From critical points to A_∞-categories

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Outline

I. From dynamical systems to A_∞-products

II. Constructing A_∞-categories via idempotents

Dynamical systems

A general non-linear dynamical system is given by a system of DEs

$$\left.\begin{aligned} \dot{x}_1 &= F_1(x_1, \dots, x_n) \\ \dot{x}_2 &= F_2(x_1, \dots, x_n) \\ &\vdots \\ \dot{x}_n &= F_n(x_1, \dots, x_n) \end{aligned}\right\} \qquad \begin{aligned} \dot{\underline{x}} &= F(\underline{x}) \\ F &: \mathbb{R}^n \longrightarrow \mathbb{R}^n \end{aligned}$$

An important class of dynamical systems are those which are conservative, in the sense that there is a <u>scalar</u> <u>potential</u> $f : U \longrightarrow \mathbb{R}$ with $U \subseteq \mathbb{R}^n$,

$$F = \nabla f.$$

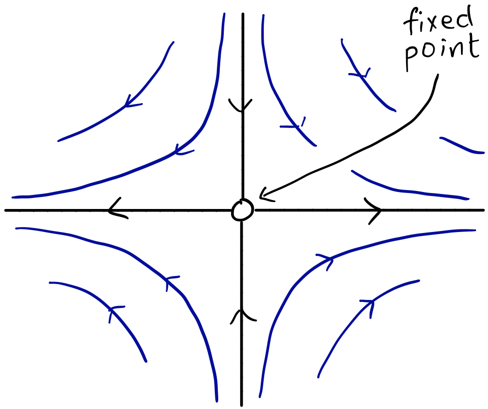
$$\{\text{fixed points of system}\} = \{\text{critical points of } f\}$$

$$\nabla f(\underline{x}) = 0$$

<u>Example</u> Consider the system

$$\dot{x}_1 = x_1 \qquad\qquad \dot{\underline{x}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \underline{x}$$
$$\dot{x}_2 = -x_2$$

Solution trajectories look like $\underline{x}(t) = (Ae^{t}, Be^{-t})$ for any $A, B \in \mathbb{R}$.



Phase portrait

The scalar potential governing this system is

$$f = \tfrac{1}{2}x_1^2 - \tfrac{1}{2}x_2^2$$

$$\nabla f = \begin{pmatrix} x_1 & 0 \\ 0 & -x_2 \end{pmatrix}$$

$$H_f = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Dynamical systems

To understand the dynamics near an isolated critical point of f we need to analyse the Hessian of f, i.e.

$$H_f := \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{1 \leq i,j \leq n},$$

its eigenvectors and eigenvalues. Actually the right way to think of this data is as a symmetric bilinear form on the tangent space $T_{\underline{x}^*}U$ at a critical point $\underline{x}^* \in U$, i.e.

$$\left(T_{\underline{x}^*}U, \langle , \rangle \right) \quad \text{where} \quad \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle = \left. \frac{\partial^2 f}{\partial x_i \partial x_j} \right|_{\underline{x} = \underline{x}^*}$$

with $\underline{u} = \underline{x} - \underline{x}^*$,

$$\underbrace{\dot{\underline{u}} = H_f\big|_{\underline{x}^*} \underline{u}}_{\text{linear system}} + \text{quadratic terms in } \underline{u} \text{ involving higher derivatives of } f$$

<u>Morse Lemma</u> If $H_f\big|_{\underline{x}^*}$ is invertible (i.e. the corresponding bilinear form is nondegenerate) for an isolated critical pt. $\underline{x}^*$ then there is a coordinate neighborhood around $\underline{x}^*$ where

$$f = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_n^2$$

so that in those coordinates

$$H_f\big|_{\underline{x}^*} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{pmatrix}$$
$$\qquad\qquad\qquad\qquad \underbrace{\qquad\quad}_{p} \; \underbrace{\qquad\quad}_{q}$$

<u>Defn</u> A critical point $\underline{x}^*$ is <u>nondegenerate</u> if $H_f\big|_{\underline{x}^*}$ is invertible.

$\therefore$ locally $\qquad \dot{\underline{u}} = H_f\big|_{\underline{x}^*} \underline{u} \qquad\qquad \underline{u} = \underline{x} - \underline{x}^*$

Quadratic spaces

Def^n The category $\mathcal{Q}$ of quadratic spaces over $\mathbb{R}$ has

- objects are f.d. vector spaces equipped with a nondegenerate symmetric bilinear form.
- morphisms $\mathcal{Q}(V,W) = \{ T : V \to W \text{ linear} \mid \langle Tu, Tv\rangle = \langle u,v\rangle \ \forall u,v \}$.

Example

- $X_{p,q} := \left(\mathbb{R}^{\oplus p} \oplus \mathbb{R}^{\oplus q}, \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \right)$ is a representative set of objects
(Sylvester's law of inertia)
- $X_{1,0} = (\mathbb{R}, (1)) \xrightarrow{\binom{1}{0}} \left(\mathbb{R} \oplus \mathbb{R}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) = X_{1,1}$ is a morphism.
- $(T_{\underline{x}^*} U, \langle , \rangle)$ $\quad \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle = \left. \frac{\partial^2 f}{\partial x_i \partial x_j} \right|_{\underline{x} = \underline{x}^*}$ at a nondeg. critical pt. $\underline{x}^*$.

Lemma $\mathcal{Q}$ is a symmetric monoidal category under direct sum of v. spaces.

Clifford algebras

Associated to each quadratic space V is an algebra $C(V)$, the Clifford algebra which is universal among $\mathbb{R}$-algebras C (associative and unital) equipped with a linear map $\iota: V \longrightarrow C$ satisfying

$$\iota(v)\iota(w) + \iota(w)\iota(v) = 2\langle v,w\rangle \cdot 1_C .$$

(so e.g. $\iota(v)^2 = \langle v,v\rangle \cdot 1_C$)

This thing exists, is naturally $\mathbb{Z}_2$-graded, $V \hookrightarrow C(V)^1$ is injective and $C(V)$ is $2^{\dim(V)}$ dimensional.

