\oplus 2} \xrightarrow{\begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}} R^{\oplus 2} \xrightarrow{\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}} R^{\oplus 2} \xrightarrow{\begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}} \underset{-1}{R^{\oplus 2}} \xrightarrow{(u \ v)} \underset{0}{R}}_{F} \longrightarrow M \longrightarrow 0$$

To check this note that $fu + gv = 0$ in $R = \mathbb{C}[u,v]/(uv)$ implies $fu + gv \in (uv)$ in $\mathbb{C}[u,v]$ so $f \in (v)$, $g \in (u)$. For exactness at the other positions we need $fv \in (uv) \Rightarrow f \in (u)$ and $gu \in (uv) \Rightarrow g \in (v)$.

- The syzygies in the infinite periodic part are $R/v \oplus R/u$. These are <u>maximal Cohen–Macaulay $R$-modules</u> (MCM) which means $\operatorname{depth}(M) = \dim(R)$. Recall over a Noetherian local ring $(R, \mathfrak{m})$

$$\begin{aligned} \operatorname{depth}(M) &= \text{common length of maximal regular sequence in } M \\ &= \inf\{ i \mid \operatorname{Ext}^i(R/\mathfrak{m}, M) \neq 0 \} \\ &\leq \dim(M) \leq \dim(R) \end{aligned}$$

Krull dimension of $R/\operatorname{Ann}(M)$

We can compute in our example (i.e. $M = \mathbb{C}$, $R = \mathbb{C}[u,v]/uv$)

$$\operatorname{Ext}^i(R/\mathfrak{m}, M) = H^i \operatorname{Hom}_R(F, M) = H^i\left( \mathbb{C} \xrightarrow{0} \mathbb{C}^{\oplus 2} \xrightarrow{0} \mathbb{C}^{\oplus 2} \xrightarrow{0} \cdots \right)$$

Hence $\operatorname{depth}(M) = 0 = \dim(M) < \dim(R) = 1$ so $M$ is <u>not</u> MCM. However

$$\operatorname{Ext}^i(R/\mathfrak{m}, R/u) = H^i \operatorname{Hom}_R(F, R/u)$$

$$= H^i\left( R/u \xrightarrow{\begin{pmatrix} 0 \\ v \end{pmatrix}} (R/u)^{\oplus 2} \xrightarrow{\begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix}} (R/u)^{\oplus 2} \xrightarrow{\begin{pmatrix} 0 & 0 \\ 0 & v \end{pmatrix}} (R/u)^{\oplus 2} \to \cdots \right)$$

$$\cong H^i\left( \mathbb{C}[[v]] \xrightarrow{\begin{pmatrix} 0 \\ v \end{pmatrix}} \mathbb{C}[[v]]^{\oplus 2} \xrightarrow{\begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix}} \mathbb{C}[[v]]^{\oplus 2} \xrightarrow{\begin{pmatrix} 0 & 0 \\ 0 & v \end{pmatrix}} \cdots \right)$$

$$\begin{array}{cccc} = & 0 & \mathbb{C}[[v]]/(v) & \underbrace{\sim\sim\sim\sim\sim}_{\text{irrelevant}} \\ & i=0 & i=1 & \end{array}$$

Hence $\operatorname{depth}_R(R/u) = 1 = \dim(R)$ so $R/u$ is MCM, and similarly $R/v$.

There is a general phenomenon at work here:

<u>Lemma</u> If $R = \mathbb{C}[[x_1, \dots, x_n]]/f$ is a hypersurface ring, and $N$ is a f.g. $R$-module with free resolution

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[rr] \arrow[dr] & & F_2 \arrow[rr] \arrow[dr] & & F_1 \arrow[rr] \arrow[dr] & & F_0 \arrow[r] & N \arrow[r] & 0 \\
& S_2 \arrow[ur] & & S_1 \arrow[ur] & & S_0 \arrow[ur] & & &
\end{tikzcd}$$

then for $j \geq \dim(R) - 1$, the module $S_j$ is MCM.

