Matrix factorisations and Quantum Error Correcting Codes

An important conceptual insight from modern logic is that there is a dynamical process "lying behind" function composition. This process is called <u>computation</u>. Taking our cue from the logicians we should, as mathematicians, attempt to find some subatomic structure inside the "atomic" operation $(g,f) \longmapsto g \circ f$. Today I'll explain one approach to doing just that in the context of hypersurface singularities.

1. Matrix factorisations

Let k be a commutative ring. A polynomial $W \in k[\underline{x}] = k[x_1, \dots, x_n]$ is a <u>potential</u> if

(i) $\partial_{x_1} W, \dots, \partial_{x_n} W$ is a quasi-regular sequence

(ii) $k[\underline{x}]/(\partial_{x_1} W, \dots, \partial_{x_n} W)$ is a f.g. free k-module

(iii) $H^i(K(\partial_{x_1} W, \dots, \partial_{x_n} W)) = 0 \quad i \neq 0$ (note (iii) $\Rightarrow$ (i))

<u>Example</u> $W \in \mathbb{C}[x_1, \dots, x_n]$ with $\dim\left(\mathbb{C}[\underline{x}]/(\partial_{x_1} W, \dots, \partial_{x_n} W)\right) < \infty$,
e.g. any isolated hypersurface singularity.

<u>Remark</u> If $(k[\underline{x}], W), (k[\underline{y}], V)$ are potentials so are $(k[\underline{x}], -W), (k[\underline{x}, \underline{y}], W + V)$.

<u>Def^n</u> (Eisenbud) A <u>matrix factorisation</u> (MF) of $W \in k[\underline{x}]$ is a $\mathbb{Z}_2$-graded free $k[\underline{x}]$-module (possibly infinite rank) $X = X^0 \oplus X^1$ and a $k[\underline{x}]$-linear odd map $d_X : X \longrightarrow X$ such that $d_X^2 = W \cdot 1_X$

$$d_X = \begin{array}{cc} X^0 & X^1 \\ \end{array} \hspace{-4em} \\ d_X = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \qquad AB = BA = W \cdot I$$

A <u>morphism</u> $f : (X, d_X) \longrightarrow (Y, d_Y)$ is a $k[\underline{x}]$-linear even map s.t. $d_Y f = f d_X$.

There are some triangulated categories associated to a potential

$$\begin{tikzcd}
\mathcal{T} = \mathrm{HMF}(k[\underline{x}], W) & \\
\mathcal{T}^c = \mathrm{hmf}(k[\underline{x}], W)^{\oplus} \arrow[u, hook] & \\
& \mathcal{S}^{\omega} \arrow[ul, "\cong"'] \\
\mathcal{S} = \mathrm{hmf}(k[\underline{x}], W) \arrow[uu, hook] \arrow[ur] &
\end{tikzcd}$$

$\mathcal{T}$: Homotopy category of <u>all</u> MFs

$\mathcal{T}^c$: Full subcategory of direct summands of $Y \in \mathrm{hmf}$

$\mathcal{S}^{\omega}$: Karoubi completion

$\mathcal{S}$: Homotopy category of f. rank MFs

<u>Def^n</u> The bicategory $\mathcal{LG}_k$ has potentials as objects and

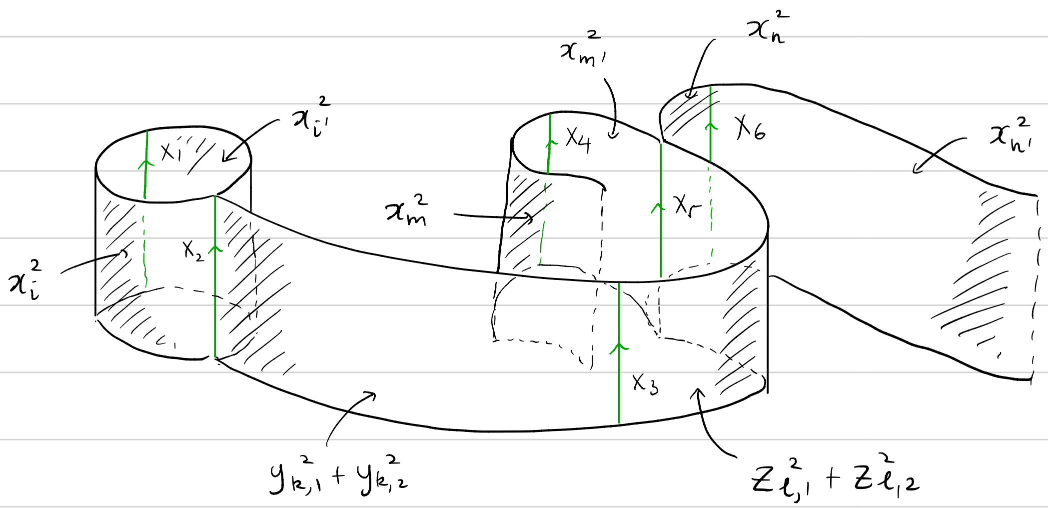
[studied by Lazaroiu-McNamee, Khovanov-Rozansky]