Examples $\quad C(X_{0,0}) \cong \mathbb{R}, \quad C(X_{0,1}) \cong \mathbb{C}, \quad C(X_{0,2}) \cong \mathbb{H}$

<u>Lemma</u> $C(-)$ is a strong monoidal functor $Q \longrightarrow \mathrm{Alg}_{\mathbb{R}}^{\mathbb{Z}_2}$, i.e. there are natural isomorphisms $C(0) \cong \mathbb{R}$ and

$$C(\underbrace{V \otimes W}_{\text{really direct sum!}}) \cong C(V) \otimes_{\mathbb{R}} C(W).$$

critical point x^* of f $\rightsquigarrow$ quadratic space $(T_{\underline{x}^*}U, Hf|_{\underline{x}^*})$

$\rightsquigarrow$ Clifford algebra $C(T_{\underline{x}^*}U, Hf|_{\underline{x}^*})$

$\rightsquigarrow$ Abelian category $\mathrm{Mod}^{\mathbb{Z}_2} C(T_{\underline{x}^*}U, Hf|_{\underline{x}^*})$.

finite-dimensional $\mathbb{Z}_2$-graded modules

Def^n Nondegenerate isolated critical points form a bicategory $\mathcal{C}rit_{\mathbb{R}}^{ndg}$

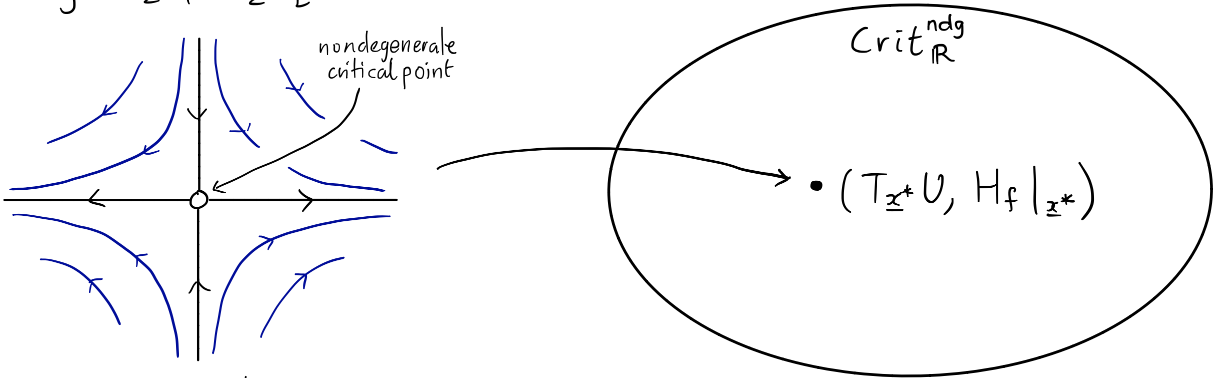
- objects quadratic spaces V
- 1-morphisms $V \longrightarrow W$ are $\mathbb{Z}_2$-graded finite-dimensional $C(W)-C(V)$-bimodules.
- 2-morphisms are bimodule homomorphisms.

Proposition $\mathcal{C}rit_{\mathbb{R}}^{ndg}$ is a symmetric monoidal bicategory in which every object is fully dualisable. (duals for objects and 1-morphisms)

Example

- $\mathcal{C}rit_{\mathbb{R}}^{ndg}(0, V) = \mathrm{Mod}^{\mathbb{Z}_2} C(V)$. $\qquad (0 = X_{0,0} = \mathbb{1})$
- $X_{0,1}^{\otimes 8} \cong \mathbb{1}$ (Bott periodicity)

$$f = \tfrac{1}{2}x_1^2 - \tfrac{1}{2}x_2^2$$



Phase portrait

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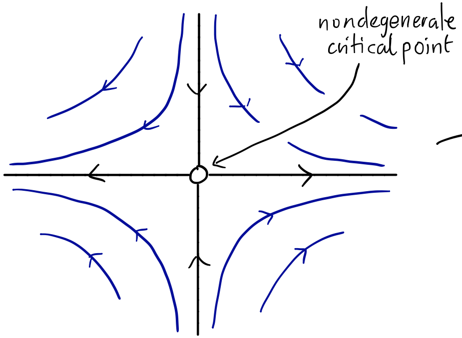

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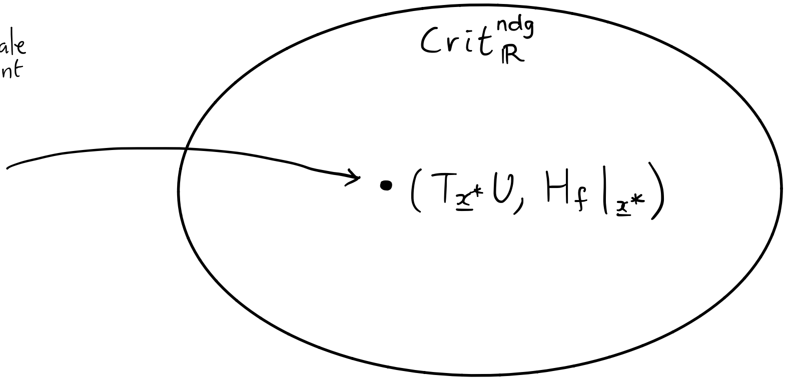
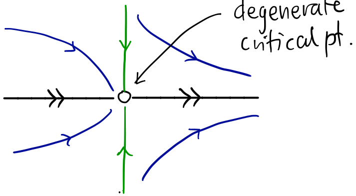


$$f = \tfrac{1}{3}x_1^3 - \tfrac{1}{2}x_2^2$$



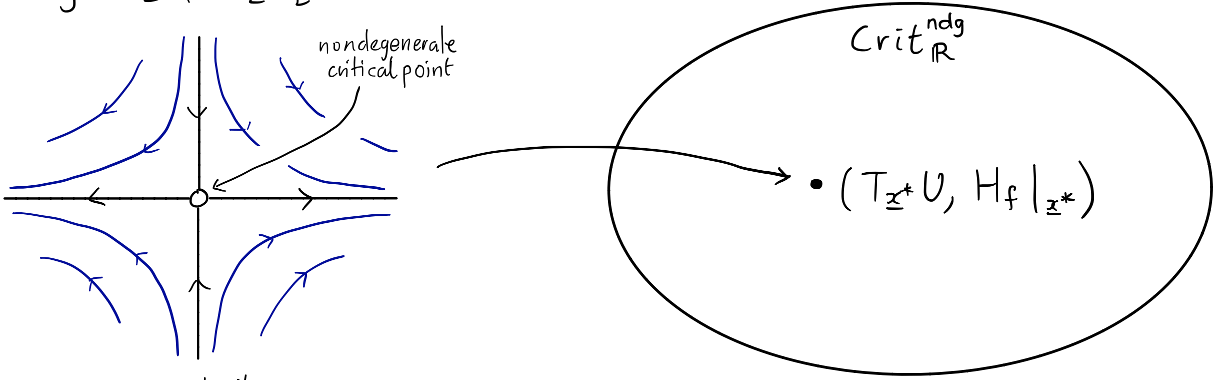
$$\dot{x}_1 = x_1^2$$
$$\dot{x}_2 = -x_2$$

Around an isolated (degenerate) critical point $\underline{x}^*$

$$\underbrace{\dot{\underline{u}} = H_f|_{\underline{x}^*}\,\underline{u}}_{\text{linear system}} + \text{quadratic terms in } \underline{u} \text{ involving higher derivatives of the potential } f.$$

where $\underline{u} = \underline{x} - \underline{x}^*$, the dynamics <u>do</u> depend on the higher derivatives of f.

$$f = \tfrac{1}{2}x_1^2 - \tfrac{1}{2}x_2^2$$



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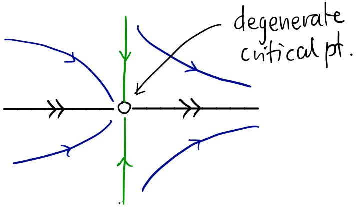

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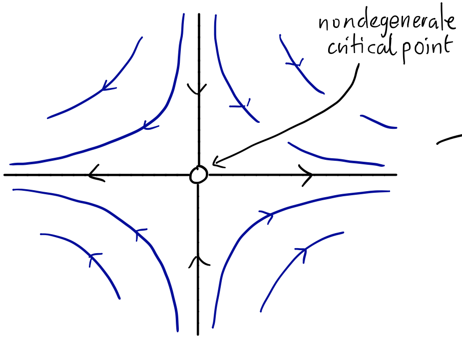
$$\underbrace{\dot{\underline{u}} = H_f|_{\underline{x}^*}\,\underline{u}}_{\text{linear system}} + \text{ quadratic terms in } \underline{u} \text{ involving higher derivatives of the potential } f.$$

<u>Question</u> What algebra to associate to $(f, \underline{x}^*)$?