<u>Proof</u> Since $R$ is Cohen-Macaulay $\operatorname{depth}(R) = \dim(R)$ so

$$\operatorname{Ext}^{<\dim(R)}(R/\mathfrak{m}, R) = 0.$$

The depths of the $S_j$ <u>increase</u>, since for $i+1 < \dim(R)$

$$\mathrm{Ext}^i(R/m, S_j) \longrightarrow \mathrm{Ext}^{i+1}(R/m, S_{j+1}) \longrightarrow \mathrm{Ext}^{i+1}(R/m, F_{j+1})$$

is exact, so $\mathrm{depth}(S_j) = k < \dim(R) \Rightarrow \mathrm{Ext}^{<k}(R/m, S_j) = 0$

$$\Rightarrow \mathrm{Ext}^{\leq k}(R/m, S_{j+1}) = 0$$
$$\Rightarrow \mathrm{depth}(S_{j+1}) > k$$

Hence as long as $\mathrm{depth}(S_{j-1}) < \dim(R)$

$$\underset{\geqslant 0}{\mathrm{depth}(N)} < \underset{\geqslant 1}{\mathrm{depth}(S_0)} < \underset{\geqslant 2}{\mathrm{depth}(S_1)} < \cdots < \underset{\geqslant j+1}{\mathrm{depth}(S_j)}$$

and so clearly $S_j$ is MCM for $j \geqslant \dim(R) - 1$. $\square$

<u>Remark</u> This means that we have an exact sequence of complexes $(j \geqslant \dim(R)-1)$

$$0 \longrightarrow F_{\leq j+1} \longrightarrow F \longrightarrow F_{\geqslant j} \longrightarrow 0$$

quasi-isomorphic to $S_j[j]$ quasi-iso to $N$ bounded cpx of free modules

hence a triangle $S_j[j] \longrightarrow N \longrightarrow F_{\geqslant j} \xrightarrow{+}$ in $\mathbb{D}^b(\mathrm{mod} R)$
and hence with $\mathrm{Perf}(R) = \{$bd. cpxs of f.g. proj. $R$-modules$\}$

$$\boxed{N \cong S_j[j] \text{ in } \mathbb{D}^b(\mathrm{mod} R)/\mathrm{Perf}(R)}$$

**Example** $R = \mathbb{C}[[u,v]]/(uv)$

$$\mathbb{C} \cong S_1[1] = R/u[1] \oplus R/v[1] \quad \text{in} \quad \mathbb{D}^b(\mathrm{mod} R)/\mathrm{Perf}(R)$$

But we can do better, observe that

$$\cdots \xrightarrow{\bar{B}} R^{\oplus 2} \xrightarrow{\bar{A} = \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}} R^{\oplus 2} \xrightarrow{\bar{B} = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}} R^{\oplus 2} \xrightarrow{\bar{A}} \cdots$$

is an infinite acyclic (zero cohomology) complex, with all syzygies $R/u \oplus R/v$. Hence by the same argument as above $(R/u \oplus R/v)[1] \cong R/u \oplus R/v$ in the Verdier quotient, hence

$$\mathbb{C} \cong R/u \oplus R/v \quad \text{in} \quad \mathbb{D}^b(\mathrm{mod} R)/\mathrm{Perf}(R)$$

Note these $R$-modules are not isomorphic in $\mathrm{mod}(R)$!

**Exercise** $R/u \cong R/v[1]$ in $\mathbb{D}^b(\mathrm{mod} R)/\mathrm{Perf}(R)$.

**Def$^{n}$** (Buchweitz, Orlov) The singularity category of $R$ is $\mathbb{D}^b_{sg}(R) := \mathbb{D}^b(\mathrm{mod} R)/\mathrm{Perf}(R)$.

**Def$^{n}$** The stable category of MCM R-modules $\underline{\mathrm{MCM}}(R)$ has MCM $R$-modules as objects and morphisms denoted

$$\underline{\mathrm{Hom}}_R(M,N) = \mathrm{Hom}_R(M,N) \Big/ \left\{ \begin{array}{l} \varphi \text{ factoring as } M \to P \to N \\ \text{with } P \text{ projective} \end{array} \right\}$$