$$\mathcal{LG}_k\big((k[\underline{x}], W), (k[\underline{y}], V) \big) := \mathrm{hmf}(k[\underline{x}, \underline{y}], V - W)^{\oplus}$$

<u>Theorem</u> The bicategory $\mathcal{LG}_k$

- has adjoints for 1-morphisms (Carqueville-Murfet '16)
- is symmetric monoidal with duals (Carqueville-Montoya '18)
- is a pivotal superbicategory (Godfrey-Murfet '18, '22)

This means $\mathcal{LG}_k$ determines a 2D TQFT of a kind which can evaluated bordisms like:



2. Composition

What has this got to do with "subatomic" structure in composition? Like any bicategory $\mathcal{LG}_k$ has composition $(Y, X) \longmapsto Y \circ X$ for 1-morphisms. Given potentials $W(\underline{x}), V(\underline{y}), U(\underline{z})$

$$\begin{tikzcd}
\overset{(Y,X)}{\mathrm{hmf}(k[\underline{y},\underline{z}], U-V) \times \mathrm{hmf}(k[\underline{x},\underline{y}], V-W)} \arrow[r, "?"] \arrow[d] \arrow[dr, dashed] & \mathrm{hmf}(k[\underline{x},\underline{z}], U-W) \arrow[d] \\
\underset{\mathcal{LG}(V,U) \quad\quad \mathcal{LG}(W,V)}{\mathrm{hmf}(k[\underline{y},\underline{z}], U-V)^{\oplus} \times \mathrm{hmf}(k[\underline{x},\underline{y}], V-W)^{\oplus}} \arrow[d] & \underset{\mathcal{LG}(W,U)}{\mathrm{hmf}(k[\underline{x},\underline{z}], U-W)^{\oplus}} \arrow[d] \\
\mathrm{HMF}(k[\underline{y},\underline{z}], U-V) \times \mathrm{HMF}(k[\underline{x},\underline{y}], V-W) \arrow[r] & \mathrm{HMF}(k[\underline{x},\underline{z}], U-W)
\end{tikzcd}$$

$$(Y, X) \qquad\qquad (Y \otimes_{k[\underline{y}]} X,\ d_Y \otimes 1 + 1 \otimes d_X) = Y \otimes X$$

We see that $(Y, X) \longmapsto Y \circ X$ is not so simple, if we want a finite-rank answer. It involves

1. Write down $Y \otimes X$

2. Invert the equivalence

$$\begin{array}{ccc}
\mathrm{hmf}(k[\underline{x},\underline{z}], U-W)^{\omega} & \xrightarrow{\ \simeq\ } & \mathrm{hmf}(k[\underline{x},\underline{z}], U-W)^{\oplus} \\
\Psi & & \Psi \\
(Z, e) & & Y \otimes X
\end{array}$$

3. Try to split e in finite rank MFs

$$\begin{tikzcd}
Z \arrow[loop left, "e"] \arrow[r, shift left, "F"] \arrow[d, shift right, "f"'] & Y \otimes X \arrow[l, shift left, "G"] \arrow[dl, dashed, "\cong"] \\
Y \circ X \arrow[u, shift right, "g"'] &
\end{tikzcd}$$

$GF = e, \ FG = 1$

$gf = e, \ fg = 1.$

hmf | HMF

What is this?

3. Cut

A *supercategory* is a category $\mathcal{T}$ and functor $F : \mathcal{T} \longrightarrow \mathcal{T}$ with $F^2 \cong \mathrm{id}$ satisfying some axioms. Any hmf or HMF is a supercategory with $F(X) = X[1]$. For $n \geq 0$ let C_n denote the Clifford algebra generated by $\gamma_1, \dots, \gamma_n, \gamma_1^\dagger, \dots, \gamma_n^\dagger$ of odd degree s.t.