- reduce to $C(T_{\underline{x}^*}U, H_f|_{\underline{x}^*})$ in the nondeg. case
- form a symmetric monoidal bicategory

$$f = \tfrac{1}{2}x_1^2 - \tfrac{1}{2}x_2^2$$



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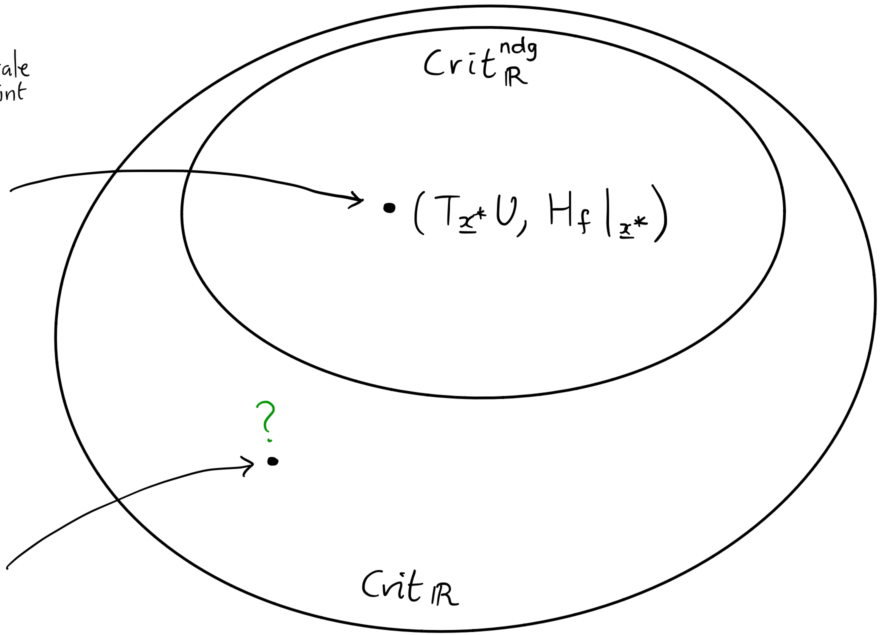

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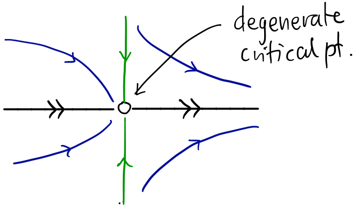

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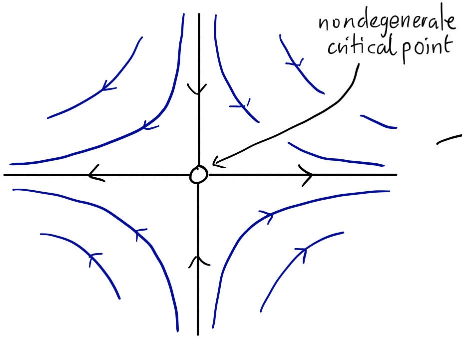

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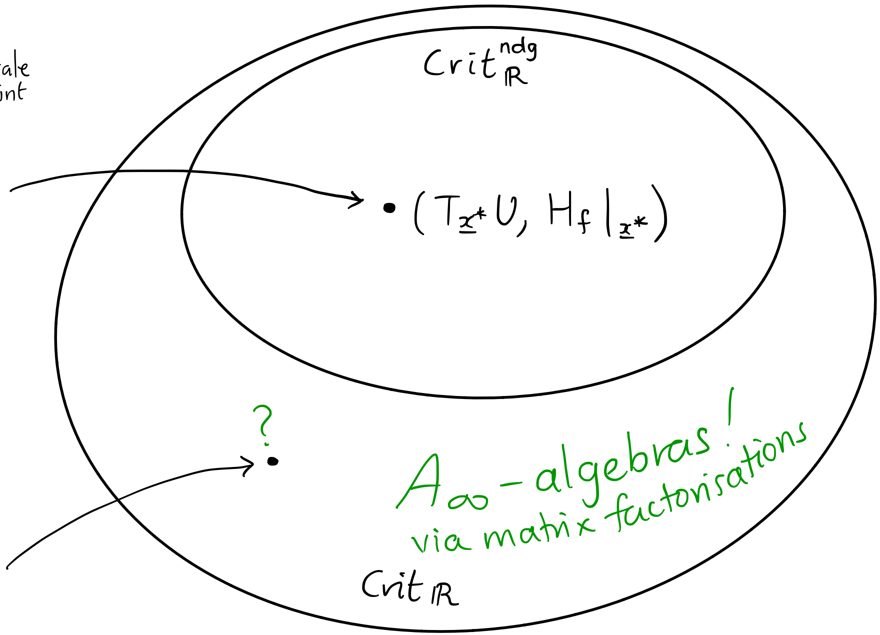

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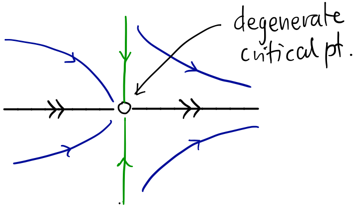

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Matrix factorisations

Let X be a $\mathbb{Z}_2$-graded f.d. module over the Clifford algebra

$C(X_{p,q})$: generated by $\gamma_1, \dots, \gamma_{p+q}$ subject to

$$\gamma_1^2 = \dots = \gamma_p^2 = 1$$
$$\gamma_{p+1}^2 = \dots = \gamma_{p+q}^2 = -1$$
$$\gamma_i \gamma_j + \gamma_j \gamma_i = 0 \qquad i \neq j$$

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Dirac's idea

Set $A = \mathbb{R}[x_1, \dots, x_{p+q}]$, and

$$X \otimes_{\mathbb{R}} A \circlearrowleft \partial = \sum_{i=1}^{n} x_i \gamma_i$$

$\mathbb{Z}_2$-graded free A-module

$$\partial^2 = \sum_{i,j} x_i x_j \gamma_i \gamma_j$$
$$= \sum_i x_i^2 \gamma_i^2$$
$$= \underbrace{x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2}_{\text{acting on } X \otimes_{\mathbb{R}} A}$$

Potentials Let k be a commutative $\mathbb{Q}$-algebra, then $f \in R = k[x_1, \dots, x_n]$ is called a potential if

(i) $\partial_{x_1} f, \dots, \partial_{x_n} f$ is quasi-regular

(ii) $R/(\partial_{x_1} f, \dots, \partial_{x_n} f)$ is a f.g. free k-module

(iii) the Koszul complex of $\partial_{x_1} f, \dots, \partial_{x_n} f$ is exact outside deg. 0.