**Theorem** (Buchweitz, Orlov) Every object of $\mathbb{D}^b_{sg}(R)$, for a hypersurface singularity $R = \mathbb{C}[[\underline{x}]]/f$, is isomorphic to an MCM module and the canonical functor

$$\mathrm{MCM}(R) \hookrightarrow \mathrm{mod}(R) \hookrightarrow \mathbb{D}^b(\mathrm{mod}R) \twoheadrightarrow \mathbb{D}^b_{sg}(R)$$

factors via an equivalence of triangulated categories

$$\underline{\mathrm{MCM}}(R) \xrightarrow[\Phi]{\cong} \mathbb{D}^b_{sg}(R).$$

**Example** $R = \mathbb{C}[[u,v]]/(uv)$ we have computed $\Phi^{-1}(\mathbb{C}) \cong R/u \oplus R/v$.
Actually we have an equivalence

$$\mathrm{Vect}^{\mathbb{Z}_2}_{\mathbb{C}} \cong \underline{\mathrm{MCM}}(R) \cong \mathbb{D}^b_{sg}(R)$$

| | | |
|---|---|---|
| $\mathbb{C}$ | $R/u$ | $R/u$ |
| $\mathbb{C}[1]$ | $R/v$ | $R/v$ |
| $\mathbb{C} \oplus \mathbb{C}[1]$ | $R/u \oplus R/v$ | $\mathbb{C}$ |

Exercise: why is this asymmetry between $u,v$ not a problem?

**Theorem** (Eisenbud) Over a hypersurface ring $R = \mathbb{C}[[\underline{x}]]/f$ the minimal free resolution of every f.g. $R$-module $N$ is eventually $\mathbb{Z}$-periodic, that is, of the form

$$\cdots \underbrace{R^{\oplus d} \xrightarrow{\bar{A}} R^{\oplus d} \xrightarrow{\bar{B}} R^{\oplus d}}_{\text{repeats}} \longrightarrow \cdots \longrightarrow N \longrightarrow 0$$

where $A, B \in M_d(\mathbb{C}[[\underline{x}]])$ are polynomial matrices satisfying $AB = f \cdot Id$, $BA = f \cdot Id$. We call $(A,B)$ a matrix factorisation of $f$.

<u>Example</u> $f = uv$ (i) $A = (u)$, $B = (v)$ (call this $X$)

(ii) $A = (v)$, $A = (u)$ (call this $Y$)

(iii) $A = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$, $B = \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}$ (call this $Z$)

<u>Def$^n$</u> The <u>homotopy category of matrix factorisations</u> $hmf(\mathbb{C}[[x]], f)$ has

- objects are MFs $(A, B)$ (square matrices of the same size)

- morphisms $(A, B) \xrightarrow{(\varphi, \psi)} (A', B')$ are commutative diagrams (writing $S = \mathbb{C}[[x]]$, all maps $S$-linear)

$$\begin{tikzcd}
S^{\oplus d} \arrow[r, "A"] \arrow[d, "\varphi"'] & S^{\oplus d} \arrow[r, "B"] \arrow[d, "\psi"] & S^{\oplus d} \arrow[d, "\varphi"] \\
S^{\oplus d} \arrow[r, "A'"'] & S^{\oplus d} \arrow[r, "B'"'] & S^{\oplus d}
\end{tikzcd}$$

modulo the <u>homotopy</u> relation $(\varphi, \psi) \sim (\alpha, \beta)$ if there exist $g, h$ such that $A'g + hB = \psi - \beta$, $B'h + gA = \varphi - \alpha$.

- triangulated structure with shift $(A, B)[1] := (-B, -A)$, so $[2] = Id$. and $(A, B) \oplus (A', B') = (A \oplus A', B \oplus B')$.

<u>Example</u> In $hmf(\mathbb{C}[[u, v]], uv)$, $Y \cong X[1]$ and $Z \cong X \oplus X[1]$.

<u>Theorem</u> For any hypersurface ring there are equivalences of triangulated categories

$$\mathrm{hmf}(\mathbb{C}[[\underline{x}]], f) \xrightarrow{\Lambda} \underline{\mathrm{MCM}}(R) \xrightarrow{\Phi} \mathbb{D}^b_{sg}(R)$$

shift here is "take syzygy"

where $\Lambda(A,B) = \operatorname{coker} A$.