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 0, \quad \gamma_i^\dagger\gamma_j^\dagger + \gamma_j^\dagger\gamma_i^\dagger = 0, \quad \gamma_i\gamma_j^\dagger + \gamma_j^\dagger\gamma_i = \delta_{ij}.$$

Example (i) $S_n := \bigwedge(k\theta_1 \oplus \cdots \oplus k\theta_n)$ is a left C_n-module, γ_i acts as $\theta_i^* \lrcorner (-)$, $\gamma_i^\dagger$ acts as $\theta_i \wedge (-)$. Also $C_n \cong \mathrm{End}_k(S_n)$.

(ii) $S_{m,n} := S_m \otimes_k S_n^*$ is a C_m-C_n-bimodule

Def^n Let $\mathcal{T}$ be an idempotent complete supercategory. The *Clifford thickening* $\mathcal{T}^\bullet$ of $\mathcal{T}$ has as objects tuples (X, n, ρ) consisting of $X \in \mathrm{ob}(\mathcal{T})$ and a left C_n-module structure $\rho : C_n \otimes_k X \longrightarrow X$. A morphism $(X, n, \rho) \longrightarrow (Y, m, \tau)$ is a morphism of C_m-modules $S_{m,n} \otimes_{C_n} X \longrightarrow Y$.

Lemma $\mathcal{T} \xrightarrow{\cong} \mathcal{T}^\bullet$, $X \longmapsto (X, 0, \mathrm{id})$.

Proof C_n is Morita trivial, so if $(X, n, \rho) \in \mathcal{T}^\bullet$ there exists $\tilde{X} \in \mathcal{T}$ with $X \cong S_n \otimes_k \tilde{X}$ as C_n-modules. Then $(X, n, \rho) \cong (\tilde{X}, 0, \mathrm{id})$ in $\mathcal{T}^\bullet$ as

$$(\tilde{X}, 0, \mathrm{id}) \longrightarrow (X, n, \rho)$$

is $\quad S_n \otimes_k \tilde{X} \longrightarrow X$. $\square$

An object X of $\mathcal{T}^\bullet$ specifies an object $\tilde{X}$ of $\mathcal{T}$ *indirectly* via $\{\gamma_i, \gamma_i^\dagger\}_{i=1}^n$. You may need to *compute* in order to actually extract $\tilde{X}$.

Def$^{\underline{n}}$ The superbicategory $\mathcal{C}$ has the same objects as $\mathcal{LG}$ and

$$\mathcal{C}\big((k[\underline{x}], W), (k[\underline{y}], V)\big) := \big(\underbrace{hmf(k[\underline{x},\underline{y}], V - W)^{\omega}}_{\text{finite rank MF + structure}}\big)^{\bullet}$$

with an <u>explicit</u> composition

$$\mathcal{C}(V, U) \times \mathcal{C}(W, V) \longrightarrow \mathcal{C}(W, U) \qquad I = (\partial_{y_1} V, \dots, \partial_{y_m} V)$$

$$(Y, X) \longmapsto \big(Y|X := {}^{Y \otimes_{k[\underline{y}]} X} \big/ {}_{I (Y \otimes_{k[\underline{y}]} X)}, \ \{\gamma_i, \gamma_i^{\dagger}\}_{i=1}^{n} \big)$$

we call this the <u>cut</u> operation

$$\gamma_i = [d_Y + d_X, \partial_{\partial y_i V}] =: At_i$$

$$\gamma_i^{\dagger} = -\partial_{y_i}(d_X) - \tfrac{1}{2} \sum_q \partial_{y_q} \partial_{y_i}(V) \, At_q .$$

<u>Theorem</u> There are isomorphisms of Clifford modules (M '18)

$$\underset{\circlearrowleft \gamma_i, \gamma_i^{\dagger}}{Y|X} \cong \underset{\circlearrowleft \theta_i^{*}, \theta_i}{\Lambda(k\theta_1 \oplus \cdots \oplus k\theta_m)} \otimes_k (Y \otimes_{k[\underline{y}]} X)$$

and an equivalence $\mathcal{LG} \cong \mathcal{C}$.

$$\begin{array}{ccc} & & hmf(V-W) \\ & \swarrow & \downarrow \\ hmf^{\omega}(V-W) & \xrightarrow{\cong} & hmf^{\oplus}(V-W) \\ \cong \downarrow & & \| \\ \mathcal{C}(W,V) := hmf^{\omega}(V-W)^{\bullet} & \dashleftarrow & \mathcal{LG}(W,V) \end{array}$$