Example $f \in \mathbb{C}[x_1, \dots, x_n]$ such that $\dim_{\mathbb{C}} \mathbb{C}[x_1, \dots, x_n] / (\partial_{x_1} f, \dots, \partial_{x_n} f) < \infty$.

(isolated critical points)

Def^n The DG-category $\mathcal{A} = mf(R, f)$ has

- **objects** f. rank matrix factorisations of f, i.e. X & $d_X^2 = f \cdot 1_X$.
- **morphisms** $\mathcal{A}(X, Y) = \left(\mathrm{Hom}_R(X, Y),\ \alpha \mapsto d_Y \alpha - (-1)^{|\alpha|} \alpha d_X \right)$.

This is a $\mathbb{Z}_2$-graded DG-category over R.

<u>Remarks</u>

- $hmf(R, f) := H^0 mf(R, f)$ is triangulated (Calabi-Yau).

- Given a quadratic space V with associated quadratic $f \in Sym(V^*)$

$$\mathrm{Mod}^{\mathbb{Z}_2}_{f.d.} C(V) \cong hmf(Sym(V^*), f)^{\omega}$$

(Buchweitz–Eisenbud–Herzog)

Remarks

- $\mathrm{hmf}(R,f) := H^0\mathrm{mf}(R,f)$ is triangulated (Calabi-Yau).

- Given a quadratic space V with associated quadratic $f \in \mathrm{Sym}(V^*)$

$$\mathrm{Mod}^{\mathbb{Z}_2}_{f.d.} C(V) \cong \mathrm{hmf}(\mathrm{Sym}(V^*), f)^\omega$$

(Buchweitz–Eisenbud–Herzog)

From a potential f to an A_∞-algebra A_f

Assume k is a field and $\mathrm{Sing}(f) = \{0\}$. Then there is a standard generator

$$\mathrm{thick}(G) = \mathrm{hmf}(R,f)^\omega$$

$$\mathrm{perf}\,\mathrm{End}_R(G) \cong \mathrm{hmf}(R,f)^\omega \qquad \text{(Keller–Lefevre)}$$

A_∞-transfer (minimal model theorem)

$$\mathrm{perf}_\infty \underbrace{H^*\mathrm{End}_R(G)} \cong \mathrm{hmf}(R,f)^\omega$$

A_∞-algebra A_f, is a Clifford algebra for quadratic f.
A_∞-products package higher derivatives of f.

Let $\mathcal{C}rit_{\mathbb{R}}$ be the bicategory of A_∞-algebras $A_{(f,x^*)}$ associated to isolated critical points and their A_∞-bimodules ($\cong$ the bicategory $\mathcal{LG}_{\mathbb{R}}$).

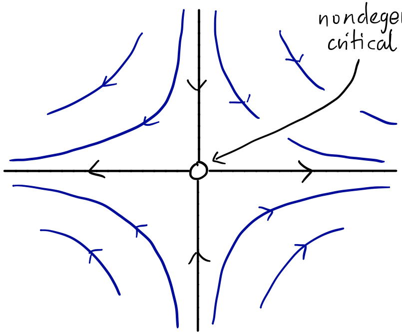
<u>Theorem</u> (Carqueville – Montoya '18) $\mathcal{C}rit_{\mathbb{R}}$ is a symmetric monoidal bicategory in which every object is fully dualisable, and therefore determines an extended 2D framed TFT

$$\mathrm{Bord}^{fr}_{2,1,0} \longrightarrow \mathcal{C}rit_{\mathbb{R}}.$$

Moreover $\mathcal{C}rit^{ndg}_{\mathbb{R}} \subset \mathcal{C}rit_{\mathbb{R}}$.

essentially due to Buchweitz – Eisenbud – Herzog.

$$f = \tfrac{1}{2}x_1^2 - \tfrac{1}{2}x_2^2$$



Phase portrait

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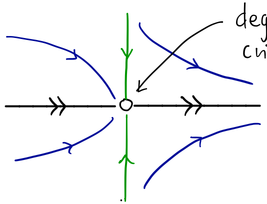

$$\dot{x}_1 = x_1^2$$
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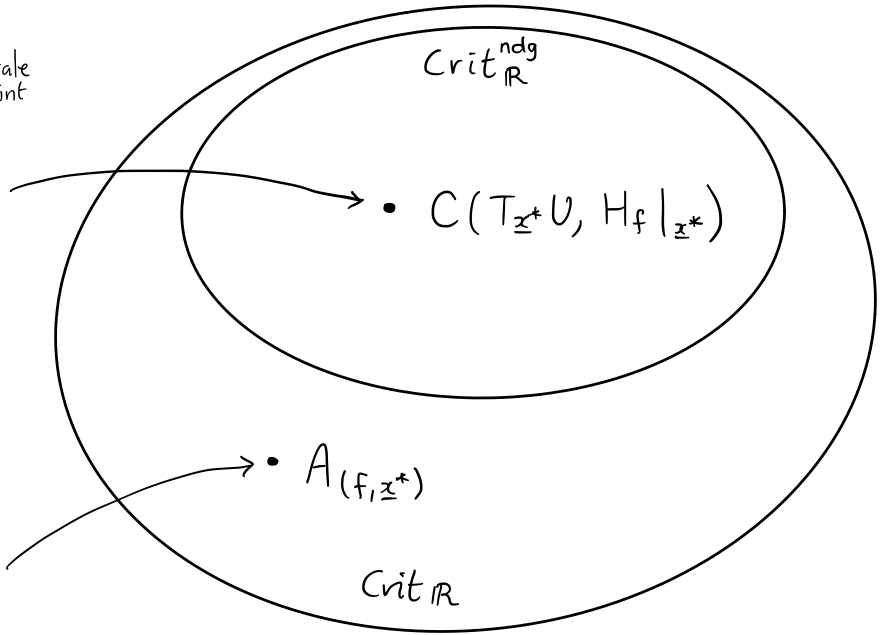

<u>Question</u> What algebra to associate to $(f, \underline{x}^*)$?

- reduce to $C(T_{\underline{x}^*}U, H_f|_{\underline{x}^*})$ in the nondeg. case ✓
- form a symmetric monoidal bicategory ✓

II. Constructing A_∞-categories

Throughout k is a commutative $\mathbb{Q}$-algebra and $W \in R = k[x_1, \ldots, x_n]$ a potential

QUESTION What is the geometric content of the A_∞-products on $\mathrm{hmf}(R, W)$?
(not just the generator)

References

- T. Dyckerhoff, D.M., "Pushing forward matrix factorisations" Duke. J. 2013
- D.M., "The cut operation on matrix factorisations" JPAA 2018.
- D.M., "Constructing A_∞-categories of matrix factorisations" arXiv: 1903.07211. (see also therisingsea.org for working notes).