<u>Proof sketch</u> Observe that $A : S^{\oplus d} \longrightarrow S^{\oplus d}$ has cokernel $N$, and given $x \in N$ with $x = \overline{y}$, $y \in S^{\oplus d}$

$$fx = \overline{fy} = \overline{ABy} = 0$$

Hence $N$ is an $R = \mathbb{C}[[\underline{x}]]/f$-module. To see $N$ is MCM we prove that the infinite complex

$$\begin{tikzcd} \cdots \arrow[r, "\bar{B}"] & R^{\oplus d} \arrow[r, "\bar{A}"] & R^{\oplus d} \arrow[r, "\bar{B}"] \arrow[dr, two heads] & R^{\oplus d} \arrow[r, "\bar{A}"] & \cdots \\ & & & N \arrow[u, hook] & \end{tikzcd}$$

is acyclic, with syzygy $N$. Suppose $\bar{A}x = 0$, and $x = \overline{y}$. Then $\overline{Ay} = 0$ that is, $Ay = (a_1 f, \dots, a_n f)^T$ some $a_i \in S$. But then writing $a = (a_1, \dots, a_n)$

$$\begin{aligned} Ay = fa &\Rightarrow Ay = ABa \\ &\Rightarrow y = Ba \\ &\Rightarrow x = \bar{B}\bar{a}. \end{aligned}$$

($A$ is injective, as $f : S^{\oplus d} \to S^{\oplus d}$ is and $f = BA$)

By the earlier depth arguments we may conclude $N$ is MCM. Fully-faithfulness requires a bit more "Ext work". $\square$

Alternative def$^N$ A matrix factorisation of $f \in \mathbb{C}[[x]]$ is a $\mathbb{Z}_2$-graded f.g. free $S = \mathbb{C}[[x]]$-module $X = X_0 \oplus X_1$ with an odd $S$-linear map $d_X : X \longrightarrow X$ such that $d_X^2 = f \cdot 1_X$.

$$d_X = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} : X_0 \oplus X_1 \longrightarrow X_0 \oplus X_1.$$

$$\therefore \quad d_X^2 = \begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix}$$

In conclusion, for our original example $R = \mathbb{C}[[u,v]]/(uv)$

$$\begin{array}{ccccccc} \mathrm{Vect}_{\mathbb{C}}^{\mathbb{Z}_2} & \cong & \mathrm{hmf}(\mathbb{C}[[u,v]], uv) & \cong & \underline{\mathrm{MCM}}(R) & \cong & \mathbb{D}^b_{sg}(R) \\ \mathbb{C} & & \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix} & & R/u & & R/u \\ \mathbb{C}[1] & & \begin{pmatrix} 0 & v \\ u & 0 \end{pmatrix} & & R/v & & R/v \\ \mathbb{C} \oplus \mathbb{C}[1] & & \begin{pmatrix} 0 & 0 & u & 0 \\ 0 & 0 & 0 & v \\ v & 0 & 0 & 0 \\ 0 & u & 0 & 0 \end{pmatrix} & & R/u \oplus R/v & & \mathbb{C} \end{array}$$

Theorem (Knörrer periodicity) For any $f \in \mathbb{C}[[x_1, \dots, x_n]]$ with an isolated singularity

$$\mathrm{hmf}(\mathbb{C}[[\underline{x}]], f) \cong \mathrm{hmf}(\mathbb{C}[[\underline{x}, u, v]], f + uv)$$

Example $\mathrm{hmf}(\mathbb{C}[[u,v]], uv) \cong \mathrm{hmf}(\mathbb{C}, 0) = \mathrm{Vect}_{\mathbb{C}}^{\mathbb{Z}_2}$.

Def$^n$ We say $f \in \mathbb{C}[[x_1, \dots, x_n]]$ has an isolated singularity if

$$\dim_{\mathbb{C}} \left( \mathbb{C}[[\underline{x}]] / (\partial_{x_1} f, \dots, \partial_{x_n} f) \right) < \infty.$$

**Theorem** Let $f \in \mathbb{C}[\![x]\!]$ have an isolated singularity. Then with $\mathcal{T} = \mathrm{hmf}(\mathbb{C}[\![x]\!], f)$,

- $\mathcal{T}$ has finite-dimensional Hom-spaces
- $\mathcal{T}$ is idempotent complete
- $\mathcal{T}$ is Krull-Remak-Schmidt, i.e.
  - every object is a direct sum of indecomposables
  - if $\bigoplus_{i=1}^n N_i \cong \bigoplus_{j=1}^m M_j$ with $N_i, M_j$ all indecomposable then $m = n$ and after renumbering $N_i \cong M_i$ for all $i$.