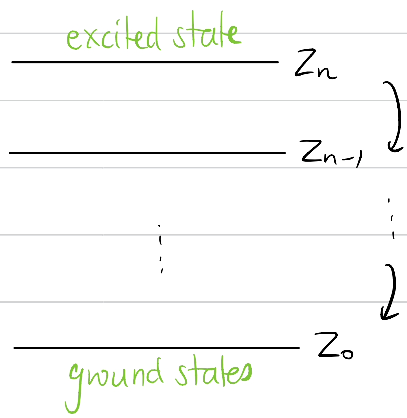
$$e = \gamma_1 \cdots \gamma_n \gamma_n^{\dagger} \cdots \gamma_1^{\dagger}$$

$$\circlearrowleft \ Y|X \rightleftarrows Y \otimes X$$

$$Y|X \cong Z_n, \qquad \gamma_1 \cdots \gamma_{n-1} \gamma_{n-1}^{\dagger} \cdots \gamma_1^{\dagger} \circlearrowleft Z_{n-1}, \quad \vdots, \quad Y \circ X \cong Z_0$$

hmf | HMF

Inside the composition is a picture of pumping energy out of a fermionic system:



4. Quantum codes

To ascertain how seriously we should take the analogy to pumping energy out of a physical system, we can consider the simplest possible example: composition of a chain of identity 1-cells on $(\mathbb{C}[x], x^2) \in \mathcal{LG}_{\mathbb{C}}$

$$x_n^2 \xleftarrow{\Delta_n} x_{n-1}^2 \longleftarrow \cdots \quad x_2^2 \xleftarrow{\Delta_2} x_1^2$$

$$\Delta_i \in \mathrm{hmf}(\mathbb{C}[x_i, x_{i-1}], x_i^2 - x_{i-1}^2)$$

$$\Delta_i = \Lambda(\mathbb{C}\psi_i) \otimes_{\mathbb{C}} \mathbb{C}[\underline{x}] = \mathbb{C}[\underline{x}]1 \oplus \mathbb{C}[\underline{x}]\psi_i$$

$$d_{\Delta_i} = (x_i - x_{i-1})\psi_i^* + (x_i + x_{i-1})\psi_i$$

$$= \begin{array}{cc} 1 & \psi_i \end{array}$$
$$\begin{pmatrix} 0 & x_i - x_{i-1} \\ x_i + x_{i-1} & 0 \end{pmatrix}$$

We have, since $\mathbb{C}[x]/(\partial_x(x^2)) = \mathbb{C}$

$$\underset{\gamma, \gamma^\dagger}{\Delta_n \mid} \Delta_{n-1} \mid \cdots \mid \Delta_2 = \Delta_n \otimes_{\mathbb{C}[x_{n-1}]} \mathbb{C} \otimes_{\mathbb{C}[x_{n-1}]} \Delta_{n-1} \otimes \cdots$$
$$\cdots \otimes_{\mathbb{C}[x_2]} \mathbb{C} \otimes_{\mathbb{C}[x_2]} \Delta_2$$

$$\cong \mathbb{C}[x_n, x_1] \otimes_{\mathbb{C}} \left(\Lambda k\psi_n \otimes_{\mathbb{C}} \Lambda k\psi_{n-1} \otimes \cdots \otimes \Lambda k\psi_2 \right)$$

$$\cong \mathbb{C}[x_n, x_1] \otimes_{\mathbb{C}} \Lambda(k\psi_n \oplus \cdots \oplus k\psi_2)$$

For each cut we have Clifford operators $\gamma, \gamma^\dagger$. For the Δ_n / Δ_{n-1} cut

$$\gamma = -\tfrac{1}{2}(\psi_{n-1}^* + \psi_{n-1} + \psi_n - \psi_n^*)$$
$$\gamma^\dagger = -\tfrac{1}{2}(\psi_{n-1}^* + \psi_{n-1} - \psi_n - \psi_n^*)$$

notice these are $\underline{x}$-free

meaning $\psi_{n-1} \wedge (-)$

meaning $\psi_{n-1}^* \lrcorner (-)$

According to our earlier prescription the process of computing $\Delta_n \circ \cdots \circ \Delta_2$ is

$$\gamma, \gamma^\dagger \circlearrowleft \Delta_n \,|\, \Delta_{n-1} \,|\, \cdots \,|\, \Delta_2 = Z_n$$

$$\uparrow\downarrow$$

$$\mathrm{Im}(\gamma\gamma^\dagger) = \mathrm{Ker}(\gamma) = \mathrm{Im}(\gamma^\dagger) = Z_{n-1} \tag{7.1}$$

$$\uparrow\downarrow$$

$$\vdots \quad \vdots$$

$$\Delta_n \circ \cdots \circ \Delta_2 \quad (= \Delta \text{ again})$$

So what is $\mathrm{Ker}(\gamma)$?