Preliminaries

Def^n A small $\mathbb{Z}_2$-graded A_∞-category $\mathcal{B}$ over k has a set $ob(\mathcal{B})$ of objects, and $\mathbb{Z}_2$-graded k-modules $\mathcal{B}(a,b)$ for all $a,b \in ob(\mathcal{B})$ equipped with suspended forward compositions which are odd linear maps

$$r_{a_0,\dots,a_n} : \mathcal{B}(a_0,a_1)[1] \otimes \cdots \otimes \mathcal{B}(a_{n-1},a_n)[1] \longrightarrow \mathcal{B}(a_0,a_n)[1]$$

r_n

satisfying the A_∞-constraints (without explicit signs)

$$\sum_{\substack{i \geq 0,\, j \geq 1 \\ 1 \leq i+j \leq n}} r_{a_0,\dots,a_i,a_{i+j},\dots,a_n} \circ \left(id_{a_0 a_1} \otimes \cdots \otimes r_{a_i,\dots,a_{i+j}} \otimes \cdots \otimes id_{a_{n-1},a_n} \right) = 0$$

Example Any $\mathbb{Z}_2$-graded DG-category, $r_n = 0$ for $n \geq 3$.

Finite A_∞-model

Let $\varphi : k \longrightarrow R$ be a morphism of commutative rings, $\mathcal{A}$ a DG-category over R

Restriction of scalars gives a functor

$$
\begin{tikzcd}
A_\infty\text{-cat}(R) \arrow[d, "\varphi_*"] \\
A_\infty\text{-cat}(k)
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{A} \arrow[d, mapsto] & \\
\varphi_*(\mathcal{A}) \arrow[r, shift left, "F"] & \mathcal{B} \arrow[l, shift left, "G"]
\end{tikzcd}
$$

may have $r_1 \neq 0$

<u>Def^n</u> A <u>finite A_∞-model</u> of $\mathcal{A}$ over k is an A_∞-category $\mathcal{B}$ over k with all Hom-spaces f.g. projective$/k$, A_∞-functors F, G and A_∞-homotopies $F \circ G \overset{\infty}{\simeq} 1$, $G \circ F \overset{\infty}{\simeq} 1$.

Minimal A_∞-model

Let $\varphi : k \longrightarrow R$ be a morphism of commutative rings, $\mathcal{A}$ a DG-category over R

Restriction of scalars gives a functor

$$\begin{array}{ccc} A_\infty\text{-cat}(R) & & \mathcal{A} \\ \big\downarrow \varphi_* & & \big\downarrow \\ A_\infty\text{-cat}(k) & & \varphi_*(\mathcal{A}) \underset{G}{\overset{F}{\rightleftarrows}} \left(H^*(\mathcal{A}), \{r_n\}_{n\geq 2}\right) \end{array}$$

<u>Def^n</u> A <u>minimal A_∞-model</u> of $\mathcal{A}$ over k is an A_∞-structure $\{r_n\}_{n\geq 1}$ on $H^*(\mathcal{A})$ with $r_1 = 0$, r_2 induced by composition, and A_∞-functors F, G and A_∞-homotopies $F \circ G \overset{\infty}{\simeq} 1$, $G \circ F \overset{\infty}{\simeq} 1$.

c.f. Remark 1.13 Seidel's book on Fukaya categories.

Idempotent finite A_∞-models

Let $\varphi : k \longrightarrow R$ be a morphism of commutative rings, $\mathcal{A}$ a DG-category over R

Restriction of scalars gives a functor

$$
\begin{tikzcd}
A_\infty\text{-cat}(R) \arrow[d, "\varphi_*"] \\
A_\infty\text{-cat}(k)
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{A} \arrow[d, mapsto] & & \\
\varphi_*(\mathcal{A}) \arrow[rr, shift left, "F"] & & \mathcal{B} \arrow[ll, shift left, "G"] \arrow[loop right, "E"]
\end{tikzcd}
$$

may have $r_1 \neq 0$.

<u>Def^n</u> An <u>idempotent finite A_∞-model</u> of $\mathcal{A}$ over k is an A_∞-category $\mathcal{B}$ with all Hom-spaces f.g. projective$/k$, A_∞-functors F, G, E as above and A_∞-homotopies $F \circ G \overset{\infty}{\simeq} E$, $G \circ F \overset{\infty}{\simeq} 1$. ($E = 1$ gives finite models)

Why finite models?

idempotent finite model	finite model	minimal model
$(\mathcal{B}, E_1, E_2, \ldots, r_1, r_2, r_3, \ldots)$	$(\mathcal{B}, r_1, r_2, r_3, \ldots)$	$(H^*(\mathcal{A}), r_2, r_3, \ldots)$
12 $A_\infty\text{-cat}(k)^\omega$	12	12
$(\mathcal{A}, 1, r_1, r_2)$	$(\mathcal{A}, r_1, r_2)$	$(\mathcal{A}, r_1, r_2)$

- String field theory (A_∞) vs. topological field theory (Δed).
  $(H^*(A), r_2, r_3, \ldots)$ $\qquad$ $(H^*(A), r_2)$

- The information in higher products is important (e.g. for studying moduli).

The question is: which kind of finite model best packages this information?

Physics refs. Lazaroiu (JHEP 2001), Lazaroiu-Roiban (JHEP 2002), Lazaroiu (2006), Carqueville-Dowdy-Recknagel (JHEP 2012), Carqueville-Kay (CMP 2012), Baumgartl-Brunner-Gaberdiel (JHEP 2007), Baumgartl-Wood (JHEP 2009), Knapp-Omer (JHEP 2006).

Melbourne

<u>idempotent finite model</u>

$$(\mathcal{B}, E_1, E_2, \ldots, r_1, r_2, r_3, \ldots)$$

$$\mathrm{I2}$$

$$(\mathcal{A}, 1, r_1, r_2)$$

<u>minimal model</u>

$$(H^*(\mathcal{A}), r_2, r_3, \ldots)$$

$$\mathrm{I2}$$

$$(\mathcal{A}, r_1, r_2)$$

k a field

<u>idempotent finite model</u>

$(\mathcal{B}, E_1, E_2, \dots, r_1, r_2, r_3, \dots)$

$\wr\wr$

$(\mathcal{A}, 1, r_1, r_2)$

<u>minimal model</u>

$(H^*(\mathcal{A}), r_2, r_3, \dots)$

$\wr\wr$

$(\mathcal{A}, r_1, r_2)$

Choose k-linear homotopy equivalences

$$\mathcal{A}(a,b) \underset{g}{\overset{f}{\rightleftarrows}} H^*\mathcal{A}(a,b)$$

$gf = 1 - [d_{\mathcal{A}}, H]$ $\qquad$ $fg = 1$

and transfer A_∞-structure to $H^*(\mathcal{A})$

- useful for special objects (e.g. k^{stab}) (Seidel, Dyckerhoff, Efimov, Sheridan)
- depends on k being a field.

idempotent finite model

$$(\mathcal{B}, E_1, E_2, \ldots, r_1, r_2, r_3, \ldots)$$

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$$(\mathcal{A}, 1, r_1, r_2)$$

- Exists for all of $\mathcal{A} = mf(W)$
- Constructive when Gröbner methods are available (e.g. k a field or poly. ring).
- Downside: not minimal. However, we know TFT formulas (HRR, Kapustin–Li) can be derived directly from $\mathcal{B}$, E_1.
- For special objects can split E.
- Key point: first enlarge $\mathcal{A}$!

minimal model

$$(H^*(\mathcal{A}), r_2, r_3, \ldots)$$

12

$$(\mathcal{A}, r_1, r_2)$$

Choose k-linear homotopy equivalences

$$\mathcal{A}(a,b) \underset{g}{\overset{f}{\rightleftarrows}} H^*\mathcal{A}(a,b)$$

and transfer A_∞-structure to $H^*(\mathcal{A})$

- useful for special objects (e.g. k^{stab})
  (Seidel, Dyckerhoff, Efimov, Sheridan, Tu)
- depends on k being a field.