**Example** $\mathrm{hmf}(\mathbb{C}[\![x,y,z]\!], x^{n+1} + y^2 + z^2) \cong \mathrm{hmf}(\mathbb{C}[\![x,u,v]\!], x^{n+1} + uv)$

$A_n$ surface singularity

$$\overset{\text{Knörrer}}{\cong} \mathrm{hmf}(\mathbb{C}[\![x]\!], x^{n+1})$$

$$\overset{\text{Buchweitz-Orlov}}{\cong} \underline{\mathrm{MCM}}(\mathbb{C}[\![x]\!]/x^{n+1})$$

But $R = \mathbb{C}[\![x]\!]/x^{n+1}$ has $\dim(R) = 0$, so <u>every</u> f.g. $R$-module $M$ is MCM. By the fundamental theorem for modules over a PID, we have in $\mathrm{mod}(R)$

$$M \cong R^{\oplus a} \oplus (R/x)^{\oplus a_1} \oplus \cdots \oplus (R/x^n)^{\oplus a_n}$$

Hence in $\underline{\mathrm{MCM}}(R)$, $M \cong \bigoplus_{i=1}^n (R/x^i)^{\oplus a_i}$, so the indecomposables are $R/x^i$ for $1 \leq i \leq n$. Observe that we have an exact sequence over $R$

$$\begin{tikzcd}
\cdots \arrow[r, "x^{n+1-i}"] & R \arrow[rr, "x^i"] \arrow[dr] & & R \arrow[rr, "x^{n+1-i}"] \arrow[dr] & & R \arrow[r, "x^i"] & \cdots \\
& & R/x^{n+1-i} \arrow[ur] & & R/x^i \arrow[ur] & &
\end{tikzcd}$$

Hence a triangle $R/x^{n+1-i} \longrightarrow R \longrightarrow R/x^i \longrightarrow R/x^{n+1-i}[1]$ in $\mathbb{D}^b(\mathrm{mod} R)$, hence

$$R/x^i \cong R/x^{n+1-i}[1] \text{ in } \mathbb{D}^b_{sg}(R)$$

We can now complete the earlier table for $f = x^{n+1}$, $R = \mathbb{C}[[x]]/x^{n+1}$

| $\mathrm{hmf}(\mathbb{C}[[x]], x^{n+1}) \cong$ | $\underline{\mathrm{MCM}}(R) \cong$ | $\mathbb{D}^b_{sg}(R)$ | |
|---|---|---|---|
| $Y_i := \begin{pmatrix} 0 & x^i \\ x^{n+1-i} & 0 \end{pmatrix}$ | $R/x^i$ | $R/x^i$ | $1 \leq i \leq n$ |

<u>Example</u> Let us compute Homs in the different categories, $S = \mathbb{C}[[x]]$

$$\mathrm{Hom}_{\mathrm{hmf}}(Y_i, Y_j) = \left\{ \begin{tikzcd} S \arrow[r, "x^i"] \arrow[d, "\varphi"] & S \arrow[r, "x^{n+1-i}"] \arrow[d, "\psi"] & S \arrow[d, "\varphi"] \\ S \arrow[r, "x^j"'] & S \arrow[r, "x^{n+1-j}"'] & S \end{tikzcd} \right\} / \mathrm{htpy}$$

$i \leq j$

$$= \left\{ \varphi, \psi \in \mathbb{C}[[x]] \;\middle|\; \begin{array}{l} x^{n+1-i}\varphi = x^{n+1-j}\psi \\ x^j\varphi = x^i\psi \end{array} \right\} / \mathrm{htpy}$$

$$= \{ \varphi, \psi \in \mathbb{C}[[x]] \mid \psi = x^{j-i}\varphi \} / \mathrm{htpy}$$

where $(\varphi, x^{j-i}\varphi) \sim (\varphi', x^{j-i}\varphi')$ iff. there exist $g, h \in S$ such that $x^{j-i}(\varphi - \varphi') = x^j g + h x^{n+1-i}$, $\varphi - \varphi' = g x^i + x^{n+1-j} h$. Now $j \leq n$ so that $j - i < n+1-i$ and these conditions are equivalent to $\varphi - \varphi' = x^i g + h x^{n+1-j}$ some $g, h$. Hence to

$$\varphi - \varphi' \in (x^i, x^{n+1-j}) = (x^{\min(i, n+1-j)})$$

$$\therefore \; \mathrm{Hom}(Y_i, Y_j) \cong \mathbb{C}[[x]]/(x^{\min(i, n+1-j)})$$