$$-2\gamma = \psi_{n-1}^* + \psi_{n-1} + \psi_n - \psi_n^* \circlearrowleft \Lambda k\psi_n \otimes \Lambda k\psi_{n-1}$$

$$\begin{aligned} \therefore \quad -2\gamma(1 + \psi_n\psi_{n-1}) &= (\psi_{n-1} + \psi_n)(1) + (\psi_{n-1}^* - \psi_n^*)(\psi_n\psi_{n-1}) \\ &= \psi_{n-1} + \psi_n + [-\psi_n - \psi_{n-1}] = 0 \qquad (*) \end{aligned}$$

$$-2\gamma(\psi_n + \psi_{n-1}) = 1 + \psi_{n-1}\psi_n + \psi_n\psi_{n-1} - 1 = 0$$

By dimension count $\dim \mathrm{Ker}(\gamma \circlearrowleft \Lambda k\psi_n \otimes \Lambda k\psi_{n-1}) = 2$ so this is it. But what is going on here? And how do we continue this to compute $\Delta_n \circ \cdots \circ \Delta_2$? Physicists recognise $(*)$ as Bell states or maximally entangled states. They write $\Lambda \mathbb{C}\psi = \mathbb{C}1 \oplus \mathbb{C}\psi = \mathbb{C}|0\rangle \oplus \mathbb{C}|1\rangle = \mathbb{C}^2$ and

$$\begin{aligned} 1 + \psi_n\psi_{n-1} \quad &\text{as} \quad |00\rangle + |11\rangle \\ \psi_n + \psi_{n-1} \quad &\text{as} \quad |10\rangle + |01\rangle \end{aligned}$$

Computing $\Delta_n \circ \Delta_{n-1}$ from $\Delta_n | \Delta_{n-1}$ consists in projecting onto these entangled states.

To cut a long story short, the procedure (7.1) rediscovers the error correction process for a particular kind of quantum error correcting code on $\mathcal{H} = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$ ($n-1$ copies) called a <u>stabiliser code</u>. With $X|0\rangle = |1\rangle$, $X|1\rangle = |0\rangle$ and $X_i := 1 \otimes \cdots \otimes \underset{i}{X} \otimes \cdots \otimes 1$ the code is

$$S = \{ X_n X_{n-1}, X_{n-1}X_{n-2}, \ldots, X_3 X_2 \} \subseteq \mathrm{Aut}_{\mathbb{C}}(\mathcal{H})$$

and the <u>codespace</u> C is the joint $+1$-eigenspace of all operators in S. This code is sometimes called a <u>quantum wire</u> (or Majorana chain) and it is closely related to the Ising model.

<u>Lemma</u> The idempotent e computing $\Delta_n \circ \cdots \circ \Delta_1$ from $\Delta_n / \cdots / \Delta_1$ is

$$P = \prod_{i=2}^{n-1} \frac{1 + X_{i+1} X_i}{2} \qquad \gamma\gamma^\dagger = \frac{1 + X_n X_{n-1}}{2}$$

the standard projector for the stabiliser code, hence

$$\Delta_n \circ \cdots \circ \Delta_1 \cong \mathbb{C}[x_n, x_1] \otimes_{\mathbb{C}} C \quad \text{codespace}$$

where a $\mathbb{C}$-basis for C is given by the entangled states

$$|+ \cdots +\rangle \pm |- \cdots -\rangle \qquad |\pm\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$$

To return to our original theme: the <u>cut operation</u> $(Y,X) \longmapsto Y/X$ followed by "eliminating" the Clifford action is our answer to the "subatomic" structure lying inside the atomic operation $(Y,X) \longmapsto Y \circ X$. In the simplest case this elimination process is identical with the process of error-correction in a particular well-known stabiliser code, which can be viewed as "pumping out entropy".

We expect that following this recipe for equivariant MFs will be a new source of error correcting codes.

<u>Theorem</u> Suppose $G \curvearrowright k[\underline{y}] = k[y_1, \dots, y_m]$ is a finite group acting so V is G-invariant and $g \cdot \partial y_i V \in \mathrm{span}_k\{\partial y_j V\}_{j=1}^{m}$.
Let F_θ be the $\mathbb{Z}_2$-graded G-rep $(I = (\partial y_1 V, \dots, \partial y_m V))$

$$F_\theta \otimes_k k[\underline{y}]/I \cong I/I^2[1]$$

↑ conormal bundle Crit$W$

Then for G-equivariant Y, X

$$(Y|X)^G \cong \left[\Lambda F_\theta \otimes_k (Y \otimes_{k[\underline{y}]} X)\right]^G$$

$$\cup$$

$$(Y \otimes_{k[\underline{y}]} X)^G$$