An idempotent finite A_∞-model of mf

DG $\qquad A = mf(R, W) \qquad R = k[x_1, \dots, x_n] \qquad t_1 = \partial_{x_1} W, \dots, t_n = \partial_{x_n} W$

DG $\qquad A_\theta = \bigwedge F_\theta \otimes_k mf(R, W) \otimes_R \hat{R} \qquad F_\theta = k\theta_1 \oplus \cdots \oplus k\theta_n$

A_∞ TBD $\qquad \mathcal{B} = R/I \otimes_R mf(R, W)$

A_∞-homotopy equivalence $/k$

$G \circ F \simeq 1$

using perturbation

$$
\begin{tikzcd}
A \arrow[r] & A \otimes_R \hat{R} \arrow[r, hook] & A_\theta \arrow[r, shift left, "F"] \arrow[loop below, "e"] & \mathcal{B} \arrow[l, shift left, "G"] \arrow[loop below, "E"]
\end{tikzcd}
$$

homotopy equiv.

$e(\theta_i) = 0$

$E \leftarrow FeG$

<u>Theorem</u> $(\mathcal{B}, E)$ is an idempotent finite A_∞-model of $A \otimes_R \hat{R}$.

Connections and Residues

Let k be a commutative $\mathbb{Q}$-algebra, R a k-algebra, and $t_1, \dots, t_n$ a quasi-regular sequence in R such that R/I is f.g. projective over k, $I = (t_1, \dots, t_n)$.

Lemma (Formal tubular neighbourhood) *Any k-linear section σ of $R \longrightarrow R/I$ induces an isomorphism of $k[[t_1, \dots, t_n]]$-modules*

$$\sigma^* : R/I \otimes_k k[[t_1, \dots, t_n]] \longrightarrow \hat{R}$$

defined by

$$(\sigma^*)^{-1}(r) = \sum_{M \in \mathbb{N}^n} r_M \otimes t^M$$

where the $r_M \in R/I$ are unique such that in $\hat{R}$ we have

$$r = \sum_{M \in \mathbb{N}^n} \sigma(r_M)\, t^M.$$

Connections and Residues

Let k be a commutative $\mathbb{Q}$-algebra, R a k-algebra, and $t_1, \dots, t_n$ a quasi-regular sequence in R such that R/I is f.g. projective over k, $I = (t_1, \dots, t_n)$.

<u>Upshot</u> If k is a field, $R = k[x_1, \dots, x_n]$, σ^* may be computed by Gröbner methods.

In general,

$$\sigma^* : R/I \otimes_k k[[t_1, \dots, t_n]] \xrightarrow{\cong} \hat{R} \qquad (k[[\underline{t}]]\text{-linear})$$

$$\sum_{M \in \mathbb{N}^n} \sigma(r_M)\, t^M = r$$

There is a k-linear connection $\nabla : \hat{R} \longrightarrow \hat{R} \otimes_{k[\underline{t}]} \Omega^1_{k[\underline{t}]/k}$, and

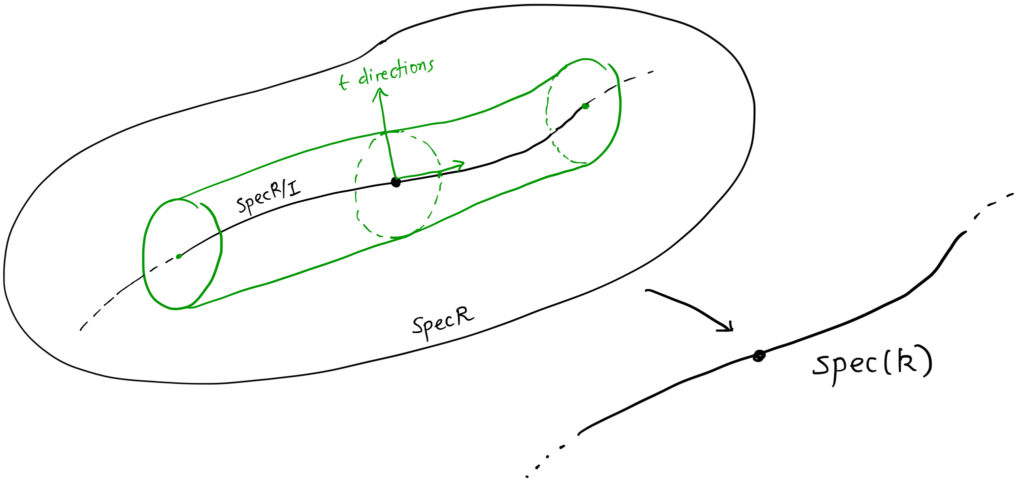
<u>Theorem</u> (Lipman, M) $\operatorname{Res}_{R/k}\begin{bmatrix} r\, dr_1 \cdots dr_n \\ t_1, \dots, t_n \end{bmatrix} = \operatorname{tr}_{R/I}\left(r[\nabla, r_1] \cdots [\nabla, r_n] \right)$

Connections and Residues

$$\sigma^* : R/I \otimes_k k[[t_1,\dots,t_n]] \xrightarrow{\cong} \hat{R}$$

$$\nabla : \hat{R} \longrightarrow \hat{R} \otimes_{k[\underline{t}]} \Omega^1_{k[\underline{t}]/k}$$

$$\mathrm{Res}_{R/k}\begin{bmatrix} \dfrac{r\,dr_1 \cdots dr_n}{t_1,\dots,t_n} \end{bmatrix} = \mathrm{tr}_{R/I}\Big(r[\nabla,r_1]\cdots[\nabla,r_n] \Big)$$



$\mathcal{A} = mf(R,W)$ $\quad R = k[x_1,\dots,x_n]$ $\quad d_{\mathcal{A}}, m_2$

$\mathcal{A}_\theta = \bigwedge F_\theta \otimes_k mf(R,W) \otimes_R \hat{R}$ $\quad d_{\mathcal{A}}, m_2$

$\mathcal{B} = R/I \otimes_R mf(R,W)$ $\quad \overline{d_{\mathcal{A}}}, m_2, m_3, \dots$

$$\mathcal{A} \longrightarrow \mathcal{A} \otimes_R \hat{R} \hookrightarrow \mathcal{A}_\theta \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$$

$$(\mathcal{A}_\theta(X,Y), d_{\mathcal{A}}) \overset{h.e.}{\rightleftarrows} (\mathcal{B}(X,Y), \overline{d_{\mathcal{A}}})$$

transfer A_∞-structure

$$\mathcal{A}_\theta(X,Y) \cong \bigwedge F_\theta \otimes_k \mathcal{B}(X,Y) \otimes_k k[[t_1,\dots,t_n]] \supset \mathcal{B}(X,Y)$$

$$\nabla = \sum_i \theta_i \frac{\partial}{\partial t_i} \qquad \zeta(\omega \otimes \alpha \otimes f) = \frac{1}{|\omega| + |f|}\, \omega \otimes \alpha \otimes f.$$

$$\mathrm{At}_A := [\nabla, d_A]$$

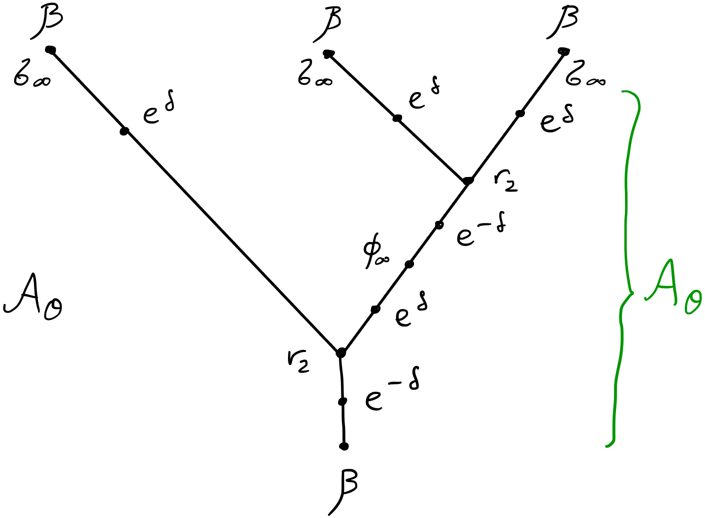
(Atiyah class of A)

$$\sigma_\infty = \sum_{m \geq 0} (-1)^m (\zeta \mathrm{At}_A)^m \sigma \; : \; \mathcal{B} \longrightarrow \mathcal{A}_\theta$$

$$\phi_\infty = \sum_{m \geq 0} (-1)^m (\zeta \mathrm{At}_A)^m \zeta \nabla \; : \; \mathcal{A}_\theta \longrightarrow \mathcal{A}_\theta$$

$$\delta = \sum_{m \geq 0} \lambda_i \theta_i^* \; : \; \mathcal{A}_\theta \longrightarrow \mathcal{A}_\theta$$

At_A, δ rewritten using σ^*



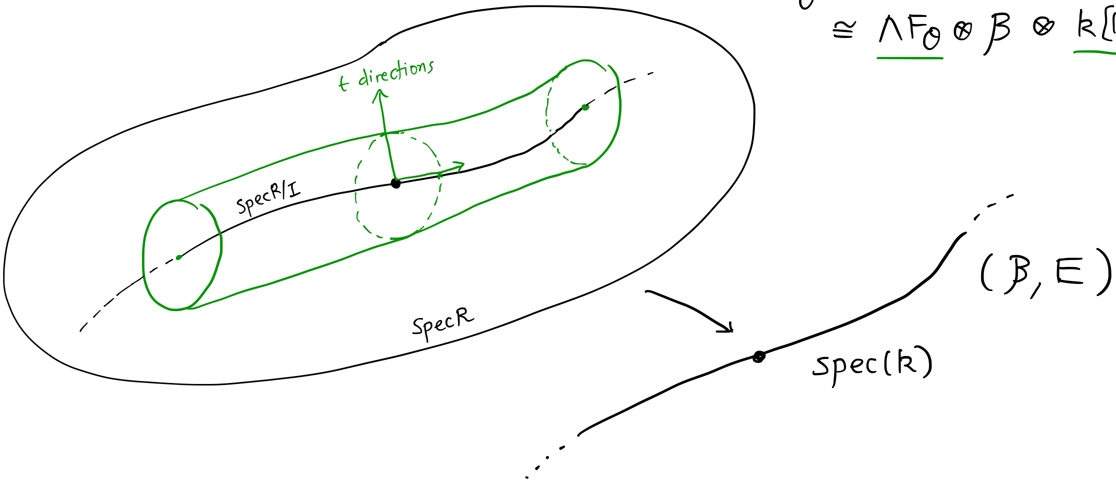
$$r_3 : \mathcal{B}[1]^{\otimes 3} \longrightarrow \mathcal{B}[1]$$

$$\sigma^* : R/I \otimes_k k[[t_1, \dots, t_n]] \xrightarrow{\cong} \hat{R}$$

$$\nabla : \hat{R} \longrightarrow \hat{R} \otimes_{k[[t]]} \Omega^1_{k[[t]]/k}$$

$$\operatorname{Res}_{R/k}\begin{bmatrix} r\, dr_1 \cdots dr_n \\ t_1, \dots, t_n \end{bmatrix} = \operatorname{tr}_{R/I}\Big(r[\nabla, r_1] \cdots [\nabla, r_n] \Big)$$

$$\mathcal{A}_\theta = \Lambda F_\theta \otimes \mathcal{A} \otimes_R \hat{R}$$
$$\cong \underline{\Lambda F_\theta} \otimes \mathcal{B} \otimes \underline{k[[t]]}$$



$$r_m^{\mathcal{B}} = r_m^{\mathcal{B}}\big([\nabla, d_A], \lambda_1, \dots, \lambda_n, \zeta \big)$$

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- P. Seidel, "Homological mirror symmetry for the genus two curve" arXiv: 0812.1171.
- T. Dyckerhoff, "Compact generators in categories of matrix factorisations" Duke Math. J. 2011.
- A. Efimov, "Homological mirror symmetry for curves of higher genus" Adv. Math. 2012.
- N. Sheridan, "Homological mirror symmetry for Calabi-Yau hypersurfaces in projective space" Inventiones 2015.
- D. Shklyarov, "Calabi-Yau structures on categories of matrix factorisations" J. of Geometry and Physics 2017.
- J. Tu, "Categorical Saito theory I: a comparison result" arXiv: 1902.04596.

Prove $\mathcal{J} \cong \mathcal{J}'$ by finding generators G, G' and A_∞-iso $\mathrm{End}(G) \cong \mathrm{End}(G')$.

Some optional slides

Proof sketch

$$\mathcal{A} = mf(R,W) \qquad R = k[x_1,\dots,x_n]$$
$$\mathcal{A}_\theta = \Lambda F_\theta \otimes_k mf(R,W) \otimes_R \hat{R}$$
$$\mathcal{B} = R/I \otimes_R mf(R,W)$$
$$\mathcal{A} \longrightarrow \mathcal{A}\otimes_R \hat{R} \hookrightarrow \mathcal{A}_\theta \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$$

Choose homotopies λ_i such that

$$[d_\mathcal{A}, \lambda_i] = t_i .$$

There is a strict homotopy retraction of complexes over k

$$\begin{tikzcd}
(\mathcal{A}_\theta(X,Y), d_\mathcal{A}) = (\Lambda F_\theta \otimes_k \mathrm{Hom}_R(X,Y) \otimes_R \hat{R}, d_\mathcal{A}) \arrow[d, shift left, "e^{-\delta}"] \\
(\Lambda F_\theta \otimes_k \mathrm{Hom}_R(X,Y) \otimes_R \hat{R}, d_\mathcal{A} + \sum_i t_i \theta_i^*) \arrow[u, shift left, "e^{\delta}"] \arrow[d, shift left, "\pi"] \\
(\mathcal{B}(X,Y), \overline{d_\mathcal{A}}) = (R/I \otimes_R \mathrm{Hom}_R(X,Y), \overline{d_\mathcal{A}}) \arrow[u, shift left, "\sigma_\infty"]
\end{tikzcd}$$

$$\delta = \textstyle\sum_i \lambda_i \theta_i$$

by homological perturbation using connection ∇ $\longrightarrow$ σ_∞

π $\longleftarrow$ canonical projection

Proof sketch

$$\mathcal{A} = mf(R,W) \qquad R = k[x_1,\dots,x_n]$$
$$\mathcal{A}_\theta = \Lambda F_\theta \otimes_k mf(R,W) \otimes_R \hat{R}$$
$$\mathcal{B} = R/I \otimes_R mf(R,W)$$
$$\mathcal{A} \longrightarrow \mathcal{A} \otimes_R \hat{R} \hookrightarrow \mathcal{A}_\theta \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$$

$$\begin{tikzcd}
(\mathcal{A}_\theta(X,Y), d_{\mathcal{A}}) \arrow[d, shift left, "\overset{\Phi}{\pi e^{-\delta}}"] \arrow[d, phantom, "\text{h.e.}"] \\
(\mathcal{B}(X,Y), \overline{d_{\mathcal{A}}}) \arrow[u, shift left, "\overset{\Phi^{-1}}{e^{\delta}\sigma_\infty}"]
\end{tikzcd}$$

$$\Phi \circ \Phi^{-1} = 1, \quad \Phi^{-1} \circ \Phi = 1 - [d_{\mathcal{A}}, H]$$

The A_∞-transfer (minimal model) theorem (Kadeshvili, Merkulov, Kontsevich–Soibelman and for our purposes Markl) constructs A_∞-products on $\mathcal{B}$ and A_∞-homotopy equivalences F, G

$$\mathcal{A}_\theta \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$$

$$F_1 = \Phi, \; G_1 = \Phi^{-1}, \; G \circ F \overset{\infty}{\simeq} 1$$

$r_1^{\mathcal{B}}, r_2^{\mathcal{B}}$ induced from $r_1^{\mathcal{A}}, r_2^{\mathcal{A}}$. $\square$

$$\mathcal{A} = mf(R,W) \qquad R = k[x_1,\dots,x_n]$$
$$\mathcal{A}_\theta = \Lambda F_\theta \otimes_k mf(R,W) \otimes_R \hat{R}$$
$$\mathcal{B} = R/I \otimes_R mf(R,W)$$
$$\mathcal{A} \longrightarrow \mathcal{A}\otimes_R \hat{R} \hookrightarrow \mathcal{A}_\theta \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$$

$$\left(\mathcal{A}_\theta(X,Y),\, d_{\mathcal{A}}\right) \underset{\Phi^{-1}}{\overset{\Phi}{\underset{\text{h.e.}}{\rightleftarrows}}} \left(\mathcal{B}(X,Y),\, \overline{d_{\mathcal{A}}}\right)$$

transfer A_∞-structure

$$\mathcal{A}_\theta(X,Y) = \Lambda F_\theta \otimes_k \mathrm{Hom}_R(X,Y) \otimes_R \hat{R}$$
$$\cong \Lambda F_\theta \otimes_k \mathrm{Hom}_k(\tilde{X},\tilde{Y}) \otimes_k \hat{R}$$

(choose bases for X, Y i.e. $X \cong \tilde{X} \otimes_k R$)

$$\cong \Lambda F_\theta \otimes_k \mathrm{Hom}_k(\tilde{X},\tilde{Y}) \otimes_k R/I \otimes_k k[[t_1,\dots,t_n]]$$
$$\cong \Lambda F_\theta \otimes_k \mathcal{B}(X,Y) \otimes_k k[[t_1,\dots,t_n]] \supset \mathcal{B}(X,Y)$$

$$\nabla = \sum_i \theta_i \frac{\partial}{\partial t_i} \qquad \zeta(\omega \otimes \alpha \otimes f) = \frac{1}{|\omega| + |f|}\, \omega \otimes \alpha \otimes f.$$

Feynman diagrams

Suppose $X = \bigwedge F_\xi \otimes_k R$, $Y = \bigwedge F_\eta \otimes_k R$ are Koszul-type MFs.

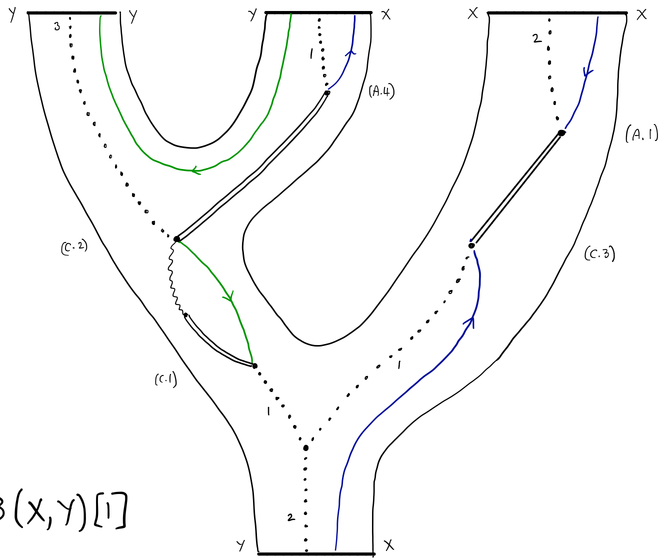
$$\underbrace{\bigwedge\left(F_\theta \oplus F_\xi^* \oplus F_\eta\right) \otimes_k R/I \otimes_k k[\![t]\!]}_{A_\theta(X,Y),\ \text{interior of trees}} \supset \underbrace{\bigwedge\left(F_\xi^* \oplus F_\eta\right) \otimes_k R/I}_{\mathcal{B}(X,Y),\ \text{exterior}}$$

- Apart from ζ all operators involved in computing A_∞-products can be written as polynomials in creation and annihilation operators.

- Feynman diagrams organise reduction of such trees to normal form.

Example One contribution for $W = \frac{1}{5}x^5$ to

$$r_3 : \mathcal{B}(X,X)[1] \otimes \mathcal{B}(X,Y)[1] \otimes \mathcal{B}(Y,Y)[1] \to \mathcal{B}(X,Y)[1]$$



$r_3\left(x^2\xi \otimes x\eta\xi^* \otimes x^3\eta^*\right)$